Holography and conformal YM equation

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California meeting

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Introduction and Motivations

Plan of the talk:

- Compactification
- Conformal densities and tractor calculus
- Holography: a toy model
- The YM model
- Some (rather technical :( ) result
- Contact geometry

Why (conformally) compactify?

M_+ a complete non compact pseudo Riem. manifold. Add a boundary at infinity ∂M and exploit it.

This structure pops up in many contexts (some key words)

- Q curvature and its generalizations
- Fefferman-Graham program
- Maldacena Holography and AdS/CFT correspondence in string theory
- Scattering and PDE boundary problem
- Renormalised volume Wilmore energies and Weyl anomalies

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Alberto Cattaneo research program on BV-BFV quantization on mfd with boundary DC11

Why YM?

YM is the theory of connections on principal bundles, namely 1 forms taking value in some (semisimple) Lie algebra or the endomorphism of a vector bundle.

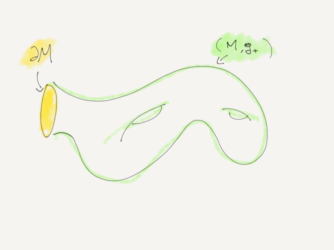
- That's the mathematical model for gauge propation (e.g. photons) and interactions in particle physics
- It is conformal invariant BUT only in dimension 4
- Interesting topological aspects relating 2+1 YM theory and Jones polynomial
- Instantons and Donaldson theory to study the topology of four manifolds

The geometrical setting

Definition

Conformal compactification of a Riemannian manifold (M_+, g_+) is a manifold M with a boundary ∂M s.t.

- $\exists\ g$ on M with
- $g_+ = g/r^2$ where r is a defining function for the boundary



Comment: Given g_+ (a fixed data) and founded g and r, if we rescale the defining function $\hat{r} = \Omega r$, we obtain another defining function and thus another "good" metric $\hat{g}$ on M conformally related to g.

$\Rightarrow$ We have induced on the boundary a conformal structure

$$(\partial M, [\bar{g} := g|_{\partial M}])$$

Poincaré-Einstein, or just PE, when g_+ is Einstein

The geometrical setting

Escher's cricle limit

$M = \mathbb{H}^2 + S^1$

$M_+ = \mathbb{H}^2$ embedded conformally in $\mathbb{E}^2$

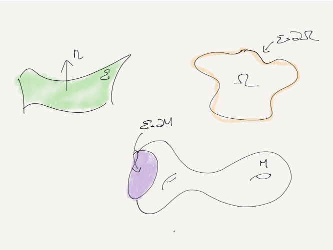
$g_+ = \frac{4}{(1-|x|^2)}(dx_1^2 + dx_2^2)$

$S^1 = \partial M$

$[(dx_1^2 + dx_2^2)|_{S^1}]$ conformal boundary structure

The geometrical setting

To better understand the geometry of the ("compactified") boundary it might be useful to be a bit more open mind. More generally one may want to study hypersurfaces Σ embedded in a conformal manifold $(M, [g])$



In this setting it is natural to ask how to make invariants and invariant operators kind of natural for the embedding, or study conformal boundary problem

Conformal Geometry

A conformal manifold is a smooth manifold M equipped with an equivalence class of pseudo-Riemannian metric $[g]$ where

$$g \sim \hat{g} \leftrightarrow \hat{g} = \Omega^2 g$$

Because there is no distinguished metric a key object in this game is the (conformal) density bundle

$$\mathcal{E}[w] := (\Lambda^d TM)^w$$

that can be understood as an equivalence class of (metrics, functions) pairs, i.e. $(g, f) \sim (\hat{g}, \hat{f}) := (\Omega^2 g, \Omega^w f)$.

Tensor bundles $\mathcal{S}$ can be twisted by $\mathcal{E}M[w]$ inducing $\mathcal{S}[w] := S \otimes \mathcal{E}[w]$

Conformal Geometry

A simple BUT relevant example: The Laplacian Δ^g in not covariant under conformal rescaling BUT when acting on densities of weight $1-\frac{d}{2}$

$$\underbrace{\left(\Delta^{\hat{g}} + (1-\frac{d}{2})J^{\hat{g}}\right)}_{Y^{\hat{g}}}\hat{f} = \Omega^{-1-\frac{d}{2}}\underbrace{\left(\Delta^{g} + (1-\frac{d}{2})J^{g}\right)}_{Y^{g}} f$$

where J^g is the trace of the Schouten tensor P^g_{ab} that is a trace adjusted Ricci.

