Clarke's generalized gradient of locally Lipschitz integral functional: Dushnik sense

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In this contribuition we define nonsmooth analysis concept to the integral functional

$$\mathcal{L}_{\beta,f}[x] = \int_a^b \cdot \, d\,\beta(t) \cdot f(t, x(t)) \,, \tag{1}$$

on the Banach space of all regulated functions $x : [a, b] \to X$, with values in Banach space X, where the integral is in Dushnik sense (see [3],[4],[5]). We start by studying the Lipschitz properties of regulated integrands and show how those can be translated into Lipschitz properties of the integral functional. The following definition is essential:

Definition 1. We will say that a regulated family $f : [a, b] \times X \to X$ is

(a) uniformly Lipschitz of rank $K \geq 0$ near a point x if, for some $\epsilon > 0$, $\forall\, t \in [a, b]$,

$$\|f(t, y) - f(t, z)\| \leq K\, \|y - z\| \,,\ \forall\, y, z \in B(x; \epsilon)$$

(b) pointwise Lipschitz of rank $k \geq 0$ near a point x if, for some $\epsilon > 0$ and some regulated function $k : [a, b] \to \mathbb{R}$, for every $t \in [a, b]$,

$$\|f(t, y) - f(t, z)\| \leq k(t)\, \|y - z\| \,,\ \forall\, y, z \in B(x; \epsilon)$$

Our first result shows that the Lipschitz property of the integrand is transferred to integral functional (1) and so we can use the notions of Nonsmooth Analysis.

Theorem 1. *Let $f : [a, b] \times X \to X$ be a regulated family satisfying one of the following conditions*

a *uniformly Lipschitz of rank K near a given point $y \in X$;*

b *pointwise Lipschitz of rank $k \geq 0$ near a point $y \in X$.*

Suppose $x \in G([a, b], X)$ and $\beta \in BV([a, b], X^*)$. Then the integral functional $\mathcal{L}_{\beta,f} : G^-([a, b], X) \to \mathbb{R}$ given by

$$\mathcal{L}_{\beta,f}[x] = \int_a^b \cdot \, d\,\beta(t) \cdot f(t, x(t))$$

is Lipschitz of rank $V[\beta]\ K$ near x.

We recall that the *generalized directional derivative* of the locally Lipschitz function $f : X \to \mathbb{R}$ at x, in direction v, is defined as

$$f^0(x;v) = \limsup_{\substack{\lambda \downarrow 0 \\ y \to x}} \frac{f(y+\lambda\, v) - f(y)}{\lambda}$$

and that the *generalized gradient* of f at x is the nonempty $\partial f(x) \subset X^*$ whose support function is $f^0(x;\cdot)$. For details see [1] [2]. Finally, we present the format of generalized directional derivative and gradient in Clarke's sense of $\mathcal{L}_{\beta,f}$. Appealing to Representation Theorem for linear functionals we conclude that $\forall\, v \in G^-([a,b],X)$,

$$\zeta \in \partial\mathcal{L}_{\beta,f}(x) \Longleftrightarrow \mathcal{L}^o_{\beta,f}(x;v) \geq \int_a^b \cdot\, d\alpha_\zeta(t) \cdot v(t)\ ,$$

characterizing the vectors belonging to the generalized gradient of (1). We have then that

Theorem 2. *A necessary and sufficient condition for* $\zeta \in \partial\mathcal{L}_{\beta,f}[x]$ *is that for all* $v \in G^-([a,b],X)$ *we have*

$$\limsup_{\substack{\lambda \downarrow 0 \\ y \to x}} \int_a^b \cdot\, d_s\ \beta(s) \cdot \frac{f(s,(y+\lambda\, v)(s)) - f(s,y(s))}{\lambda} \geq \int_a^b \cdot\, d\alpha_\zeta(t) \cdot v(t).$$

Remark 1. In this paper we extend some concepts on Nonsmooth Analysis to the context of regulated functions and Dushnik integrals. Our purpose is to derived optimality conditions for infinite dimensional optimization problems and in this respect this work is only the first step.

References

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