A new framework for limits in double categories

B. Clarke

Nathanael Arkor (`https://arkor.co`)
Tallinn University of Technology, Estonia

Bryce Clarke (`https://bryceclarke.github.io`)
Tallinn University of Technology, Estonia

Abstract. Robert Paré introduced the notion of limit in a double category at the International Category Theory Conference in 1989, showing that weighted limits in a 2-category arise as a special case. The subsequent paper with Marco Grandis [1] presents an elegant theory of double limits, and proves that the limit of any double functor $F\colon \mathbb{J} \to \mathbb{D}$ exists if and only if the double category $\mathbb{D}$ has small double products, double equalisers, and *tabulators*. Many double categories admit all limits and colimits in this sense, including the double category $\mathbb{S}\mathrm{pan}$ of sets, functions, and spans, and the double category $\mathbb{D}\mathrm{ist}$ of categories, functors, and distributors (a.k.a. profunctors).

Among the most fundamental concepts in double category theory are those of *companion* and *conjoint*, together with closely related notions of *restriction* and *corestriction*. A double category has companions and conjoints if and only if it has restrictions if and only if it has corestrictions—in this case, it is called an *equipment* or *framed bicategory* [2]. Another important concept is a *local (co)limit* in a double category [3]. For example, local coequalisers are required to construct the double category $\mathbb{M}\mathrm{od}(\mathbb{D})$ of monads, monad morphisms, and bimodules in a double category $\mathbb{D}$. Despite the central role they play, neither restrictions (thus companions and conjoints) nor local limits are captured by the limit of a double functor, which is defined as an *object* rather than a *loose morphism* in the double category.

In this talk, we introduce a new framework for limits in double categories which extends the work of Grandis and Paré, and captures companions, conjoints, restrictions, and local limits as examples. The novel aspect of this framework is to use a *loose distributor* $J\colon \mathbb{S} \nrightarrow \mathbb{T}$ between double categories as the *shape* of a diagram. Our main theorem characterises when a double category admits all limits of diagrams in this new sense; examples include both $\mathbb{S}\mathrm{pan}$ and $\mathbb{D}\mathrm{ist}$.

References

[1] Marco Grandis & Robert Paré, *Limits in double categories*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, Vol. 40 (1999).

[2] Michael Shulman, *Framed bicategories and monoidal fibrations*, Theory and Applications of Categories, Vol. 20 (2008).

[3] Robert Paré, *Composition of modules for lax functors*, Theory and Applications of Categories, Vol. 27 (2013).