Exploring dualities beyond sound doctrines

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Abstract.

Several dualities in categorical logic (including Gabriel-Ulmer duality) admit a unified treatment in terms of the kinds of limits required to exist in the relevant "theories" (finite limits in the Gabriel-Ulmer case). Given a collection of small categories Φ (a "doctrine"), we may regard categories $\mathcal{C}$ having all limits of all shapes $\mathcal{J} \in \Phi$ as syntax-independent avatars for certain essentially algebraic theories, possibly with infinitary operations; **Set**-models of $\mathcal{C}$ are precisely the Φ-limit-preserving functors $\mathcal{C} \to \mathbf{Set}$.

A technical condition on Φ – that of being *sound* (a notion introduced in [1]) – guarantees a duality between the 2-category **Φ-cat** of small Φ-complete categories and that of locally Φ-presentable categories (defined much like locally finitely presentable categories, with all instances of "finite limits" swapped with "Φ-limits"). The bi-equivalence is given in one direction by taking models in **Set**, and in the other by taking (the opposite category of) Φ-presentable objects. The general theorem was proved in [2].

In this talk, we will investigate the extent to which soundness is necessary to obtain some form of duality $\Phi[-, \mathbf{Set}] \colon \mathbf{\Phi\text{-}Cat} \to \mathcal{K}^{\mathrm{op}}$. When $\mathcal{K}$ is the 2-category of locally Φ-presentable categories and the inverse 2-functor is that of taking Φ-presentable objects, it quickly follows that Φ is sound. We first drop the second requirement, showing that if $\Phi[-, \mathbf{Set}]$ is an equivalence with $\mathcal{K} = \mathbf{L\Phi P}$ then Φ must still necessarily be sound. This is done by exhibiting a (relative) right adjoint to $\Phi[-, \mathbf{Set}]$ for an arbitrary doctrine (essentially via the cartesian closedness of **Cat**). We then drop the assumption that $\mathcal{K}$ consists of the locally Φ-presentable categories. In this general situation, the existence of a duality (potentially "abstract", i.e. with $\mathcal{K}$ not defined explicitly) for Φ turns out to depend *precisely* on whether there are limits not present in Φ-complete categories that are nonetheless preserved by Φ-continuous functors (much like the existence of limits preserved by all functors). Indeed, the relevant condition on **Φ-Cat** concerns the nature of the *lax epimorphisms* therein, which in **Cat** are shown in [3] to be precisely the absolutely (co)dense functors.

References

[1] J. Adámek, F. Borceux, S. Lack, J. Rosický, *A classification of accessible categories*, 2002.

[2] C. Centazzo, E. Vitale, *A duality relative to a limit doctrine*, 2002.

[3] J. Adámek, R. El Bashir, M. Sobral, J. Velebil, *On functors which are lax epimorphisms*, 2001.