The interpretation is that we have a conformal covariant Laplace operator

$$Y : \mathcal{E}[1-\frac{d}{2}] \to \mathcal{E}[-1-\frac{d}{2}]$$

In conformal geometry we don't have a metric but a more general object called the **conformal metric**

$$\mathbf{g} \in Sym^2(T^*M)[2] \Rightarrow g^{\sigma} := \sigma^{-2}\mathbf{g} \text{ with } \sigma \in \mathcal{E}_+[1] \text{ the scale}$$

Tractor calculus

Conformal manifolds is one of the most important example of parabolic geometry

Definition

A parabolic geometry modelled on $G \to G/P$ with P some parabolic subgroup of G, is the data of

- a P principal bundle $\mathcal{G} \to M$
- a Cartan connection $A \in \Omega^1(\mathcal{G}, \mathfrak{g})$ with $LieG = \mathfrak{g} = \mathfrak{g}_{-1} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1}_{\mathfrak{p}}$.

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Buachalla and Strung (DC4 and DC5) for quantum flags. Fioresi and Lledo (DC6 and DC9) for quantum super flags.

Tractor calculus

Given a finite dimensional representation V of G one can construct $\mathcal{G} \times_P V = \mathbb{V}$ that is named tractor bundle naturally equipped with a linear "tractor" connection.

Going back to the conformal case one has $G = SO(p+1, q+1)$ and the following graded Lie algebra

$$\mathfrak{so}(p+1, q+1) = (\mathbb{R}^{p+q})^* \oplus \underbrace{\mathfrak{co}(p,q) \oplus \mathbb{R}^{p+q}}_{\mathfrak{p}}$$

and taking $V = \mathbb{R}^{p+q+2}$ one gets the standard tractor bundle:

$$\mathcal{T} := \mathcal{G} \times_P \mathbb{R}^{p+q+2} \quad \stackrel{g}{\Rightarrow} \quad \mathcal{T} \stackrel{g}{=} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1]$$

Explicitly in a scale we write a section of this bundle as $T_A \stackrel{g}{=} (\sigma, t_a, \rho)$.

California link

Neusser and Slovak DC1 and DC2 for tractor calculus. Also Waldron and Gover that are involved in this network.

Tractor calculus

We have the (conformal) tractor connection $\nabla_a^{\mathcal{T}}$ acting on $\mathcal{T}$

$$\nabla_a^{\mathcal{T}} \underbrace{(\sigma, t_a, \rho)}_{T_B} \stackrel{g}{=} (\nabla_a \sigma, \nabla_a t_b + P_{ab}\sigma + g_{ab}\rho, \nabla_a \rho - P_{ab} t^b)$$

∇_a the L.C. connection of the chosen metric g in the equivalence class
Let's play with that structure:

$$\begin{aligned} D_A : \mathcal{E}[w] &\to \mathcal{T}[w-1] \\ f &\to ((d+2w-2)wf, (d+2w-2)\nabla_a f, -(\Delta + wJ)f) \end{aligned}$$

Note: if $f \in \mathcal{E}[1-\frac{d}{2}]$ then

$$D_A f = (0, 0, -Yf)$$

Comment: we can make D_A to any tractor bundle simply by $\nabla_a \to \nabla_a^{\mathcal{T}}$

The defining density

Definition

A *conformally compactified manifold* is the data $(M, \mathbf{g}, \sigma)$ with

- M a (conformal) d-manifold with boundary ∂M
- A defining density i.e. a non-negative $\sigma \in \mathcal{E}[1]$ (i.e. $\sigma = [(g, r)]$) s.t. $\Sigma := \partial M = \sigma^{-1}(0)$ and for any LC connection in the conformal class,

$$\nabla\sigma|_\Sigma \neq 0\,,$$

The scale tractor: if we take instead $f = \sigma \in \mathcal{E}[1]$

$$I_A = \frac{1}{d} D_A \sigma \stackrel{g}{=} (\sigma, n_a := \nabla_a \sigma, -\frac{1}{d}(\Delta + J)\sigma)$$

observations $M_+ := M \backslash \Sigma$ is equipped with the distinguished metric

$$g_+ := \mathbf{g}/\sigma^2\,.$$

IdotD

On the tractor bundle we have a natural parallel metric

$$h_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

we use to construct

$$I^A D_A = \begin{cases} -\Delta^{g_+} + .. & \text{in the interior } M_+ \\ \delta^{(1)} := \nabla_n + wH & \text{along } \Sigma \end{cases}$$

$\delta^{(1)} : \mathcal{E}M[w] \to \mathcal{E}\Sigma[w-1]$ is the famous Robin operator, very important in the context of DtoN maps.

