Sturmian theory for abnormal linear Hamiltonian systems

Roman Šimon Hilscher

Brno, Czech Republic

Sturmian theory for differential equations is a classical topic in the literature with active research and generalizations in various directions. Classical results of this type, the Sturmian comparison and separation theorems, relate the number of zeros or the focal points of two solutions of one differential equation or two different differential equations. For higher order Sturm–Liouville differential equations or systems, the theory has been established in [3, 6]. Here we consider the linear Hamiltonian system

$$x' = A(t)\,x + B(t)\,u, \quad u' = C(t)\,x - A^T(t)\,u, \quad t \in [a,b]. \tag{H}$$

where $n \in \mathbb{N}$ is given a dimension, $[a,b]$ with $a < b$ is a fixed real interval, and the coefficients $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ are real piecewise continuous $n \times n$ matrix functions on $[a,b]$ such that $B(t)$ and $C(t)$ are symmetric.

A crucial assumption in the existing Sturmian theory for system (H) is the complete controllability or equivalently identical normality condition:

$$\left.\begin{array}{l}\text{if } (x(\cdot) \equiv 0, u) \text{ is a solution of system (H) on a nondegenerate} \\ \text{subinterval of } [a,b], \text{ then also } u(\cdot) \equiv 0.\end{array}\right\} \tag{1}$$

If the Legendre condition

$$B(t) \geq 0 \quad \text{for all } t \in [a,b] \tag{2}$$

is satisfied, then it is known in [3, Theorem 4.1.3] that the controllability condition (1) is equivalent to the fact that conjoined bases of system (H) have $X(t)$ invertible on $[a,b]$ except possibly at isolated points $t_0 \in [a,b]$. Such an isolated point t_0 where $X(t_0)$ is singular is then called to be a focal point of (X,U), and the defect of $X(t_0)$, i.e., the number $m(t_0) := \operatorname{def} X(t_0) = \dim \operatorname{Ker} X(t_0)$, is its multiplicity. Conditions (1) and (2) then yield through this theorem that every conjoined basis of (H) has finitely many focal points in the interval $[a,b]$. The classical Sturmian separation theorem for controllable Hamiltonian systems (H) says that the numbers of focal points of any two conjoined bases of (H) differ by at most n, where each of the focal points is counted according to its multiplicity, see e.g. [6, Corollary 1, pg. 366].

Following the discrete time results in [1, 2], we develop the Sturmian theory for systems (H) when the controllability assumption (1) is removed. In this case, conjoined bases of (H) may have $X(t)$ singular on a nondegenerate interval or even throughout $[a,b]$. Therefore, the above traditional focal point notion is not suitable for the Sturmian theory in the abnormal case. The study of abnormal linear Hamiltonian systems has been revived with the paper [4] by W. Kratz, in which it is shown that under the Legendre condition (2) the kernel of $X(\cdot)$ is piecewise constant on $[a,b]$. In this case the Moore–Penrose generalized inverse

$X^\dagger(\cdot)$ of $X(\cdot)$ should be used to develop the theory. In particular, the matrix function $X^\dagger(\cdot)$ is differentiable on intervals where the kernel of $X(\cdot)$ is constant. The notion of a proper focal point and its multiplicity for the abnormal case was introduced in [8]. Under condition (2), a point $t_0 \in (a, b]$ is called a proper focal point of a conjoined basis (X, U) of (H) if $\operatorname{Ker} X(t_0^-) \subsetneqq \operatorname{Ker} X(t_0)$, and then the number $m(t_0) := \operatorname{def} X(t_0) - \operatorname{def} X(t_0^-) = \dim \big([\operatorname{Ker} X(t_0^-)]^\perp \cap \operatorname{Ker} X(t_0) \big)$ is its multiplicity. If the system (H) is completely controllable and condition (2) holds, then the above notion reduces to the traditional multiplicity of a focal point.

Quoting the results from [7], we establish generalizations of the classical Sturmian separation and comparison theorems to possibly abnormal linear Hamiltonian systems. As an example of a general result we give the following theorem, which can be found under assumption (1) in [6, Corollary 1, pg. 366].

Theorem 1 (Sturmian separation theorem)**.** *Assume* (2). *Then the difference between the numbers of proper proper focal points in* $(a, b]$ *of any two conjoined bases of* (H) *is at most* n.

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References

[1] M. Bohner, O. Došlý, W. Kratz, *Sturmian and spectral theory for discrete symplectic systems*, Trans. Amer. Math. Soc. **361** (2009), no. 6, 3109–3123.

[2] O. Došlý, W. Kratz, *A Sturmian separation theorem for symplectic difference systems*, J. Math. Anal. Appl. **325** (2007), no. 1, 333–341.

[3] W. Kratz, *Quadratic Functionals in Variational Analysis and Control Theory*, Akademie Verlag, Berlin, 1995.

[4] W. Kratz, *Definiteness of quadratic functionals*, Analysis (Munich) **23** (2003), no. 2, 163–183.

[5] W. Kratz, R. Šimon Hilscher, *Rayleigh principle for linear Hamiltonian systems without controllability*, submitted (2010).

[6] W. T. Reid, *Ordinary Differential Equations*, Wiley, New York, 1971.

[7] R. Šimon Hilscher, *Sturmian theory for linear Hamiltonian systems without controllability*, submitted (2010).

[8] M. Wahrheit, *Eigenvalue problems and oscillation of linear Hamiltonian systems*, Internat. J. Difference Equ. **2** (2007), no. 2, 221–244.