Discontinuous differential equations with deviated arguments depending on the unknown

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We will present a coupled fixed points theorem for a meaningful class of mixed monotone multivalued operators and then we use it to derive some results on existence of quasisolutions and unique solutions to first-order functional differential equations with state-dependent deviating arguments.

Definition 1. A partially ordered metric space X is an ordered metric space if the intervals

$$[x) = \{y \in X \, : \, x \leq y\}, \; (x] = \{y \in X \, : \, y \leq x\}$$

are closed for every $x \in X$.

Definition 2. Let P be a subset of an ordered metric space. An operator $A : P \times P \longrightarrow P$ is mixed monotone if $A(\cdot, z)$ is nondecreasing and $A(z, \cdot)$ is nonincreasing for each $z \in P$.

We say that A satisfies the mixed monotone convergence property if $(A(v_j, w_j))_{j=1}^{\infty}$ converges whenever $(v_j)_{j=1}^{\infty}$ and $(w_j)_{j=1}^{\infty}$ are sequences in P, one being nondecreasing and the other nonincreasing.

Definition 3. Let $\mathcal{A} : P \times P \longrightarrow 2^P \backslash \emptyset$ be a multivalued operator. We say that two elements $v, w \in P$ are coupled fixed points (CFP) of $\mathcal{A}$ if

$$v \in \mathcal{A}(v, w), \quad w \in \mathcal{A}(w, v), \tag{1}$$

and we say that v_*, w^* are extremal coupled fixed points (ECFP) of $\mathcal{A}$ if v_*, w^* satisfy (1) and

$$v, w \in P \text{ satisfy (1)} \Longrightarrow v_* \leq v, \quad w \leq w^*.$$

MAIN RESULT 1

Theorem 1. *Let Y be a subset of an ordered metric space X, $[\alpha, \beta]$ a nonempty closed interval in Y, and*

$$\mathcal{A} : [\alpha, \beta] \times [\alpha, \beta] \longrightarrow 2^{[\alpha, \beta]} \backslash \emptyset$$

a multivalued operator.
Assume that for all $v, w \in [\alpha, \beta]$ there exist

$$A_-(v, w) = \min \mathcal{A}(v, w), \; A_+(v, w) = \max \mathcal{A}(v, w),$$

and these operators are mixed monotone and satisfy the mixed monotone convergence property. Then $\mathcal{A}$ has the extremal coupled fixed points in $[\alpha, \beta]$.

The previous result is now used in order to guarantee the existence of quasisolutions and unique solutions for the boundary value problem

$$\left\{ \begin{array}{l} x'(t) = f(t, x(t), x(\tau(t, x(t), x)), x) \text{ a.e. on } I_+, \\ \\ x(t) = \Lambda(x_{|I_+}) + k(t), \; t \in I_-, \end{array} \right. \tag{2}$$

where $\Lambda : AC(I_+) \longrightarrow \mathbb{R}$ and $k \in \mathcal{C}(I_-)$.

MAIN RESULT 2

Theorem 2. *Assume that the following conditions hold:*

(H_1) *There exist $\alpha, \beta \in \mathcal{C}(I_\pm)$, $\alpha \le \beta$, coupled lower and upper solutions for problem (2).*

(H_2) *There exist $\psi_m, \psi_M \in L^1(I_+)$ such that*

$$0 \le \psi_m(t) \le f(t, x, \gamma_2(\tau(t, x, \gamma_1)), \gamma_1) \le \psi_M(t),$$

for $\alpha \le \gamma_1, \gamma_2 \le \beta$, and $f(\cdot, x, \gamma_2(\tau(\cdot, x, \gamma_1)), \gamma_1)$ is measurable.

(H_3) *The set of discontinuity points of the function $f(t, \cdot, \gamma_2(\tau(t, \cdot, \gamma_1)), \gamma_1)$ is either unviable or resolvent for every nondecreasing functions $\gamma_1, \gamma_2 \in [\alpha, \beta]$.*

(H_4) (On monotonicities)

$$f(t, x, \stackrel{\downarrow}{\cdot}, \stackrel{\uparrow}{\cdot}) \quad \tau(t, x, \stackrel{\downarrow}{\cdot}) \quad \Lambda(\stackrel{\uparrow}{\cdot}) \quad k(\stackrel{\uparrow}{\cdot})$$

In these conditions, problem (2) has the extremal (coupled) quasisolutions between α and β.

Remark 1. With adequate Lipschitz conditions, it is proven that problem (2) has a unique solution between α and β.

References

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