Comment: For PE structure ($I^2 = 1$) in the interior we get $-\Delta^{g_+} + s(d-1-s)$ with $s = (d+w-1)$ being **the spectral parameter** and this is the scattering problem (Mazzeo-Melrose and Graham-Zworski)

The beginning of holography

THE PROBLEM:

Given $f|_\Sigma$ and an arbitrary extension $f_0 \in \mathcal{E}[w_0]$ on M find

$$f^{(\ell)} := f_0 + \sigma f_1 + ..$$

solving formally $I \cdot Df^{(\ell)} = O(\sigma^\ell)$ for ℓ as high as possible

Assuming $I \cdot Df^{(\ell)} = O(\sigma^\ell)$ for some ℓ, and thanks to

$$I \cdot Df^{(\ell+1)} = I \cdot Df^{(\ell)} - \sigma^\ell(\ell+1)(d + 2w_0 - \ell - 2)f_{\ell+1} + O(\sigma^{\ell+1})$$

we can formally solve this problem for $\ell = \infty$ WHEN $d + 2w_0 - \ell - 2 \neq 0$

BUT...

The beginning of holography

WHEN $d + 2w_0 - \ell - 2 = 0$ the solution is obstructed by $I \cdot Df^{(\ell)}$

$$I \cdot Df^{(\ell)} = I \cdot D \underbrace{\left(f^{(\ell)} + \sigma^{\ell+1} f_{\ell+1}\right)}_{f^{\ell}+1} \text{ modulo } O(\sigma^{\ell+1})$$

Definition

We say that an operator O is tangential if $O(h + \sigma \tilde{h}) = Oh + \sigma \tilde{O}\tilde{h}$ Thus along Σ is insensitive to how the function is extended off the boundary $\Rightarrow$ there is a formula on Σ involving only derivatives tang. to Σ

OBSERVE at the special $\ell = d + 2w_0 - 2$ then$I \cdot Df^{(\ell)}$ is tangential

$$f_0|_\Sigma \to (I \cdot D)^{d+2w_0-1} f_0|_\Sigma$$

is conformal invariant, $\ell + 1$ is even $\Rightarrow$ holographic GJMS operators

$$P^{\ell+1} : \mathcal{E}\Sigma[\tfrac{\ell+1-d}{2}] \to \mathcal{E}\Sigma[\tfrac{-\ell-1-d}{2}]$$

$$f \to \bar{\Delta}^{\frac{\ell+1}{2}} f + \ldots$$

COMMENT 1 The solution could be prolonged to all order by adding a log term

COMMENT 2 There is a second solution of the problem of the form

$$\tilde{f} = \sigma^{d+2w_0-1}(\tilde{f}_0 + \sigma\tilde{f}_1 + ..)$$

that is not obstructed.

COMMENT 3 Call $F_1 = f$ and $F_2 = \sigma^{1-d-2w_0}\tilde{f}$ we can combine this two solutions to get a general one

$$F_1 + \sigma^{d+2w_0-1}F_2$$

For global solutions for the the scattering problem

YM theory

- YM is the theory of connections on a principal G bundle $E \to M$ or equivalently an equivalence class of $A \in \Omega^1(M, \mathfrak{g})$ with

$$A \sim A' = \underbrace{\mu^{-1} A \mu + \mu^{-1} d\mu}_{\text{Gauge transformation}} \qquad \mu : M \to G$$

- The YM curvature is then defined by

$$\Omega^2(M, ad(\mathfrak{g})) \ni F[A] = dA + [A, A]$$

  under gauge transformation $F[A'] = \mu^{-1} F[A] \mu$ where

- In general on can be more open mind and consider $F \in \Omega^2(M, End\mathcal{V}M)$ with $\mathcal{V}M$ some vector bundle over M

YM theory

The action functional

$$S[A] := -\frac{1}{4}\int_M \mathrm{dVol}(g)\, Tr(g^{ab}g^{cd}F_{ac}\circ F_{bd})\,,$$

induce the YM equation

$$\jmath[A] := \delta^A F = 0 = g^{ac}\nabla^A_c F_{ab}\,.$$

(Every covariant derivative is now twisted also by the YM connection even if not specified.)

