The second eigenfunction of the p–Laplacian on the planar disc

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We will present results from the joint paper with Jiří Benedikt and Petr Girg [3], where we have used computer to give a part of rigorous proof of the fact that the *second eigenfunction* of the p–Laplacian on the disc is *not radially symmetric*.

The role of computers in mathematics had become very important from the very beginning, when the first computers were invented and spread among universities and research institutions all over the world. In particular, young people are very skilful and capable to use computers in many different contexts.

In mathematics and other sciences as well, computer is an important tool which helps to perform the research on higher level and allows more deep insight in many problems. However, it should be emphasized that the research is always performed by the scientist but not by the computer itself.

There are basically *three different ways of using computers in mathematics*:

1. *To calculate numerically or symbolically (and to plot graphically if necessary) the solutions of real world problems with concrete data and inputs*. This is usually done when the solution is known to exist and the error estimate can be given. *Numerical mathematics* deals with problems of this kind.

2. *To look for solutions the existence of which is not a priori known*. To perform numerical simulations and experiments with various data and inputs. Such a "laboratory approach" can be very useful step towards the rigorous mathematical proof of mathematical statement the right formulation of which is one looking for. This way of using computers is sometimes called *numerical experiments.*

3. *To use computer performed steps as a part of rigorous mathematical proof*. There are several examples of this approach, maybe the most famous one from the four colour problem in the graph theory. Nevertheless, it seems that such *computer aided proofs* are still not much recognized in the literature and even not acceptable for some mathematicians at all.

We are convinced that for proving assertion similar to Theorem 1 in [3], computer aided proof represents one of the possible tools beside the direct or indirect proof or the mathematical induction.

Let $D \subset \mathbb{R}^2$ be the open unit disc centered at the origin. We consider the following eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1}$$

where $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian, $p > 1$, and λ is the spectral parameter. It is a well-known fact that the principal eigenfunction of (1) (corresponding to the least eigenvalue λ_1 of (1)) is a radial function which does not change the sign in D (see, e.g., Kawohl, Fridman [6]) and it is unique up to a multiple by a nonzero real number. The existence of sign changing radial eigenfunctions associated with higher eigenvalues was shown in Walter [9] (cf. also

Brown, Reichel [4]). Note that the radial eigenfunctions of (1) are determined by nonzero solutions $u = u(r)$ of the ordinary differential equation

$$-\left(r|u'|^{p-2}u'\right)' = \mu r|u|^{p-2}u \quad \text{in } (0,1) \tag{2}$$

subject to the boundary conditions

$$u'(0) = 0, \quad u(1) = 0. \tag{3}$$

It is also well-known that there is the second eigenvalue of (1), $\lambda_2 > \lambda_1$. There are no eigenvalues of (1) in (λ_1, λ_2), and an eigenfunction associated with λ_2 changes the sign exactly once in D (see, e.g., Anane, Tsouli [2]). Note that the structure of the set of all eigenvalues of (1) ($p \neq 2$) beyond λ_2 seems to be an interesting open problem.

The main result of [3] is the following statement.

Theorem 1. *An eigenfunction associated with λ_2 is not radial for all $p \in (1, +\infty)$.*

For the case $p = 2$, this fact follows from Payne [8]. In [3] we present a different argument to prove this fact and generalize it for arbitrary $p > 1$. It is important to note that the result from Theorem 1 for p sufficiently close to 1 follows from Parini [7, Thm. 6.1]. The proof of Parini's Theorem 6.1 is based on Cheeger's inequality and implies that a second eigenfunction of (1) is not radial provided $1 < p < p_0$, where p_0 is sufficiently close to 1. However, the value of p_0 is not quantified in [7].

We quantify the constant p_0 from [7] in our paper [3]. The proof of the statement for $p \geq p_0$ is a combination of asymptotic analysis for $p \to +\infty$ and the application of interval arithmetic.

To be more specific, in [3], we prove that $p_0 \geq 1.1$. This follows from an upper estimate of the Rayleigh quotient for the principal eigenvalue of (1) restricted to the upper half–disc. We also show analytically that the result is true for all $p \geq 226$. This is done by means of the variational characterization of the second eigenvalue of (1) (see, e.g., [5]) combined with the monotone dependence of the principal eigenvalue of the p–Laplacian on the domain and with the estimates for the first two zeros of the solution to the initial value problem

$$\begin{cases} -\left(r|u'|^{p-2}u'\right)' = r|u|^{p-2}u \quad \text{in } (0,+\infty), \\ u(0) = 1, \quad u'(0) = 0. \end{cases} \tag{4}$$

Note that for $p = 2$ the solution of (4) coincides with the Bessel function $J_0 = J_0(r)$, see Abramowitz, Stegun [1]. Our estimate reflects the fact that the shape of the solution to (4) approaches uniformly the piecewise linear "saw type" function as $p \to +\infty$ and, as a consequence, the second zero is almost three times as big as the first one. The proof of the statement for $1.01 \leq p \leq 226$ is based on self-validated numerical computation. Besides the proof a short overview of the interval arithmetic and an explanation of its role in the computer aided mathematical proofs is given in [3]. We also describe an implementation of some key functions in *Mathematica*®.

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