Induced representations and the (super) Poincaré group

María A. Lledó

Kick off meeting of the Doctoral Network CaLIFoRNIA

Brno, January 2025

Table of contents

Introduction and motivation

The method of induced representations of a group G starts with a representation of a closed subgroup H which will 'induce' a representation of G. It is not difficult to see how this happens in general.

Suppose that we have a representation of H, say σ, in some vector space V. One important object is the coset space G/H. The other is a vector bundle constructed in the following way: take the Cartesian product $G \times V$ and define the following equivalence relation:

$$(g, v) \sim (g', v') \text{ if } g' = gh \text{ and } v' = \sigma(h^{-1})v,$$

for some $h \in H$, where $v, v' \in V$. The quotient set is denoted as

$$E = G \times_H V.$$

Introduction and motivation

E is a vector bundle with basis G/H and fiber V.

The space of sections of E is a vector space that carries a representation of G, induced by the representation of H.

If G is a Lie group and H a Lie subgroup, the representation of H can be finite dimensional, but the representation of G on E will have infinite dimension.

Under certain conditions, if the representation of H is unitary, the representation of G will also be unitary.

Introduction and motivation

The method of induced representations dates back to Frobenius (late XIXth century), who proposed it for finite groups. But the application to physics that we will see today is for Lie groups

Its use in physics is very important, because it produces all the unitary representations of the Poincaré group, starting from a representation of a certain subgroup. It is due to the work of Wigner and Mackey (middle XXth century).

Under the correspondence

Unitary representation of Poincaré $\quad\leftrightarrow\quad$ Particle

one can classify *kinematically* all the possible particles.

The Poincaré group

The Poincaré group is the set of transformations of the Minkowski spacetime that leave invariant the metric

$$\eta = \mathrm{diag}(+1, -1, -1, -1).$$

It acts on Minkowski spacetime as

$$x^\mu \xrightarrow{(t,\Lambda)} \Lambda^\nu_\mu x^\mu + t^\nu, \qquad \mu = 0, \dots,$$

where Λ is a Lorentz transformation and t is a translation.

The Poincaré group

It is a semidirect product of the Lorentz group times the translations in spacetime. We will represent it as $P = T^4 \rtimes L$.

What does it mean?

For $(t, \Lambda), (t', \Lambda') \in P$, the group law is

$$(t, \Lambda)(t', \Lambda') = (t + \Lambda t', \Lambda\Lambda').$$

We are assuming that there is an action of L on $T^4 \cong \mathbb{R}^4$:

$$(\Lambda t)^\mu = \Lambda^\mu_\nu t^\nu.$$

Mackey's theorem for the Poincaré group

Let N be an abelian group. A character of N is an homomorphism

$$\chi : N \to S^1.$$

For example, take the translations, $T^4 \cong \mathbb{R}^4$.

$$\chi_p(x) = \exp \mathrm{i}\langle p, x\rangle, \qquad p \in {T^4}^* \cong (\mathbb{R}^4)^*, \; x \in T^4.$$

Mackey's theorem for the Poincaré group

The set of characters of N is also an abelian group denoted by $\hat{N}$. On $\hat{T}^4$ we have the dual action of L, given by the matrices Λ^{t-1}. We have, in fact, an action of the whole Poincaré on $\hat{T}^4$, but the translations act trivially.

For this action, and given a character $p \in \hat{T}^4$, we have:

- the isotropy group,

$$P_p = \{g \in P \mid gp = p\},$$

- the orbit,

$$\mathcal{O}_p = \{gp \quad \forall g \in P\},$$

- the little group,

$$H_p = L \cap P_p.$$

Mackey's theorem for the Poincaré group

Let p be a character of T^4. We are going to induce a representation of the whole P starting from a representation of the isotropy group.

Not all p's are the same, not all the little groups are the same!

The translations are represented by the character $\mathrm{e}^{\mathrm{i}\langle p,t\rangle}$, $t \in T^4$, as a global factor. It remains to see the fate of the little group.