- YM functional and equations satisfy a conformal covariance in dimension 4
- What about conformal YM in other dimensions??
- IDEA!!!! use holography

YM theory

In our setting consider the YM equation constructed out of the singular metric on M_+

$$g_+^{ab}\nabla_a F_{bc} = 0$$

Observe that

$$g_+^{ab}\nabla_a F_{bc} = \sigma(\underbrace{\sigma \mathbf{g}^{ab}\nabla_a F_{bc} - (d-4)\mathbf{g}^{ab} n_a F_{bc}}_{j_b})$$

$\Rightarrow$ Note the confromally compact YM current it does make sense everywhere on M (even along the boundary) and we want to study

$$j_b[A] = 0\,, \quad \text{on } M$$

YM theory

THE MAGNETIC PROBLEM:

Given a conformally compact structure and a connection $\nabla^{\bar{A}}$ along Σ find a smooth connection ∇^A on M such that

$$\nabla^A_X \stackrel{\Sigma}{=} \nabla^{\bar{A}}_X\,, \qquad j[A] = \mathcal{O}(\sigma^\ell)\,, X \in T\Sigma$$

Theorem (Gover,Waldron, E.L., Zhang)

When $d \geq 4$ there exists a solution of the magnetic problem to order $\ell = d - 4$. When $d = 3$ there exists an order $\ell = \infty$ solution. Moreover two solutions are "gauge" equivalent

∇^A on a $d \geq 4$ conformally compact structure s.t.

$$j[A] = \sigma^{d-4} k\,,$$

for $k \in T^*M[3-d] \otimes End\mathcal{V}M$, is named *asymptotic Yang–Mills connection*

YM theory

An important corollary of the above theorem concerns the uniqueness of $\bar{k} := k|_\Sigma$

Corollary

Let $(M, \mathbf{g}, \sigma)$ be a conformally compact structure and $\bar{A}$ a connection on $\Sigma \Rightarrow$ a canonical map

$$(M, \mathbf{g}, \sigma, \bar{A}) \mapsto \bar{k} \in T^*\Sigma[3-d] \otimes End\mathcal{V}\Sigma \, .$$

By construction $\bar{k}$ is conformal and gauge invariant thus we interpret it as higher YM equation.
Questions:

- is it variational?
- how explicit can we write it?

Renormalized action

Integrals over conformally compact manifolds are Il-defined. Nonetheless, useful information can often be extracted.

Take $\tau \in (\mathcal{E}_+ M[1])$ namely a true scale for the conformal manifold (note $\tau|_\Sigma$ is a scale for the boundary conformal structure $(\partial\Sigma, [\bar{g}])$)

$$M^\varepsilon = \{p \in M \,|\, \sigma(p)/\tau(p) > \varepsilon\} \subset M_+ = M \backslash \partial M \,.$$

and the renormalized functional

$$S^\varepsilon_{\mathrm{YM}}[A] := \int_{M^\varepsilon} \mathrm{Vol}(g_+)\, \langle F[A], F[A] \rangle_{g_+} \,.$$

Theorem (Gover,Waldron, E.L., Zhang)

$$S^\varepsilon_{\mathrm{YM}}[A] = \frac{v_{d-5}}{(d-5)\varepsilon^{d-5}} + \frac{v_{d-6}}{(d-6)\varepsilon^{d-6}} + \cdots + \frac{v_1}{\varepsilon} + \mathrm{En}[A] log \frac{1}{\varepsilon} + S^{\mathrm{ren}}_{\mathrm{YM}}[A] + \mathcal{O}(\varepsilon) \,,$$

Moreover, the energy En *is independent of the choice of regulator* τ*.*

Renormalized energy

We have the following relevant result

Theorem (Gover,Waldron, E.L., Zhang)

$$\begin{aligned} S^{\mathrm{ren}}_{YM}[A;\lambda\tau] - S^{\mathrm{ren}}_{YM}[A;\tau] &= \int_\Sigma Vol(\bar{g})\sqrt{I^2}\left(\tfrac{1}{I^2} I.D\right)^{d-5}\left(\tfrac{\langle F[A],F[A]\rangle}{I^2}\lambda\right) \\ &= \lambda E[A] \end{aligned}$$

with λ some constant. When ∇^A is asymptotically YM the above depends on the boundary connection only. Moreover the functional gradient of the energy is a non-zero multiple of the obstruction current $\bar{k}$. On PE

