On two-dimensional nonabsolute integration

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The problem of generalizing the Lebesgue integral in order to obtain a large class of functions $F : [a, b] \to \mathbb{R}$ which satisfy

$$\int_a^b F' = F(b) - F(a). \tag{1}$$

is naturally extended to the case where the domain has two dimensions. It is desirable to define an integration process in some domain $U \subset \mathbb{R}^2$ which satisfy the following version of the divergence theorem:

$$\int_U divF = \int_{\partial U} F \cdot N, \text{for each differentiable } F : \mathbb{R}^2 \to \mathbb{R}^2. \tag{2}$$

It is known that for the one-dimensional case we obtain full generality (that is, (1) holds for each differentiable F) when we consider the equivalent Denjoy, Perron or Henstock-Kurzweil integrals (see e.g. [1]). The natural extension of the definition of the Henstock-Kurzweil integral for the two-dimensional case does not satisfy (2) even when U is an interval (that is, the cartesian product of compact intervals in the real line).

The present work includes a brief discussion on previous attempts on modifying the Henstock-Kurzweil definition in order to obtain (2). The discussion is centralized on the gauge-based definitions introduced in [2], [3] and [4]. We then present some old (maybe revisited under a different perspective) and new results that can be summarized as follows:

• The M_1-integral introduced in [4] has the unpleasant property of being sensitive to rotations; this is shown using an example found in [5]. As a subproduct, we prove that the two-dimensional Henstock-Kurzweil has the same property, independently from it's equivalence to the two-dimensional Perron integral.

• Through an example from [4], we generalize a result from this article and show that *any* attepmt on generalizing the two-dimensional Riemann integral in the interval in order to obtain (2) necessarily results in an integration process which fails to satisfy Fubini's theorem. (The incompatibility, at some level, between the divergence theorem and Fubini's theorem is well known and pointed out for example in [10])

• We propose a new definition of integral, based on the mentioned M_1-integral and private communications between the author and prof. Pavel Krejčí from the Institute of Mathematics of the Academy of Sciences of the Czech Republic:

Definition 1. Let $I \subset \mathbb{R}^2$ be a triangle and δ be a gauge in I. We say that a (finite, nonempty) set of the form $\mathcal{P} = \{(I^j, t^j)\}_{j \in \Gamma}$ is a Δ-*partition* of I if $\{I^j\}_{j\in\Gamma}$ is a partition of I into triangles with $t^j \in I^j$ for each $j \in \Gamma$, and we say that it is δ-*fine* when for each $j \in \Gamma$ we have that $I^j \subset B_{\delta(t^j)}(t^j)$. The *irregularity* of $\mathcal{P}$ is defined by $irr(\mathcal{P}) \doteq \sum_{j\in\Gamma} q(I^j)$, where $q(I^j) \doteq perimeter(I^j)\, diameter(I^j)$, and for each given $C > 0$, we say that $\mathcal{P}$ is C-*regular* when $irr(\mathcal{P}) \leq C$. We say that $f : I \to \mathbb{R}$ is Δ-*integrable* if there exists $A \in \mathbb{R}$ such that, for each $\epsilon > 0$ and each sufficiently large $C > 0$, there is a gauge δ in I satisfying $|\sum_{j\in\Gamma} f(t^j)|I^j| - A| < \epsilon$ for each δ-fine, C-regular Δ-partition $\mathcal{P} = \{(I^j, t^j)\}_{j\in\Gamma}$ of I.

This way we obtain an integral which is strictly less general than the M_1-integral and admits a change of variables formula valid up to affine tranformations. We prove basic and convergence properties for the Δ-integral, study it's relation with the Lebesgue integral, and propose some problems for further investigation.

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