Palatini-Cartan Gravity

The standard formulation of gravity

- The field describing gravity on a manifold M is a (pseudo)Riemannian metric g
- Classically, g must satisfy Einstein's equations

$$G + \Lambda g = 0$$

(when no matter is present)

Here:

- $G = Ric - \alpha R g$

Ricci tensor

$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$ ← Riemann curvature

scalar curvature $R = R_{\mu\nu} g^{\mu\nu}$

α s.t. $\operatorname{Tr} G = 0$

- Λ is a fixed constant (cosmological constant)

Einstein's equations are the Euler-Lagrange equations for the Einstein-Hilbert (EH) action

$$S[g] = \int_M \sqrt{g}\, R + \Lambda \sqrt{g}$$
M

$$\sqrt{g} = \sqrt{|\det g|}\, d^d x$$ the density associated to g

<u>Rem</u> The curvatures Riem, Ric, R are computed in terms of the Levi-Civita connection

Palatini considered

$$S_M[g,\Gamma] := \int \sqrt{g}\, R + \Lambda \sqrt{g}$$

where the curvature are defined in terms of a torsion-free connection Γ.

$\hookrightarrow$ EL equations:
1) Γ must be metric, so Levi-Civita
2) Einstein's equations

Cartan's formalism

Basic idea from linear algebra:

- g positive-definite symm. bil. form $\overset{\text{Gram–Schmidt}}{\rightsquigarrow}$ orthonormal basis

  $\rightsquigarrow$ $g = E^t E$

  E invertible

- conversely: E invertible $\Rightarrow$ $E^t E$ positive-definite symm. bil. form

Rem The columns of E^{-1} are an orthon. basis of $E^t E$

E^{-1} is called a frame, E is called a coframe

Therefore,

cofrances

$$\left\{ \begin{matrix} \text{positive-definite symm. bil. forms} \\ \text{on } \mathbb{R}^n \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \text{invertible matrices} \right\} \big/ O(n)$$

The "same" works w/ other signatures.

E.g. signature $(1,n)$: $\quad g = E^t \eta E, \quad \eta$ Minkowski metric

$$\left\{ \begin{matrix} \text{symm. bil. forms of signature } (1,n) \\ \text{on } \mathbb{R}^{n+1} \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \text{invertible matrices} \right\} \big/ O(1,n)$$

On manifolds To describe a (pseudo) Riemannian metric g on M, we need a coframe field e

Locally, we have a <u>frame</u> $\bar{e}^1, \dots, \bar{e}^m \in T_x M$

$m = \dim M$

which corresponds to an orthonormal basis for g_x

Equivalently, $\bar{e} : \mathbb{R}^m \xrightarrow{\sim} T_x M$

$v \mapsto \sum_i \bar{e}^i v_i = \sum_{i,\mu} \bar{e}^i_\mu v_i \, dx^\mu$

The frame is the inverse

$$e = \bar{e}^{-1} : T_x M \xrightarrow{\sim} \mathbb{R}^m$$

The metric g is recovered as before.

In components: $g_{\mu\nu} = e^a_\mu \, \eta_{ab} \, e^b_\nu$ $\qquad \eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

η = Euclidean / Minkowski metric on $\mathbb{R}^n$

Globally, we need a vector bundle $\mathcal{V}$

with Eucl/Mink metric η on fibers

and $\qquad e: TM \xrightarrow{\sim} \mathcal{V}$

Construction 1) Fix a (ps)Riem metric g_0

2) Consider the associated orthonormal frame bundle F_M
$$\begin{array}{c} F_M \\ \downarrow \\ M \end{array}$$

3) Let $V = \mathbb{R}^m$ with Eucl/Mink metric η

$$\mathcal{V} := F_M \times_{O(n)} V$$

$$\left(F_M \times_{O(1,n-1)} V \right)$$

The standard density

From $g = e^t \eta e$, we get

$$\det g = (\det e)^2 \det \eta$$

$$\Rightarrow \quad \sqrt{|\det g|} = |\det e|$$

Rem $\det e$ can be interpreted as $\underbrace{e \wedge \dots \wedge e}_{n \text{ factors}} =: e^n$

$e : TM \xrightarrow{\sim} V$

Actually $e \in \Gamma(T^*M \otimes V)$

$$\rightsquigarrow \quad e^n \in \Gamma(\underbrace{\Lambda^n T^*M \otimes \Lambda^n V}_{\overset{\text{canonically}}{\cong} \operatorname{Dens} M})$$

So $$\int_M \frac{\Lambda}{n!}\sqrt{g} = \int_M \frac{\Lambda}{n!} e^n$$

and we have one piece of EH action.

<u>Notation</u> $$\Omega^{k,l} := \Gamma\left(\Lambda^k T^*M \otimes \Lambda^l \mathcal{V}\right)$$

To get the other term, we need to introduce a connection and its curvature.

