Level é

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Abstract.

The dimension theory formulated in [1, Section II] does not identify dimensions with numbers but with *levels* (i.e. essential subtoposes), so that each topos determines a partial order of dimensions equipped with a 'next dimension' operator (the Aufhebung). This order need not be total and leaves room for 'extraordinary' dimensions such as infinitesimal ones [2] or as the one introduced below.

Partially inspired by the distinction between 'gros' and 'petit' toposes, [1, Section I] proposes a tentative clarification of the distinction and investigates how one class arises from the other: given an object X in a topos 'of spaces', what is the reasonable topos of pseudo-classical sheaves on X? Lawvere complains that he could not give a site-invariant description but, in order to clarify the problem, he proposes to study a class of toposes $\mathcal{E}$ such that, for every object X in $\mathcal{E}$, $\mathcal{E}/X$ has a well-defined QD-subtopos $\mathrm{P}X$. Additionally, he formulates a conjecture relating $\mathrm{P}X$ and the dimension of X.

Replacing QD toposes with étendues we can give a site-invariant description of a candidate 'petit topos' associated to each object, and a proof of the corresponding conjecture.

We say that a geometric morphism $g : \mathcal{G} \to \mathcal{S}$ is an *étendue* if there is a well-supported object G in $\mathcal{G}$ such that the composite

$$\mathcal{G}/G \longrightarrow \mathcal{G} \xrightarrow{g} \mathcal{S}$$

is localic, where $\mathcal{G}/G \to \mathcal{G}$ is the canonical geometric morphism induced by G.

Definition 1. A geometric morphism $f : \mathcal{F} \to \mathcal{S}$ has a *level é* if $\mathcal{F}$ has a largest level $\mathcal{L} \to \mathcal{F}$ such that $\mathcal{L} \to \mathcal{F} \to \mathcal{S}$ is an étendue.

We are interested in geometric morphisms $p : \mathcal{E} \to \mathcal{S}$ such that, for every object X in $\mathcal{E}$, the composite $\mathcal{E}/X \to \mathcal{E} \to \mathcal{S}$ has a level é, which will be denoted by $\text{É}X \to \mathcal{E}/X$. For such a geometric morphism p, the topos $\mathcal{E}$ is to be thought of as a topos 'of spaces' and the étendues $\text{É}X$ as the 'petit' toposes, one for each X in $\mathcal{E}$.

We will show that the topos of simplicial sets, and similar pre-cohesive presheaf toposes, are of the kind specified in the previous paragraph, describing the étendues $\text{É}X$ in very concrete terms. Time permitting, we will explain how $\text{É}X$ determines the dimension of X. (Details may be found in [3].)

References

[1] F. W. Lawvere, *Some thoughts on the future of category theory*, Lect. Notes Math. 1488, 1991.

[2] F. Marmolejo, M. Menni, *Level* ϵ, Cah. Topol. Géom. Différ. Catég. 60, 2019.

[3] M. Menni, *The étendue of a combinatorial space and its dimension*, Adv. Math. 459, 2024.