Cographs, twofold symmetric monoidal structures, and factorization algberas in quite a lot of generality

C. Barwick

Clark Barwick (clark.barwick@ed.ac.uk)
The University of Edinburgh

Abstract.

The observables of a quantum field theory assemble into a remarkable algebraic gadget called a *factorization algebra*. The shape of the multiplication law in a factorization algebra is controlled by the geometry of a spacetime manifold (or related space). Intuitively, one imagines objects attached to finite collections of points of the spacetime which then multiply as the points collide. In practice, the precise behavior of these objects is sensitive to the definitional apparatus one uses, which in turn often takes advantage of certain particularities of the geometric setting. For example, some definitions exploit the fact that every point of a manifold is contained in an open set homeomorphic to a euclidean space.

In the past few years, there has been interest [2, 5] in making sense of *arithmetic quantum field theories*. In such a context, spacetime would not be any sort of manifold but rather a variety over a finite field, or arithmetic scheme, or something more exotic. How could one study such quantum field theories? What sort of algebraic gadget should the observables form?

To answer these questions, in work in progress [4], I give an approach to defining factorization algebras that works with a quite general category $\mathbf{X}$ of geometric objects. On these objects, one needs a structure that will allow us to make sense of the idea that two points (or generalized points) are 'apart' – or at any rate sufficiently apart to have observations made near these points combined. This structure – which I call an *isolability structure* – is a functor to $\mathbf{X}$ from a certain category $\mathbf{D}$ of graphs called *cographs*.

This use of the prefix 'co' is unfortunate at a category theory conference. Cographs are a particular kind of graph, and they are not in any sense dual to graphs. Rather, 'co' is short for *complement reducible*, which is a way of expressing the universal property that $\mathbf{D}$ has as a *twofold symmetric monoidal category*.

Twofold monoidal structures, which were identified by Borcherds [3] and studied carefully by Balteanu–Fiedorowicz–Schwänz–Vogt [1], are pairs of monoidal structures (on the same category) along with an intertwiner map that is generally not invertible. The (graded) category of *sheaves* on an isolability object is (via Day convolution) twofold symmetric monoidal, and the category of factorization algebras is then be defined naturally in terms of this structure.

More speculatively, what this suggests is a new kind of geometry that bears the same relationship to twofold symmetric monoidal categories that algebraic geometry bears to symmetric monoidal categories. Time permitting, we will end with some thoughts in this direction.

References

[1] C. Balteanu, Z. Fiedorowicz, R. Schwänzl, R. Vogt, *Iterated monoidal categories*, Adv. Math., vol. 176 no. 2, pp. 277–349, 2003.

[2] D. Ben-Zvi, B. Sakellaridis, and A. Venkatesh, *Relative Langlands duality*, preprint arXiv:2409.04677, 2024.

[3] R. Borcherds, *Quantum vertex algebras*, in: *Taniguchi Conference on Mathematics Nara*, pp. 51–74, 2001.

[4] C. Barwick, *Factorization algebras in quite a lot of generality*, preprint `https://clarkbar.github.io/facts.pdf`, 2025.

[5] M. Kim, *Arithmetic Chern–Simons theory, I*, in: *Galois Covers, Grothendieck–Teichmüller Theory and Dessins d'Enfants*, 2020.