Dynamic equations and non-absolute integrals in Banach spaces

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Differential and difference equations in infinite dimenensional Banach spaces are intensively studied. When we consider infinite systems of differential equations, evolution equations or functional-differential equations such problems should be considered as the problems in infinite dimensional spaces. This justify still growing development of equations in Banach spaces. Moreover, in many problems arising, for instance, in the control theory or mathematical economics it is necessary to consider both continuous and discrete models which lead to superfluous duplicating of the theory. One of the procedure of avoiding such problems is based on utilization of the notion of a time scale.

We need to enlarge this procedure also for problems in infinite dimensional Banach spaces by introducing new type of integrals for vector valued functions defined on a general time scale. Such a procedure was begun by Kurzweil in 1957 for differential equations in $\mathbb{R}$. Due to equivalence of differential and integral problems it is necessary to check the properties of some new integrals and then to investigate differential Cauchy problems. This was done, in particular, in the papers or books of J. Kurzweil, Š. Schwabik or G. Ye (cf. [5], [6] or [7]).

Due to equivalence of differential or dynamic problems to the integral form we are able to fully cover all theories for differential and difference equations in Banach spaces including all types of considered solutions (cf. M. Cichoń [1]).

The notion of a time scale was introduced by S. Hilger and allows us to treat by unified manner differential equations, integral equations and difference equations. Moreover, the so-called dynamic equations cover different kind of hybrid equations which do not involve solely continuous aspects or solely discrete aspects. For instance, neither difference equations nor differential equations give a good description of most population growth or when we try to describe a population dynamics where nonoverlapping generations occur.

The main goal of this note is to make possible such an advantage of dynamic equations also for vector-valued functions i.e. for dynamic modeling in Banach spaces. To do it we define appropriate integrals on time scales and we prove their properties which are useful for solving dynamic equations. For the vector-valued functions this topic is not sufficiently investigated. Such integrals are necessary to unify theories of differential, difference, q-difference equations for vector-valued functions. Each of these theories is intensively developed, but the unification is still an open problem, mainly due to lack of research dealing with different kind of derivatives and integrals. We will deal with a new type of integrals (cf. [2]).

Let us also note, that by using our new integrals we are able to extend all existing results for dynamic equations in Banach spaces including the latest ones for other classes of solutions. In particular, for Carathéodory solutions we have the following result:

Theorem 1. *Suppose that a function $f : I_a \times B_r \to E$ is a Carathéodory function and* *there exists a constant $c > 0$ satisfying*

$$\alpha(f(I_b, X)) \leq c \cdot \alpha(X), 0 \leq ca < 1, \tag{1}$$

for each $X \subset B_r$ and for each subinterval I_b of I_a. Assume that there exist bounded, right-dense continuous functions $a, b : I_a \to E$) such that $\|f(t, x(t))\| \leq a(t) + b(t)\,\|x(t)\|$ for $(t, x) \in I_a \times B_r$. Then there exists at least one Carathéodory solution of the problem

$$\begin{array}{l} x^{\Delta}(t) = f(t, x(t)) \\ x(0) = x_0 \end{array} \ , \ \ t \in I_a, \tag{2}$$

on some subinterval $I_b \subset I_a$.

By using the properties of different non-absolute integrals we are able to prove also the result for the weak type of solutions (cf. [2]:

Theorem 2. *Suppose that a function $f : I_a \times B_r \to E$ satisfies the following conditions: (C1) $f(t, \cdot)$ is weakly–weakly sequentially continuous, for each $t \in I_a$, (C2) For each strongly absolutely continuous function $x : I_a \to E$, $f(\cdot, c)$ is weakly continuous, (C3) For any nondecreasing Kamke function $\beta(f(I_b \times X) \leq h(X)$ for each $X \subset B_r$ and for each $I_b \subset I_a$, (C4) There exist bounded, right-dense continuous functions $a, b : I_a \to E$ such that $\|f(t, x(t))\| \leq a(t) + b(t)\,\|x(t)\|$ for $(t, x) \in I_a \times B_r$.*
Then there exists at least one Δ-weak solution of the problem (2) *on some subinterval $I_b \subset I_a$.*

Let us mention, that for vector-valued dynamic problems we still have some classical problems for solving. For example, we check how dense must be a time scale in such a way that Peano's Theorem holds and we present a counterexample to Peano's Theorem on a time scale with only one right-dense point ([4]).

References

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