Autonomous functional differential equations in the frame of generalized ODEs

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The interest in studying generalized ordinary differential equations (GODEs) lies on the fact that they encompass RFDEs with or without impulses (see [1] for instance) and they are much simpler to deal with than RFDEs.

For instance, it is a well-known fact that we can approximate the Riemann integral, when it exists, of a function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ by a finite sum of type

$$\int f(s)ds \approx \sum f(\tau_i)(t_i - t_{i-1}),$$

where $a = t_0 < t_1 < \ldots < t_n = b$ is a partition of $[a, b]$, $\tau_i \in [t_{i-1}, t_i]$, for $i = 1, 2, \ldots, n$ and $\max\{t_i - t_{i-1},\ i = 1, 2, \ldots, n\}$ is sufficiently small.

It is also known that for a Banach space X with norm $\|.\|$ and a function $U : [a, b] \times [a, b] \to X$, the generalized Perron integral or Kurzweil integral can be approximated by a sum of the form

$$\int DU(\tau, t) \approx \sum [U(\tau_i, t_i) - U(\tau_i, t_{i-1})], \tag{1}$$

where $a = t_0 < t_1 < \ldots < t_n = b$ is a partition of $[a, b]$, $\tau_i \in [t_{i-1}, t_i] \subset]\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)[$, for a function $\delta : [a, b] \to (0, \infty)$. In particular, when $U(\tau, t) = H(\tau)t$, for $H : [a, b] \to X$, then (1) becomes

$$\int D[H(\tau)t] \approx \sum_{i=1}^{n} H(\tau_i)(t_i - t_{i-1}).$$

Observing these two approximations, it becomes clear that it is possible to relate autonomous RFDEs with autonomous GODEs in an easy way. Our aim in this work is therefore to prove that, in fact, it is possible to relate autonomous RFDEs with impulses with autonomous GODEs. To this end, we shall first define what is an autonomous GODE.

We denote by $G^{-}([a, b], \mathbb{R}^n)$ the Banach space of regulated functions from $[a, b] \subset \mathbb{R}$ to $\mathbb{R}^n$ with the usual supremum norm.

We consider the following initial value problem for a retarded functional differential equation:

$$\dot{y}\,(t) = f\,(y_t)\,, \quad y_{t_0} = \phi, \quad t \in [t_0, t_0 + \sigma], \tag{2}$$

where $\sigma, r > 0$, $\phi \in G^{-}([-r, 0], \mathbb{R}^n)$ and $f : G^{-}([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma]$ to $\mathbb{R}^n$ and

(A) there is a Lebesgue integrable function $M : [t_0, t_0 + \sigma] \to \mathbb{R}$ such that for all $y \in G^{-}([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ and for all $t \in [t_0, t_0 + \sigma]$,

$$|f(y_t)| \leq \int_{t_0}^{t} M\,(s)\,ds;$$

(B) there is a Lebesgue integrable function $L : [t_0, t_0 + \sigma] \to \mathbb{R}$ such that for all $x, y \in G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ and for all $t \in [t_0, t_0 + \sigma]$,

$$| [f(x_t) - f(y_t)] | \leq \int_{t_0}^{t} L(s) \|x_s - y_s\| \, ds.$$

We also consider the GODE in the following form:

$$\frac{dx}{d\tau} = D[H(\tau)t] \tag{3}$$

where $H : G^-([t_0, t_0 + \sigma], \mathbb{R}^n) \to G^-([t_0, t_0 + \sigma], \mathbb{R}^n)$.

Theorem 1. *Consider equation* (2)*, where $f : G^-([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$, $t \mapsto f(y_t)$ is Lebesgue integrable over $[t_0, t_0 + \sigma]$ and* (A), (B) *are fulfilled. Let $y(t)$ be a solution of problem* (2) *in the interval $[t_0, t_0 + \sigma]$. Given $t \in [t_0, t_0 + \sigma]$, let*

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), \ \vartheta \in [t_0 - r, t] \\ y(t), \ \vartheta \in [t, t_0 + \sigma]. \end{cases} \tag{4}$$

Then $x : [t_0, t_0 + \sigma] \to C([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ is a solution of the autonomous GODE in $[t_0, t_0 + \sigma]$.Conversely, let $x(t)$ be a solution of (3)*, with H given by $H(y)(\vartheta) = f(y_\vartheta)$, in the interval $[t_0 - r, t_0 + \sigma]$ satisfying the initial condition*

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), \ t_0 - r \leq \vartheta \leq t_0, \\ x(t_0)(t_0), \ t_0 \leq \vartheta \leq t_0 + \sigma \end{cases}$$

For every $\vartheta \in [t_0 - r, t_0 + \sigma]$, let

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), \ t_0 - r \leq \vartheta \leq t_0 \\ x(\vartheta)(\vartheta), \ \ t_0 \leq \vartheta \leq t_0 + \sigma. \end{cases} \tag{5}$$

Then $y(\vartheta)$ is a solution of the problem (2) *in $[t_0 - r, t_0 + \sigma]$.*

Acknowledgement

Joint work with M. Federson, supported by FAPESP grant 2009/06259-0.

References

[1] M. Federson and Š. Schwabik, Generalized ODEs approach to impulsive retarded differential equations. *Differential and Integral Equations*, 19 (2006), no. 11, 1201-1234.

[2] Š. Schwabik, Generalized Ordinary Differential Equations, *World Scientific, Singapore, Series in Real Anal.*, vol. 5, 1992.