Let $\mathcal{H}$ be a Hilbert space where we have a unitary representation of the little group

$$\sigma(h) : \mathcal{H} \to \mathcal{H}, \ h \in H_p.$$

Mackey's theorem for the Poincaré group

On T^4 we have also the Minkowski metric

$$\eta = \mathrm{diag}(+1, -1, -1, -1),$$

which induces a metric on $\hat{T}^4$.

Since the Lorentz group preserves the metric, the quantity

$$p^2 = p_\mu \eta^{\mu\nu} p_\nu = (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2$$

is invariant on every orbit.

It actually defines the orbits: for each value of p^2 we will have a different orbit.

Mackey's theorem for the Poincaré group

- $p^2 = m^2 > 0$. Time-like orbits. Little group SO(3). Massive particles.
- $p^2 = m^2 = 0$, $p \neq 0$. Light-like orbits. Little group $E(2) = T^2 \rtimes \mathrm{SO}(2)$. Massless particles.
- $p = 0$. The origin. Little group $\mathrm{SO}(1,3)$. Vacuum.
- $p^2 = -m^2 < 0$. Space-like orbits. Little group $\mathrm{SO}(2,1)$. Tachyons.

The Spin group

$$\mathrm{SU}(2) \xrightarrow{\varphi_1} \mathrm{SO}(3), \qquad \mathfrak{su}(2) \cong \mathfrak{so}(3).$$

$$\mathrm{SL}(2,\mathbb{C})_{\mathbb{R}} \xrightarrow{\varphi_2} \mathrm{SO}^0(1,3), \qquad \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \cong \mathfrak{so}(1,3).$$

They are $2 \to 1$ homomorphisms of groups.

$\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{C})_{\mathbb{R}}$ are double covers of $\mathrm{SO}(3)$, $\mathrm{SO}^+(1,3)$.

Physics is sensitive to the double over, so we should have started with

$$\hat{P} = T^4 \rtimes \mathrm{SL}(2,\mathbb{C})_{\mathbb{R}}.$$

This does not affect to the action on the momentum space.

Massive particles

The little group is $\mathrm{SU}(2)$, so we can put as germ of the induced representations all its (unitary) irreducible representations. These are classified by a non negative, half integer, the spin s.

- $s = 0$. Mesons, composed of a quark and an antiquark, for example $\pi^{\pm}, \pi^{0}$; Higgs. Bosons.
- $s = 1/2$. Quarks, leptons: electrons, muons, tauons and their associated neutrinos. Fermions.
- $s = 1$. $W^{\pm}, Z$. Bosons.
- $s = 3/2$. Rarita-Schwinger particle. Gravitinos (hypothetical). Fermions.

Massless particles

Little group $E(2) = T^2 \rtimes \mathrm{SO}(2)$. The Euclidean group is a semidirect product itself, so one can apply Mackey's theorem for semidirect products. We have two types of orbits:

- The origin. The isotropy group is the full $E(2)$, the little group is $\mathrm{SO}(2)$ (its double cover). Its representations are labelled by integers:

  $$\sigma(\mathrm{e}^{\mathrm{i}\alpha/2}) = \mathrm{e}^{\pm \mathrm{i} m\alpha/2}, \qquad m = 0, 1, 2, 3 \ldots$$

  The photons and gluons consists of two of these representations, $m/2 = \pm 1$. Instead of spin, helicity. Boson.

  The graviton (hypothetical) consists of $m/2 = \pm 2$. Boson.

Massless particles

- The regular orbits, circles of radius q. The isotropy group is $T^2 \rtimes \{\mathbb{1}, -\mathbb{1}\}$ and the little group is $\{\mathbb{1}, -\mathbb{1}\}$.

  The base manifold of the bundle is $\cong \mathrm{SO}(2)$ and the fiber $\mathbb{C}$.

  These representations are infinite dimensional.

  They are called 'continuous' or 'infinite' spin.

