Regularity and long-time behavior of a nonlocal phase separation system

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We will present an alternative to the Cahn-Hilliard model of phase separation for two-phase system. The model is derived from a free energy with a nonlocal term which accounts for interactions between states in both short and long scales. This phenomenon is represented by spatial convolution with a suitable kernel. Considering a binary alloy with components A and B occupying a spatial domain Ω, and denoting by u and $1-u$ the local concentrations of A and B respectively, the model in question reads:

$$u_t - \nabla \cdot (\mu \nabla v) = 0 \text{ in } (0,T) \times \Omega, \tag{1}$$

$$v = f'(u) + \int_\Omega K(|x-y|)(1-2u(t,y))dy, \quad (t,x) \in (0,T) \times \Omega, \tag{2}$$

$$\mu\nu \cdot \nabla v = 0 \text{ in } (0,T) \times \partial\Omega, \tag{3}$$

$$u(0,x) = u_0, \quad 0 \le u_0(x) \le 1, \quad 0 < \int_\Omega u_0 = u_\alpha < 1. \tag{4}$$

Here the chemical potential v is the gradient of the energy functional F,

$$F(u) = \int_\Omega \Big[f(u) + u \int_\Omega K(|x-y|)(1-u(t,y))dy \Big], \tag{5}$$

and μ denotes a suitable mobility. A natural choice is

$$\mu = \frac{a}{f''(u)}.$$

In the standard case, f is given by

$$f(u) = u \ln u + (1-u)\ln(1-u). \tag{6}$$

This implies

$$f'(u) = \ln\Big(\frac{u}{1-u}\Big), \ \mu = \frac{a}{f''(u)} = au(1-u). \tag{7}$$

Hence

$$(f')^{-1}(v-w) = \frac{1}{1+\exp(w-v)}, \ w = \int_\Omega K(|x-y|)(1-2u(t,y))dy, \tag{8}$$

which gives the *a priori* estimate

$$u \in [0,1]. \tag{9}$$

With μ as in (7), and u satisfying (9), $\mu(0,x)$ can vanish, even on a set of a positive measure, so a degeneracy in (1) and (4) is not excluded, which obviously represent a difficulty.

Gajewski and Zacharias [1] proved global existence and uniqueness of weak solutions, and showed that the $\omega-$limit sets of trajectories emanating from the initial values satisfying (4) are contained in the set of stationary solutions, which are given by the formula

$$u^*(x) = \frac{1}{1+\exp(w^*(x)-v^*)},\ \ w^*(x) = \int_\Omega K(|x-y|)(1-2u^*(t,y))dy,\ v^* = const.$$

The set of stationary solutions may have a complicated structure, and in the case where this set is not discrete, the question whether any solution stabilizes to a single equilibrium is not trivial. The convergence result was obtained in [2] by applying the generalized Łojasiewicz-Simon theorem. Because of the degenerate character of our system, and the singularity of the potential, the main difficulty was to prove the proper separation property, which is necessary for the verifying of assumptions in the Łojasiewicz type theorem. The following result was proved in [2]:

Theorem 1. *Let u be the solution of (1)-(4), Then there exist $T_0 \geq 0, k > 0$ such that*

$$k \leq u(t,x) \leq 1-k \text{ for a.a. } x \in \Omega, \text{ and } t \geq T_0, \tag{10}$$

and

$$u(t) \to u^* \ \ \textit{strongly in} \ \ L^2(\Omega) \ \ \textit{as} \ \ t \to \infty.$$

With (10) at hand, it is possible to prove that solutions are more regular, show that the trajectory is compact in $C(\Omega)$, and obtain convergence in the space of continuous functions:

$$u(t) \to u^* \ \ \text{strongly in} \ \ C(\Omega) \ \ \text{as} \ \ t \to \infty.$$

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References

[1] H. Gajewski and K. Zacharias, *On a nonlocal phase separation model*, J. Math. Anal. Appl.,**286** (2003), 11–31.

[2] S.-O. Londen, H. Petzeltová, *Convergence of solutions of a non-local phase-field system*, DCDS-S (2010), to appear.