Asymptotic properties of solutions to second order dynamic equations in the framework of regular variation with emphasis on q-calculus case

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In [10] we introduced a *regularly varying function* $f : \mathbb{T} \to (0, \infty)$ *of index* ϑ (we write $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$) as a measurable function satisfying $f(t) \sim C\alpha(t)$ as $t \to \infty$ and $\lim_{t\to\infty} t\alpha^{\Delta}(t)/\alpha(t) = \vartheta$, with $C \in (0, \infty)$ and a positive sufficiently smooth α. In [13] we established the following equivalent (under certain assumption on the graininess) Karamata characterization of $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$: $\lim_{t\to\infty} f(\tau(\lambda t))/f(t) = \lambda^{\vartheta}$ uniformly on each compact λ-set in $(0, \infty)$, where $\tau : \mathbb{R} \to \mathbb{T}$, $\tau(t) = \max\{s \in \mathbb{T} : s \le t\}$. Various equivalent representations and properties of $\mathcal{RV}_{\mathbb{T}}$ functions were obtained in [10, 13]. This theory was applied to study asymptotic properties of solutions to second order dynamic equations, see [10, 13]. For the classical continuous resp. discrete theory of regular variation see e.g. [2], resp. [4], and for its applications in the theory of differential resp. difference equations see e.g. [7, 8], resp.[9].

Soon it has turned out that it is advisable and somehow necessary to distinguish three following cases when studying regular variation on time scales: (i) $\mu(t) = o(t)$. Then we obtain a continuous like theory, which is mentioned above; details can be found in [10, 13]. The condition $\mu(t) = o(t)$ cannot be omitted. (ii) $\mu(t) = Ct$ with $C > 0$. Here we are in the q-calculus case and we obtain surprisingly simplified and powerful theory. It is briefly described below and details can be found in [11, 12]. (iii) Other cases (in particular, either a "very big" graininess or a combination of "big" and "small' graininess). Then we get no reasonable theory which would satisfy our natural requirements. There are more reasons for such a categorization: For example, we want to prove important (equivalent) characterizations of $\mathcal{RV}_{\mathbb{T}}$ functions and we want $f(t) = t^{\vartheta}$ to be an element of $\mathcal{RV}_{\mathbb{T}}(\vartheta)$, which is impossible without additional conditions on μ; in contrast to the case $\mu(t) \sim Ct$, with $C > 0$, where $\mu(t) \not\equiv Ct$, or to all other cases, the theory of regular variation in q-calculus is simple, elegant, and shows very interesting untypical features – this provides strong tools in applications.

We will focus primarily on the q-calculus case, where we work on the q-uniform lattice $q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$. For a basic material on this calculus see [1, 5, 6]. See also [3] for the calculus on time scales which somehow contains q-calculus. The theory of q-calculus is very extensive with many aspects; some people speak about different tongues of q-calculus. In our consideration we follow essentially its "time scale dialect". Concerning the definition of q-regularly varying functions, we may follow general theory and define it in terms of the Jackson derivative or to provide a Karamata type definition. But, as can be shown, we can work with much simpler (but still equivalent) characterization, which is not known in the classical continuous or the discrete case. Such a simplification is possible because of special structure of $q^{\mathbb{N}_0}$, which turns out to be very "natural environment" for the Karamata like theory. Indeed, q-regular variation can be characterized in terms of relations between $f(t)$ and $f(qt)$, which perfectly fits the discrete q-calculus, in contrast to other settings. A function $f : q^{\mathbb{N}_0} \to (0, \infty)$ is said to be *q-regularly varying of index ϑ* if $\lim_{t\to\infty} f(qt)/f(t) = q^{\vartheta}$. We are interested also in situations where the limit value attains their extremal values ∞ and 0; this leads to the concept of *q-rapid variation of index* $\pm\infty$. We also introduce the concept of *q-regular boundedness*, which can be seen as a generalization of q-regular variation in the sense that the limit may not exist, but the expression is somehow bounded. Every such a function is automatically normalized (which does not hold in the continuous case). Various characterizations and properties of these functions were established in [12, 11]. We consider the equation $D_q^2 y(t) + p(t)y(qt) = 0$ (with no sign condition on p). Linear q-difference equation were studied e.g. in [1]; see also the references therein. In [11] we established sufficient and necessary conditions for all positive solutions of this equation to behave like the above introduced functions. In contrast to the differential equations case, the conditions are not in integral form and show also some other differences. The methods in the proofs are different too. Since q-calculus is natural setting in $\mathcal{RV}$ theory, these methods seem to be very promising also for the examination of more general or other q-difference equations. Such results may then serve in other parts related to q-calculus or may serve to predict how their continuous counterpart (which can be difficult to be examined) could look like.

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