Let ω be a connection 1-form on $\begin{array}{c}\mathcal{F}_M \\ \downarrow \\ M\end{array}$

i.e., ω is an orthogonal connection on $\mathcal{V}$

Its connection 2-form $F_\omega = d\omega + \frac{1}{2}[\omega, \omega]$

can be viewed as a 2-form with values in $\mathrm{ad}\,\mathcal{F}_M$.

<u>Rem</u> $\mathfrak{so}(\cdot) \simeq \Lambda^2 V$

and $\mathrm{ad}\,\mathcal{F}_M \simeq \Lambda^2 \mathcal{V}$ using η

So $F_\omega \in \Omega^{2,2}$ $\quad e \in \Omega^{1,1}$

$\Omega^{n,n}$ invariant

Construction of the Palatini-Cartan (PC) action

Ingredients: e, F_ω

Outcome: a density, i.e., an element of $\Omega^{n,n}$

In 4d, we only have

$$e^4, \quad e^2 F_\omega, \quad F_\omega^2$$

Pontrjagin class

$$S^{PC}[e,\omega] := \int_M \frac{1}{2} e^2 F_\omega + \frac{\Lambda}{4!} e^4$$

$\Lambda \in \mathbb{R}$

Equivalence with CH

$\delta F_\omega = d_\omega \delta\omega$

EL equations

$$\frac{\delta S}{\delta \omega} = 0 \Rightarrow \frac{1}{2} d_\omega e^2 = 0 \qquad (1)$$

$$\frac{\delta S}{\delta e} = 0 \Rightarrow e F_\omega + \frac{\Lambda}{3!} e^3 = 0 \qquad (2)$$

$e : TM \xrightarrow{\sim} V$

$$(1) \Leftrightarrow e\, d_\omega e = 0 \overset{e \text{ n.d.}}{\Longleftrightarrow} d_\omega e = 0$$

Rem Given e $\exists!\, \omega_e : d_{\omega_e} e = 0$

This is the analogue of the Levi-Civita condition

More precisely, using $e : TM \xrightarrow{\sim} \mathcal{V}$

one can associate to the connection ω on $\mathcal{V}$

a connection Γ on TM

1) ω orthogonal $\Rightarrow$ Γ is metric (for $g = e^t \eta e$)

2) $d_\omega e = 0 \Leftrightarrow \Gamma$ is torsion-free

Inserting ω_e, 2) reads

$$e F_{\omega_e} + \frac{\Lambda}{3!} e^3 = 0$$

<u>Exercise</u> This is equivalent to Einstein's equations for g

Globally hyperbolic manifolds and symplectic structure on initial data

Let Σ be a closed 3-manifolds and take [Canepa-C, 2025]

$$M = \Sigma \times I \quad \leftarrow \text{an interval} \ni t \text{ "time"}$$

For simplicity now assume:
- $\Lambda = 0$
- Minkowski signature

Also write $\tilde{e}, \tilde{\omega}$ for coframe and connection on M.

$$S_{PC} = \int_M \frac{1}{2} \tilde{e}\,\tilde{e}\,F_{\tilde{\omega}}$$

Decompose $\tilde{e} = e_n dt + e$

$\tilde{\omega} = \omega_n dt + \hat{\omega}$

Here $e_n \in \Omega^{0,1}$ $\quad e \in \Omega^{1,1}$ s.t. $\mathcal{L}_{\partial_n} e = 0$, $\mathcal{L}_{\partial_n} \hat{\omega} = 0$

$\omega_n \in \Omega^{0,2}$ $\quad \hat{\omega}$ connection

$\partial_n := \frac{\partial}{\partial t}$

Then

$$S_{PC} = \int_M \Big[e_n e F_{\hat{\omega}} + \underbrace{\tfrac{1}{2} e e \partial_n \hat{\omega}}_{\text{"}p\dot{q}\text{"}} + \omega_n e d_{\hat{\omega}} e \Big] dt$$

d only along Σ

$$\left(S = \int (p \cdot \dot{q} - H(p,q)) dt \right)$$

Idea: • Fix $\varepsilon_n \in \Omega^{0,1}$ s.t. $\varepsilon_n^t \eta \varepsilon_n = -1$ time-like

• Restrict the space of fields to that for which the e's are space-like

$\Leftrightarrow$ $g := e^t \eta e$ is a Riemannian metric on $\Sigma \times \{t\}$ $\forall t$

