Feynman Diagrams

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A Feynman diagram is a visual device which graphically summarizes the mathematical representation of what happens when particles interact. The basic mathematical representation of the interaction takes the form of an integral—the *Feynman path integral*. When the path integral is expanded as an infinite series, each term of the series corresponds to a particular physical phenomenon or aspect of the interaction of the particles; and each term of the series has a graphical description in the form of a Feynman diagram. The aggregate of these diagrams provides a visual calculus of the interaction phenomena.

A Feynman path integral is concerned with the *mechanical action* of a physical system, represented as the integral of the *Lagrangian function* (= Kinetic Energy - Interaction Energy) along each of the paths (or "histories") traversable by the system. These paths correspond to the sample paths x of Brownian motion, and, given a particular path $x = (x(t))_{t \in T}$, the mechanical action of the system can be represented as the integral with respect to time t, along a traversed path x, of the Lagrangian of the system, where the intrinsic kinetic energy and externally induced potential energy or interaction energy are variable and depend on position $x(t)$ and time t at different times t. In simple systems kinetic energy at a given instant t of time may be proportional to half the square of velocity, which can be approximated as $\frac{1}{2}\left(\frac{x(t_j)-x(t_{j-1})}{t_j-t_{j-1}}\right)^2$, while the interaction energy is given by a potential energy function $V(x(t),t)$, with $t_{j-1} \leq t \leq t_j$. With $\iota = \sqrt{-1}$, the *state function* ψ for the physical system is given by the "average" of the function (exponential of ι times the mechanical action along the path x),

$$\begin{aligned} & \exp\left(\iota \sum_j \left(\frac{1}{2}\left(\frac{x(t_j)-x(t_{j-1})}{t_j-t_{j-1}}\right)^2 - V(x(t),t)\right)(t_j - t_{j-1})\right), \\ = \quad & \left(\prod_j e^{-\iota V(x(t),t)(t_j-t_{j-1})}\right)\left(\prod_j e^{\iota\left(\frac{1}{2}\left(\frac{x(t_j)-x(t_{j-1})}{t_j-t_{j-1}}\right)^2\right)(t_j-t_{j-1})}\right). \end{aligned} \tag{1}$$

We take the average or expected value of this function over the domain $\mathbf{R}^T$ of paths $x(t)$, where $t \in T$ and $T =]\tau', \tau[$ is an interval of time, with $x(\tau') = \xi' \in \mathbf{R}$ and $x(\tau) = \xi \in \mathbf{R}$. For variable $x \in \mathbf{R}^T$, the "average" is estimated by taking a sum of terms (1), each term being weighted by a factor $|I|\left(\prod_j \sqrt{2\pi\iota(t_j - t_{j-1})}\right)^{-1}$, where $|I|$ represents the "volume" of sets I which partition $\mathbf{R}^T$, with $x \in I$ for each term. **This average is the path integral**, sometimes denoted heuristically as

$$\psi = \psi(\xi', \tau'; \xi, \tau) = \int_{\mathbf{R}^T} \left(e^{-\iota \int_T V(x(t),t)dt} e^{\iota \int_T \left(\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right)dt}\right) \prod_{t \in T} \frac{\delta x(t)}{\sqrt{2\pi\iota dt}}, \tag{2}$$

where $\prod_{t\in T}\delta x(t)$ corresponds to volume $|I|$. If $e^{-\iota\int_T V(x(t),t)dt}$ is expanded as

$$\sum_{r=0}^{\infty}(r!)^{-1}\left(-\iota\int_T V(x(t),t)dt\right)^r,$$

then, by reversing the order of $\int_{\mathbf{R}^T}$ and $\sum_{r=0}^{\infty}$, the "average" (2) can be expressed as $\psi=\sum_{r=0}^{\infty}\psi_r$ where each ψ_r is given recursively by

$$\psi_r=-\iota\int_T\left(\int_{\mathbf{R}}\left(\psi_{r-1}(\xi',\tau';x_{s_r},s_r)V(x_{s_r},s_r)\psi_0(x_{s_r},s_r;\xi,\tau)\right)dx_{s_r}\right)ds_r, \qquad (3)$$

with $\tau'\leq s_1\leq s_2\leq\cdots\leq s_r\leq\tau$ and $x_{s_r}=x(s_r)$. **Each term $\psi_r=\psi_r(\xi',\tau';\xi,\tau)$ has a visual representation as a Feynman diagram.** The derivation of (3) from (2) in [2] assumed that (2) has mathematical meaning as an integral, and used arguments from the theory of integration. Both of these issues were problematic. **Is (2) really an integral? And can integration theorems be applied to it?** The answer to both questions turns out to be **Yes**. The "average" (2) can be expressed as a generalized Riemann integral $\psi=\int_{\mathbf{R}^T}f(x)dF$, where $f(x)=e^{\left(-\iota\int_T V(x(t),t)dt\right)}$,

$$F(I)=\left(\prod_{j=1}^{n}e^{\frac{\iota\left(x_{t_j}-x_{t_{j-1}}\right)^2}{2(t_j-t_{j-1})}}\right)\left(\prod_{j=1}^{n}\left(2\pi\iota(t_j-t_{j-1})\right)^{-\frac{1}{2}}\right)\left(\prod_{j=1}^{n-1}(v_j-u_j)\right),$$

$I_j=]u_j,v_j]$, and each $I=\left(\prod_{j=1}^{n-1}I_j\right)\times\mathbf{R}^{T\setminus\{t_1,\ldots,t_{n-1}\}}$ is a cylindrical interval in $\mathbf{R}^T$. Note that, from this perspective, we are calculating the "average value" of $f(x)$ with respect to weightings $F(I)$.

The generalized Riemann integral $\int_{\mathbf{R}^T}f(x)dF$ is defined, in a familiar manner, by gauge-constrained Riemann sums $\sum f(x)F(I)$ over finite partitions $\{I\}$ of $\mathbf{R}^T$. (So $\bigcup\{I\}=\mathbf{R}^T$; and each $n\to\infty$ as each $\{t_1,\ldots,t_{n-1}\}$ expands in the Riemann sums, with each $t_j-t_{j-1}\to 0$ for each j.) We then deduce (3) from (2) using theorems about integrals, such as reversing $\int_{\mathbf{R}^T}$ and $\sum_{r=0}^{\infty}$ above.

The basic physical ideas in Feynman diagrams are explained in simple terms in a lecture by Feynman in [1]—the third in a series of four lectures filmed in 1979. The derivation of (3) from (2) is given heuristically in Chapter 6 of [2]. The generalized Riemann version of the path integral is given in [3, 4].

References

[1] Feynman, R.P., *http://www.vega.org.uk/video/programme/47*

[2] Feynman, R.P., and Hibbs, A.R., Quantum Mechanics and Path Integrals, McGraw-Hill, New York, 1965.

[3] Muldowney, P., A General Theory of Integration in Function Spaces, Pitman, Harlow, 1987.

[4] Muldowney, P., *Feynman's path integrals and Henstock's non-absolute integration*, Journal of Applied Analysis, Vol. 6, No. 1, 2000, pp. 1-24.