2-classifiers for 2-algebras

M. Capucci

Matteo Capucci (`matteo.capucci@gmail.com`)
University of Strathclyde & Independent[1], Modena (IT)

David Jaz Myers (`davidjaz@topos.institute`)
Topos Research UK, Oxford (UK)

Abstract. In [2], Paré investigates the Yoneda theory of double categories, giving a construction of *lax* double functors $D(d, -) : D \to \mathbb{S}\mathbf{pan}(\mathbf{Set})$ represented by objects of double categories. The double categories of elements $\mathbf{El}(F)$ of lax functors $F : D \to \mathbb{S}\mathbf{pan}(\mathbf{Set})$ are discretely fibered over D by a *strict* double functor $\mathbf{El}(F) \to D$; as shown in [3]. This gives an equivalence between the categories of *lax* functors $D \to \mathbb{S}\mathbf{pan}(\mathbf{Set})$ and the category of *strict* discrete fibrations over D.

Motivated by applications in categorical systems theory, we generalize this equivalence to general 2-algebraic theories, showing that discrete opfibration classifiers with suitable 2-algebraic structure classify *strict* discrete opfibrations in the 2-category of algebras and lax morphisms.

More precisely, let $\mathcal{K}$ be an enhanced 2-category equipped with a *good 2-classifier* $1 \xrightarrow{\tau} \Omega$ in the sense of Mesiti [1], meaning it classifies tight discrete opfibrations. We ask: when can we lift such a structure to the enhanced 2-category of (strict) 2-algebras (with tight strict morphisms as tights and lax morphisms as looses) of an enhanced 2-monad T on $\mathcal{K}$?

We single out the following condition:

Definition 1. *An* Ω-***structure*** *on* T *is a* T*-algebra structure* $\omega : T\Omega \to \Omega$ *which classifies* Tu*, where* $u : 1/\Omega \to \Omega$ *is the free (tight) discrete opfibration.*

The theorem we prove is the following:

Theorem 1. *Let* T *be an enhanced 2-monad which moreover (1) preserves pullbacks of tight discrete opfibrations and (2) has tight-cartesian*[2] *unit and multiplication. Then an* Ω*-structure on* T *induces an enhanced 2-classifier in the enhanced 2-category of* T*-algebras and lax morphisms.*

Crucially, the construction can be easily iterated since $\mathbf{Alg}_l(\mathcal{K})$ is again an enhanced 2-category with a good 2-classifier. Indeed, our main applications are: (1) for $\mathcal{K} = \mathbf{Cat}$ and $T =$ free SMC, (2) $\mathcal{K} = \mathbf{Cat}^{\cdot \rightrightarrows \cdot}$ and $T =$ free double category. Since the free SMC construction lifts to $\mathbf{Cat}^{\cdot \rightrightarrows \cdot}$, we see how by applying Theorem 1 twice we can lift $1 \xrightarrow{\{*\}} \mathbf{Set}$ from $\mathbf{Cat}$ to double categories (recovering [2]) to symmetric monoidal double categories.

References

[1] L. Mesiti, *2-classifiers via dense generators and Hofmann-Streicher universe in stacks*, Canadian Journal of Mathematics, pp. 1-52, 2024

[2] R. Paré, *Yoneda Theory for Double Categories*, Theory and Applications of Categories, vol. 25, no. 17, pp. 436-489, Nov. 2011

[3] M. Lambert, *Discrete Double Fibrations*, Theory and Applications of Categories, vol. 37, no. 22, pp. 671-708, Jun. 2021

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[2] Meaning the naturality squares at tight maps are pullbacks.