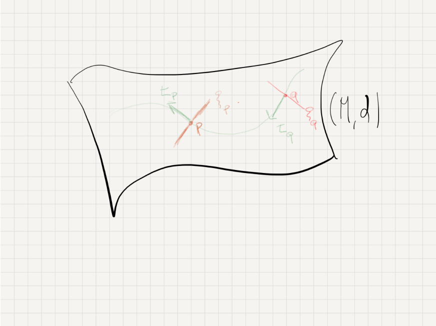
$$\bar{k}_b = \begin{cases} \bar{j}_b\,, & d=5\,, \\ \tfrac{1}{2}\bar{\nabla}^a\left(\bar{\nabla}_{[a}\bar{j}_{b]} - 4\bar{P}_{[a}{}^c\bar{F}_{b]c} - \bar{J}\bar{F}_{ab}\right) + \tfrac{1}{4}[\,\bar{j}^a,\bar{F}_{ab}]\,, & d=7\,, \\ 0\,, & d = even. \end{cases}$$

Contact geometry

Definition

A contact structure is a $2n+1$ dimensional manifold M is the data of an hyperplane distribution $\xi \subset TM$ maximally nonintegrable. Equivalently one can introduce a contact form α with the property that $Vol_\alpha = \alpha \wedge (d\alpha)^n$ is a volume form, and construct ξ as $\ker \alpha$

- $\omega_p := (d\alpha)_p$ is a symplectic form on $\xi_p \subset T_pM$
- Exists a unique vector field $t \in \Gamma(TM)$, named Reeb v.f., satisfying $\alpha(t) = 1$ and $\omega(t, \bullet) = 0$



Contact geometry

California link

Contact distributions are often equipped with a sub-Riemannian structure. Geometric control theory/deppe learning (DC2, DC9 and DC10)

Example

$M = \mathbb{R}^3$ with $\alpha = ydx - dz$ is a contact manifold with hyperplane distribution given by $\xi = \ker \alpha = span\{\partial_y, y\partial_z + \partial_x\}$ and $t = -\partial_z$

Theorem

Locally we can always find a coordinate system s.t. $\alpha = p^i dq_i - dt$

Contact geometry

Comment: Contact geometry is closely related to classical mechanics.

$$S = \int_\gamma (p\dot{q} - H(p,q,t))dt$$

that controls the dynamics of some system. This can be interpreted as local expression of the pullback on γ of the contact form $\alpha = pq - Hdt$ on some manifold M in local coordinates (p,q,t). Hamiltonian setting is "good" for quantization.

Theorem

$(M, \alpha = p_i q^i - Hdt)$ an odd dimensional manifold with a contact form and an Hilbert bundle Z that we name quantum connection. We can locally construct

$$\nabla = d + \frac{\alpha}{i\hbar} + ...$$

a linear flat connection for Z. Then the equation $\nabla\Psi = 0$ reproduces the Schroedinger equation.

Contact geometry

Theorem (Quantum Darboux theorem)

Any quantum connection ∇ is formally gauge equivalent to the Darboux one $\nabla_D = d + \frac{1}{i\hbar}(p^i dq_i - dt) + \hat{k}$ that you might think as the simplest one in town.

California link

Thi quantization procedure is directly linked to BRST quantization of first class constraint that is a special case of the BV quantization approach (see again DC11) and it might somehow be useful for quantum information geometry (see Perez-Canellas and Ercolessi DC7 and DC8).

Contact geometry

I lied to you sorry...

We have discussed so far strict contact structures (M, α) while contact geometry must be understood as $(M, [\alpha])$.
In fact when $\hat{\alpha} = \Omega^2 \alpha$ we have $\ker \alpha = \xi = \ker \hat{\alpha}$ but the other structures changes accordingly as for example the Reeb dynamics.

What about

$$(M, [g]) \longrightarrow (M, [\alpha])$$

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What about

$$(M, [g]) \longrightarrow (M, [\alpha])$$

STAY TUNED

Bibliography:

1. A. Cap, J. Slovak, *Parabolic geometries 1: Background and general theory*
2. R.Gover, A.Waldron, *Boundary calculus for conformally compact manifolds* Indiana University Mathematics Journal, 63(1): 119-163, 2014
3. Rod Gover, Emanuele Latini, and Andrew Waldron, *Poincaré-Einstein holography for forms via conformal geometry in the bulk* Mem. Amer. Math. Soc. 235 (2015)
4. A. Rod Gover, Emanuele Latini, Andrew Waldron, Yongbing Zhang, *Conformally compact and higher conformal Yang-Mills equations*, arXiv:2311.11458 [math.DG]
5. G. Herczeg, E. Latini and A. Waldron, *Contact quantization: Quantum mechanics=parallel transport*, archivum mathematicum.