New nonabsolutely convergent integrals

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The discovery of the Lebesgue integral opened a new age in the theory of integral. However, for some purposes, the class of Lebesgue integrable function is still too restrictive; namely, it may be desirable to integrate all derivatives. Denjoy [2], followed by Luzin [6] and Perron [8] found definitions of integral including both the Lebesgue integral and the "Newtonian" integral based on antiderivatives. These definitions were quite involved so not convenient for introductory courses of integration.

The elementary courses of calculus usually follow definitions based on ideas of Cauchy, Riemann and Darboux. These definition are intuitive and geometrical, but do not reach the strength of the Lebesgue definition. In fifties, Kurzweil [5] observed that a seemingly inconspicuous improvement of Riemann's definition leads to a concept of integral which is equivalent to the Perron integral. The same definition has been later rediscovered and studied by Henstock [3]. The definition is so comprehensible that it is conceivable to use it for teaching.

The definitions by Riemann, Lebesgue or Kurzweil are *constructive*, the idea is to partition the space, obtain approximate integral sums and pass to the limit. On the other hand, the idea of Newton is that an antiderivative (indefinite integral) F is associated with the given data f and the definite integral is defined as the increment of the antiderivative. Since the function F is defined by describing its properties, (typically that f is in some sense the derivative of F), such definitions are called *descriptive*. Luzin's definition is a typical example of an advanced descriptive approach.

We present a new very simple definition of integral in the strength of Perron (Kurzweil,...) integral. Whereas Kurzweil's definition is the simplest one what is known in the class of constructive definitions, our definition is a candidate to be a most simple definition in the class of descriptive definitions.

Our definition [1] is the following

Definition 1. Let $I = (a, b) \subset \mathbb{R}$ be an interval and $f, F : I \to \mathbb{R}$ be functions. We say that F is an MC*-antiderivative* (monotonically controlled derivative) of F if there exists a strictly increasing function $\varphi\colon I \to \mathbb{R}$ (the so-called *control function* to the pair (F, f)) such that

$$\lim_{y\to x} \frac{F(y) - F(x) - f(x)(y - x)}{\varphi(y) - \varphi(x)} = 0, \qquad x \in I. \tag{1}$$

We want to demonstrate how easily the calculus of integral can be developed from our definition. Let us notice that it avoids conditions of absolute continuity type. In addition, no a priori knowledge of null sets is needed. Once we show that any MC-antiderivative of a positive function is increasing, it follows that the definition of integral is correct. We can easily derive the product rule, the chain rule and the monotone convergence theorem. We can also investigate a Stieltjes version in which the numerator in (1) is replaced by $F(y) - F(x) - f(x)(G(y) - G(x))$.

The indefinite integral F can be identified with the interval function

$$\mathbf{F} : [a, b] \mapsto F(b) - F(a).$$

The language of interval function allows to generalize the integral into the multidimensional case. In this case there is a diversity of definitions in dependence on aims: some definitions support the Fubini theorem, other (never the same) integrate all derivatives. The latter purpose requires some regularity of intervals used in the limiting process, as studied by Mawhin [7], Jarník, Kurzweil and Schwabik [4] and others. Our method of monotone control can be adapted to give alternative definitions of various multidimensional integrals.

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