We are interested in the existence of positive solutions of some non-local boundary value problems (BVPs) for equations of the form

\[ u^{(4)}(t) - \omega^4 u(t) = f(t, u(t)), \text{ a.e. } t \in (0, 1), \]

for some constant \( \omega \in (0, \pi) \), subject to the following non-local boundary conditions (BCs)

\[ u(0) = \beta_1[u], \quad u''(0) + \beta_2[u] = 0, \quad u(1) = \beta_3[u], \quad u''(1) + \beta_4[u] = 0, \]

where each \( \beta_i[u] \) is a linear functional on \( C[0, 1] \), that is, is given by a Riemann-Stieltjes integral

\[ \beta_i[u] = \int_0^1 u(s) \, dB_i(s). \]

Since some of the \( \beta_i \) can be zero, while others are not, this covers many BCs in one. A distinguishing feature of our work is that each \( B_i \) is a function of bounded variation, that is, \( dB_i \) is a signed measure. Some kind of positivity on the functionals \( \beta_i \) is needed in order to have positive solutions, but we do not suppose that \( \beta_i[u] \geq 0 \) for all \( u \geq 0 \).

For (1) we can consider cases where \( f(t, u) \) is not positive for all positive \( u \) but is such that \( f(t, u) + k^4 u \geq 0 \) for \( u \geq 0 \) for some constant \( k \in (0, \omega) \). One useful motivation is that the original problem (1) with the BCs (2) may be at resonance, that is, \( \lambda = 0 \) is an eigenvalue of the linear problem \( u^{(4)} - \omega^4 u = \lambda u \) with the given BCs. In such a case we can consider the equivalent problem, which is of the same type as the original one,

\[ u^{(4)}(t) - \tilde{\omega}^4 u(t) = \tilde{f}(t, u(t)), \]

where \( \tilde{\omega}^4 := \omega^4 - k^4 \), \( \tilde{f}(t, u) := f(t, u) + k^4 u \), with the same BCs. We will show that, under natural conditions, this perturbed problem is non-resonant. In order to study the existence of positive solutions for (1)-(2) we use the method developed by Webb and Infante in [1] and [2].

References
