On the coincidence of Pettis and McShane integrals and Hilbert generated spaces

Marián Fabian,

Prague, fabian@math.cas.cz

In [DR], R. Deville and J. Rodríguez prove that *every Pettis integrable function with values in a Hilbert generated space is already McShane integrable.* In [APR], A. Avilés, G. Plebanek, and J. Rodríguez construct *a weakly compactly generated Banach space X and a scalarly null (hence Pettis integrable) function from $[0,1]$ into X, that is not McShane integrable.* In this note, we elaborate some ideas from [APR] and get a more general result, see Theorems 1 and 2 below.

A Banach space X is called *weakly compactly generated* if it contains a weakly compact set which is linearly dense in it. X is called *Hilbert generated* provided that there are a Hilbert space Y and a linear bounded mapping from Y into X whose range is dense in X. A compact space is called *Eberlein (uniform Eberlein)* if it can be continuously injected into a Banach space (into a Hilbert space) provided with the weak topology. We recall well known facts that *a compact space K is Eberlein (uniform Eberlein) if and only if the corresponding Banach space $C(K)$ is weakly compactly generated (Hilbert generated),* see [F$\sim$, Theorems 12.12, 12.17].

Let λ denote the Lebesgue measure and let $f : [0,1] \longrightarrow X$ be a function with values in a Banach space X. We say that f is *Pettis integrable* if for every $x^* \in X^*$ the composition $x^* \circ f$ is Lebesgue inegrable and for every measurable set $E \subset [0,1]$ there is $x_E \in X$ such that $x^*(x_E) = \int x^*\big(f(t)\big)\mathrm{d}\lambda(t)$. We say that f is *McShane integrable* if there exists $x \in X$ such that for every $\varepsilon > 0$ there are $\eta \in (0,1)$ and a function δ assigning to every $t \in [0,1]$ an open subset $\delta(t) \subset [0,1]$, containing t, such that: for every finite family $\mathcal{E}$ of pairwise disjoint measurable subsets of $[0,1]$, with $\lambda\big(\bigcup \mathcal{E}\big) > 1-\eta$, and for every choice of points $t_E \in [0,1]$, with $\delta(t_E) \supset E,\ E \in \mathcal{E}$, we have $\big\| \sum_{E\in\mathcal{E}} \lambda(E) f(t_E) - x \big\| < \varepsilon$.

Theorem 1. *Let K be any Eberlein compact space, of density at most* $\mathbf{c}$*, which is* `not` *uniform Eberlein. Then there exist an Eberlein compact over-space $H \supset K$, of density at most* $\mathbf{c}$*, and a scalarly null (hence Pettis integrable) $f : [0,1] \longrightarrow C(H)$ which is not McShane integrable.*

Sketch of proof: According to Amir and Lindenstrauss, we may assume that $K \subset c_0(\Gamma)^+$ where $\#\Gamma \le \mathbf{c}$. By [F, 419I], there is a partition $[0,1] = \bigcup_{\gamma\in\Gamma} Z_\gamma$ such that $\lambda^*(Z_\gamma) = 1$ for every $\gamma \in \Gamma$. For $k \in K$, for every $S \subset \operatorname{supp} k$ and for every $\gamma \in S$ pick $t_\gamma \in Z_\gamma$, and define then $h(t_\gamma) = k(\gamma)$ if $\gamma \in S$ and $h(t) = 0$ otherwise. Let H denote the space of all h's constructed this way. Note that H is and Eberlein and not uniformly Eberlein compact space. Define $f : [0,1] \longrightarrow C(H)$ by

$$f(t)(h) = h(t), \quad h \in H, \quad t \in [0,1].$$

Then use Farmaki's result [Fa] that H is a uniform Eberelin compact space if and only if for every $\varepsilon > 0$ there is a partition $[0,1] = \bigcup_{n=1}^{\infty} \Delta_n^\varepsilon$ such that

$$\forall n \in \mathbb{N}\ \forall h \in H \quad \#\big\{t \in \Delta_n^\varepsilon :\ h(t) > \varepsilon\big\} < n.$$

Question. Is it possible to take $H := K$ in Theorem 1?

Theorem 2. *Let X be a weakly compactly generated Banach space, of density at most* $\mathbf{c}$*, which is* `not` *a subspace of a Hilbert generated space. Then there exist a weakly compactly generated space Y, of density at most* $\mathbf{c}$*, whose quotient contains X, and a scalarly null (hence Pettis integrable) $f : [0,1] \longrightarrow Y$ which is not McShane integrable.*

Question. Is it possible to take $Y := X$ in Theorem 2?

Remark 1. There do exist Eberlein compact spaces built on a hereditary family of finite subsets of $[0,1]$ that are not uniform Eberlein, see [BS], [LS, Example 5.2].

Remark 2. If K is a Gul'ko and not Talagrand compact space, or K is a Talagrand and not Eberlein compact space, then such a K is also suitable for the argument proving Theorem 1.

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