  They look like an infinite tower of particles of different helicities.

  They are bosonic or fermionic.

A hint of supersymmetry

Fermions and bosons have different statistics:

- When we take a bunch of identical bosons they can all be in the same, lowest energy state.
- When we take a bunch of identical fermions, no two of them can be in the same state.

The spin-statistics theorem justifies this phenomenon.

A hint of suypersymmetry

Despite of behaving so differently, they appear together in Quantum Field Theories.

Generically, matter is made of fermions, and the particles carrying the forces between them (electro-weak and strong nuclear forces and gravity) are bosons.

And the Higgs, that plays a special role.

A hint of suypersymmetry

Remarkably, all known symmetries of QFT relate bosons with bosons and fermions with fermions. This may seem natural from the physicists point of view (we are used to it), but it is not so from a mathematical point of view.

The spin statistics theorem tells us that bosons are fields with values in an ordinary manifold, with ordinary (even) commuting coordinates f^{μ}. But fermions take 'values' in a manifold with anticommuting (odd) coordinates:

$$\psi_\alpha\psi_\beta + \psi_\beta\psi_\alpha = 0, \text{ if } \alpha \neq \beta, \text{ and } (\psi_\alpha)^2 = 0.$$

These are no ordinary numbers.

A hint of suypersymmetry

We can see the set of coordinates (f^μ, ψ_α) as generators of an associative superalgebra, and through it we define what is a supermanifold.

Supermanifolds are the adequate geometry to describe quantum field theories.

But coordinate transformations of a supermanifold naturally mix even with odd coordinates!

$$f^\mu \longrightarrow f'^\mu = f^\mu + A^{\mu\alpha\beta}\psi_\alpha\psi_\beta,$$
$$\psi_\alpha \longrightarrow \psi'_\alpha = \psi_\alpha + B^\beta_{\mu\alpha}\psi_\beta f^\mu.$$

How is it possible that they are not related in the real world?

A hint of suypersymmetry

Either there is a strong principle to keep these two subspaces separated or... there must be some kind of supersymmetry.

The Poincaré super algebra

The Poincaré Lie superalgebra is composed of:

- translation generators, p_μ, (Lorentz vector),
- Lorentz generators, $M_{\mu\nu}$, antisymmetric $\mu \leftrightarrow \nu$, (antisymmetric Lorentz tensor),
- supersymmetry charges $Q_\alpha, Q_{\dot{\alpha}}$, (spinor charges).

The Poincaré superalgebra

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^{\mu}_{\alpha\dot{\alpha}} p_\mu$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho},$$

$$[M_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu,$$

$$[M_{\mu\nu}, Q_\alpha] = \frac{1}{2} (\sigma_{\mu\nu})_\alpha^\beta Q_\alpha$$

$$[M_{\mu\nu}, \bar{Q}_{\dot{\alpha}}] = -\frac{1}{2} (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} Q_{\dot{\beta}}$$

The Poincaré superalgebra

A Lie supergroup is a functor from the category of superalgebras to the category of groups.

A representation of a super Lie group is given by a representation of the underlying Lie group (in this case it would be the Poincaré group) and a representation of the Lie superalgebra.

$$Q|\mathrm{boson}\rangle = |\mathrm{fermion}\rangle, \qquad Q|\mathrm{fermion}\rangle = |\mathrm{boson}\rangle.$$

The result is that, instead of a particle, we have multiplets of particles that include bosonic and fermionic particles, all with the same mass.

The Poincaré superalgebra

- Chiral multiplet: massive $(0, 1/2)$, for example, (selectron,electron),
- Vector multiplet: massless $(1/2, 1)$, for example, (photino, photon),
- $(1, 3/2)$: Not known theory!
- Gravity multiplet: massless $(3/2, 2)$, (gravitino, graviton).

Each multiplet contains the same number of bosonic degrees of freedom than fermionic degrees of freedom.

THANK YOU FOR YOUR ATTENTION