On the existence and uniqueness of a slowly growing solution of singular linear functional differential systems

V. Pylypenko and A. Rontó
Kiev, Ukraine, Brno, Czech Republic

This note deals with a class of linear functional differential equations which may involve singularities with respect to the independent variable. More precisely, we consider the system of linear functional differential equations

\[ x'(t) = \sum_{k=1}^{n} p_{ik}(t)x_k(\omega_{ik}(t)) + q_i(t), \quad t \in [a,b), \quad i = 1, 2, \ldots, n, \tag{1} \]

subjected to the initial conditions

\[ x_i(a) = \lambda_i, \quad i = 1, 2, \ldots, n, \tag{2} \]

where \(-\infty < a < b < \infty\) and \(\{p_{ik}, q_i \mid i, k = 1, 2, \ldots, n\} \subset L_{1; \text{loc}}([a,b), \mathbb{R})\). The argument deviations \(\omega_{k}, k = 1, 2, \ldots, n\), in (1) are arbitrary Lebesgue measurable functions that are supposed to transform the interval \([a,b)\) to itself. Similarly to [1, 2], our aim here is to find conditions sufficient for the existence and uniqueness of a slowly growing solution of the initial value problem (1), (2). The “slow growth” of a solution \(x = (x_i)_{i=1}^{n} : [a,b) \to \mathbb{R}^n\) is understood in the sense that its components satisfy the conditions

\[ \sup_{t \in [a,b]} h_i(t)|x_i(t)| < +\infty, \quad i = 1, 2, \ldots, n, \tag{3} \]

where \(h_i : [a,b) \to [0, +\infty), \quad i = 1, 2, \ldots, n\), are certain given continuous functions possessing the properties \(\lim_{t \to b^-} h_i(t) = 0, \quad i = 1, 2, \ldots, n\). In addition, we assume that the functions \(h_i : [a,b) \to [0, +\infty), \quad i = 1, 2, \ldots, n\) are non-increasing.

By a solution of the functional differential system (1), we mean a locally absolutely continuous vector function \(x = (x_i)_{i=1}^{n} : [a,b) \to \mathbb{R}^n\) with components possessing the properties \(h_i x'_i \in L_1([a,b), \mathbb{R}), \quad i = 1, 2, \ldots, n\), and satisfying equalities (1) almost everywhere on the interval \([a,b)\).

**Theorem 1.** Assume that the functions \(p_{ik}, i, k = 1, 2, \ldots, n,\) are non-negative almost everywhere on \([a,b)\). Moreover, assume that, for all \(i, k = 1, 2, \ldots, n,\)

\[ \int_{a}^{b} \frac{h_k(t)p_{ik}(t)}{h_k(\omega_{ik}(t))} dt < +\infty, \tag{4} \]

\[ \text{ess sup}_{t \in [a,b]} h_k(\omega_{ik}(t)) \sum_{j=1}^{n} \int_{a}^{\omega_{ik}(t)} \frac{p_{kj}(s)}{h_j(\omega_{kj}(s))} ds < 1. \tag{5} \]

Then problem (1), (2), (3) has a unique solution for arbitrary locally integrable functions \(q_i : [a,b) \to \mathbb{R}, \quad i = 1, 2, \ldots, n,\) possessing the property

\[ \{h_i, q_i \mid i = 1, 2, \ldots, n\} \subset L_1([a,b), \mathbb{R}). \tag{6} \]
and any \( \{ \lambda_i \mid i = 1, 2, \ldots, n \} \). Furthermore, if \( q_i \) and \( \lambda_i, i = 1, 2, \ldots, n \), for almost every \( t \in [a, b) \) satisfy the condition

\[
- \sum_{k=1}^{n} \lambda_k p_{ik}(t) \leq q_i(t), \quad i = 1, 2, \ldots, n,
\]

then the unique solution of problem (1), (2), (3) has non-negative components.

Note that condition (5) of Theorem 1 is unimprovable in the sense that it cannot be replaced by the corresponding non-strict inequality

\[
\text{ess sup}_{t \in [a, b)} h_k(\omega_{ik}(t)) \sum_{j=1}^{n} \int_{a}^{\omega_{ik}(t)} \frac{|p_{kj}(s)|}{h_j(\omega_{kj}(s))} \, ds < 1
\]

even for a single pair of indices \( i \) and \( k \), because after such a replacement the assertion of Theorem 1 is not true any more.

**Theorem 2.** Let \( p_{ik}, i, k = 1, 2, \ldots, n \), satisfy relations (4) and the condition

\[
\text{ess sup}_{t \in [a, b)} h_k(\omega_{ik}(t)) \sum_{j=1}^{n} \int_{a}^{\omega_{ik}(t)} \frac{|p_{kj}(s)|}{h_j(\omega_{kj}(s))} \, ds < 1
\]

for all \( i, k = 1, 2, \ldots, n \).

Then, for any locally integrable functions \( q_i : [a, b) \to \mathbb{R}, i = 1, 2, \ldots, n \), possessing property (6) and arbitrary real \( \lambda_i, i = 1, 2, \ldots, n \), the initial value problem (1), (2) has a unique solution possessing property (3).

It should be mentioned that, under the assumptions of the last theorem, the unique solution of problem (1), (2), (3) may not be non-negative even under condition (7). Note also that a remark similar to that on the non-strict inequality (8) is also true for the non-strict version of condition (9).

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**References**