In other words, we only allow metrics on M that make it globally hyperbolic

Now:

$$\left.\begin{array}{l} \varepsilon_n \text{ time-like} \\ e_i \text{ space-like} \\ (e_1, e_2, e_3) \text{ lin indep} \end{array}\right) \Rightarrow (\varepsilon_n, e_1, e_2, e_3) \text{ coframe}$$

$$\Rightarrow \exists!\ \mu \in C^\infty(\Sigma \times I)$$

$$\exists!\ z \in \Gamma(\Sigma \times I, T\Sigma) \qquad \text{s.t.} \quad \boxed{e_n = \mu \varepsilon_n + \iota_z e}$$

$$\left(\text{Recall} \quad \begin{array}{l} \tilde{e} = e_n dt + e \\ \tilde{\omega} = \omega_n dt + \hat{\omega} \end{array} \right)$$

Moreover (due to a thm by Canepa - C - Schiavina)

$\exists!$ splitting $\hat{\omega} = \omega + v$, with $\begin{cases} \epsilon_n d_\omega e = e\sigma & (\text{some } \sigma) \\ e v = 0 \end{cases}$

connection $\quad$ $\in \Omega^{1,2}$, $\iota_{\partial_n} v = 0$

Then

$$S_{PC} = \int_M \Big[e_n e F_\omega + \frac{1}{2} e e \partial_n \omega dt + \omega_n e d_\omega e + \iota_Z v\, e d_\omega e + \frac{1}{2} e_n e [v, v] \Big] dt$$

- $\int_M \frac{1}{2} e e \partial_n \omega dt$ analogue to $\int p \dot{q} dt$

- e_n, ω_n: Lagrange multipliers yielding the constraints $\begin{cases} e F_\omega = 0 \\ e d_\omega e = 0 \end{cases}$ on $\Sigma \times \{t\}$

Finally, set $\mathcal{W} := \omega_n + 2_z v$, z vector field
(I-dependent, in Σ-directions)

Then
$$\boxed{S_{PC} = S_{PC_{can}} + S_{aux}}$$

- $S_{PC_{can}}(e, \omega, \mu, z, \mathcal{W}) =$

$$= \int_M \left[\frac{1}{2} e e \partial_n \omega + (\mu \varepsilon_n + 2_z e) e F_\omega + \mathcal{W} e d_\omega e\right] dt$$

- $S_{aux} = \int_M \frac{1}{2} e_n e [v, v] dt$

Getting rid of v

Thm (Corollary of C-Schiavina)

For given e_n, e, the quadratic form in v

$$Q_{e,e_n}(v) = \int_M \frac{1}{2} e_n e [v, v] dt$$

is nondegenerate.

Corollary $\frac{\delta S_K}{\delta v} = 0 \iff \frac{\delta Q}{\delta v} = 0 \iff \boxed{v = 0}$

Rem/Digression In the quantum version, we can "integrate out" v

$$\int e^{\frac{i}{\hbar} S_{aux}} \, Dv = J(e_n, e) \quad \text{(some determinant)}$$

so

$$\int e^{\frac{i}{\hbar} S_{PC}} \, D\tilde{e} \, D\tilde{\omega} = \int e^{\frac{i}{\hbar} S_{PC\,can}} \, J \, De \, D\mu \, Dz \, Dw$$

Back to the analysis of $S_{PC\,can}$

$$S_{PC\,can}(e, \omega, \mu, z, w) =$$

$$= \int_M \Big[\frac{1}{2} e e \partial_n \omega + \underbrace{(\mu \varepsilon_n + z_2 e)}_{e_n} e F_\omega + w e d_\omega e\Big] dt$$

of the form $\int \alpha\, dt + \lambda^i H_i$

$\Omega = d\alpha$ symplectic form

Lagrange multipliers

That is, on Σ we have

- Space of fields $\left\{ (e,\omega) \in \Omega^{1,1} \oplus \mathrm{Conn} : \varepsilon_m d_\omega e = e\sigma \text{ for some } \sigma \right\}$

- Symplectic form $\Omega = \delta\alpha$, $\alpha = \int \frac{1}{2} e e \delta\omega$

$$\rightsquigarrow \Omega = \int e \delta e \delta \omega$$

closed, nondeg 2-form

- Constraints $\begin{cases} e d_\omega e = 0 \\ e F = 0 \end{cases}$

$\rightsquigarrow$ this leave 2 local degrees of freedom

BV-BFV extensions

- The critical locus / symmetries may be resolved cohomologically (BV)
- The constraint locus / characteristic distribution may be resolved cohologically (BFV)

Symplectic reduction

For $M = \Sigma \times I$ glob. hyp., working w/ $S_{PC_{can}}$

yields a compatible BV-BFV formulation

<u>Roughly speaking</u>

- BFV (on Σ) is a dg-symplectic structure
- BV (on $\Sigma \times I$) is a Lagrangian structure

$BV_{\Sigma \times I} \xrightarrow{\pi} BV_{\Sigma}, \Omega_{\Sigma}$

$\pi^* \Omega_{\Sigma} \sim 0$

nondeg

- The BV-BFV formalism is very flexible
  - Possible way towards quantization
  - Way to establish equivalence with other theories
    (quasi-isomorphisms)

Thanks