Generalized linear differential equations in a Banach space: Continuous dependence on a parameter

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We will present a continuous dependence result for integral equations in a Banach space $X$ of the form

$$x(t) = \tilde{x} + \int_a^t d[A(s)] x(s), \quad t \in [a, b],$$

where $-\infty < a < b < \infty$, $\tilde{x} \in X$ and $A: [a, b] \to L(X)$ has a bounded variation on $[a, b]$. The contribution is based on the joint research [2] with M. Tvrď.

Throughout these notes $X$ is a Banach space and $L(X)$ is the Banach space of bounded linear operators on $X$.

Lemma 1. If $F \in G([a, b], L(X))$ and $H \in BV([a, b], L(X))$ then

$$\sum_{t \in [a,b]} \|\Delta^+ F(t) \Delta^+ H(t)\|_{L(X)} + \sum_{t \in [a,b]} \|\Delta^- F(t) \Delta^- H(t)\|_{L(X)} \leq 2 \|F\|_\infty \text{var}^b_a H.$$ 

The integrals are the abstract Kurzweil-Stieltjes integrals defined as in [6].

The result we are about to present extends that presented for the case of a finite dimensional $X$ by Opial in [3]. The following assumption implies the existence of solution to (1) (cf. [6]) and hence it is crucial for our purposes:

$$[I - \Delta^- A(t)]^{-1} \in L(X) \quad \text{for all } t \in (a, b).$$

Theorem. Let $A, A_k \in BV([a, b], L(X))$ satisfy (2) and $\tilde{x}, \tilde{x}_k \in X$ for $k \in \mathbb{N}$. Furthermore, assume that

$$\lim_{k \to \infty} \|A_k - A\|_\infty (1 + \text{var}^b_a A_k) = 0 \quad \text{and} \quad \lim_{k \to \infty} \|\tilde{x}_k - \tilde{x}\|_X = 0.$$ (3)

Then (1) has a unique solution $x$ on $[a, b]$. Moreover, for each $k \in \mathbb{N}$ the equation

$$x_k(t) = \tilde{x}_k + \int_a^t d[A_k(s)] x_k(s), \quad t \in [a, b]$$

has a unique solution $x_k$ on $[a, b]$ and $\lim_{n \to \infty} \|x_k - x\|_\infty = 0$.

Sketch of the Proof. Denote by $x$ and $x_k$ the solutions on $[a, b]$ of (1) and (4), respectively. For $t \in [a, b]$ and $k \in \mathbb{N}$, integrating by parts (cf. [7] and [2]) and using the substitution formula (cf. [1, Proposition II.1.9]), we get

$$x_k(t) - x(t) = \tilde{x}_k - \tilde{x} + \int_a^t d[A] (x_k - x) + [A_k(t) - A(t)] x_k(t) - [A_k(a) - A(a)] \tilde{x}_k$$

$$- \int_a^t (A_k - A) d[A_k] x_k \sum_{a \leq \tau < t} [\Delta^+ (A_k - A)(\tau) \Delta^+ x_k(\tau)] - \sum_{a < \tau \leq t} [\Delta^- (A_k - A)(\tau) \Delta^- x_k(\tau)].$$
Recalling that $\Delta^\pm x_k(s) = \Delta^\pm A_k(s) x_k(s)$, by Lemma 1 we have

$$\sum_{\alpha \leq \tau < t} [\Delta^+ (A_k - A)(\tau) \Delta^+ x_k(\tau)] - \sum_{\alpha < \tau \leq t} [\Delta^- (A_k - A)(\tau) \Delta^- x_k(\tau)] \leq 2 \| A_k - A \|_\infty \text{var}^\alpha_k A_k \| x_k \|_\infty.$$ 

Having this in mind and using [5, Proposition 10] we obtain

$$\| x_k(t) - x(t) \|_X \leq \| \tilde{x}_k - \tilde{x} \|_X + \alpha_k \| x_k \|_\infty + \int_a^t \text{d} \var^\alpha_k A \| x_k(s) - x(s) \|_X,$$

where $\alpha_k := \| A_k - A \|_\infty \left(2 + 3 \text{var}^\alpha_k A_k \right)$. Hence, the generalized Gronwall inequality (cf. [4]) yields $\| x_k(t) - x(t) \|_X \leq \left(\| \tilde{x}_k - \tilde{x} \|_X + \alpha_k \| x_k \|_\infty \right) \exp(\var^\alpha_k A)$. Since $t$ was arbitrary, it follows that

$$\| x_k - x \|_\infty \leq \left(\| \tilde{x}_k - \tilde{x} \|_X + \alpha_k \| x_k \|_\infty \right) \exp(\var^\alpha_k A).$$

Note that $\alpha_k$ tends to zero if $k \to \infty$. Moreover,

$$\| x_k \|_\infty \leq \| x_k - x \|_\infty + \| x \|_\infty \leq \left(\| \tilde{x}_k - \tilde{x} \|_X + \alpha_k \| x_k \|_\infty \right) \exp(\var^\alpha_k A) + \| x \|_\infty$$

which together with (3) imply that the sequence $\| x_k \|_\infty$ is bounded and this completes the proof. \qed

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**References**


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