Profinite completions and clones

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Abstract.

The seminal work of Kennison and Gildenhuys [1], later studied and expanded by Leinster [2], is one of the cornerstones for the categorical approach to profinite structures and Stone duality. By characterizing the ultrafilter monad

$$\beta \quad : \quad \mathbf{Set} \longrightarrow \mathbf{Set}$$

as the codensity monad of the inclusion of finite sets into all sets, it paved the way for notions of profinite completions for various kinds of structures. In particular, the codensity monad of the inclusion of finite monoids into all monoids yields the familiar profinite completion of monoids.

Inspired by ideas coming from automata theory, we apply the codensity viewpoint to the study of clones, i.e. unisorted Lawvere theories. We define the profinite completion of clones

$$\widehat{(-)} \quad : \quad \mathbf{Clone} \longrightarrow \mathbf{Clone}$$

as the codensity monad induced by the full subcategory of clones that are pointwise finite, i.e. locally finite unisorted Lawvere theories.

Using the identification of the multisorted Lawvere theory of clones as a specific full subcategory of the free cartesian closed category on one object, we obtain two fully faithful functors

$$\mathrm{Cl}_{\mathbf{Set}} \quad : \quad \mathbf{Set} \longrightarrow \mathbf{Clone} \qquad \text{and} \qquad \mathrm{Cl}_{\mathbf{Mon}} \quad : \quad \mathbf{Mon} \longrightarrow \mathbf{Clone}$$

that encode sets and monoids as clones. A crucial, yet seemingly new fact, is that these two functors are parametric right adjoints. Together with the analog statement for finite structures, these observations are key ingredients in the proof of the two following theorems.

Theorem. For any set X, we have a clone isomorphism

$$\widehat{\mathrm{Cl}_{\mathbf{Set}}(X)} \quad \cong \quad \mathrm{Cl}_{\mathbf{Set}}(\beta X)$$

Theorem. For any monoid M, we have a clone isomorphism

$$\widehat{\mathrm{Cl}_{\mathbf{Mon}}(M)} \quad \cong \quad \mathrm{Cl}_{\mathbf{Mon}}(\widehat{M})$$

where the monoid $\widehat{M}$ is the profinite completion of M.

These two theorems demonstrate that the profinite completion of clones generalizes both the ultrafilter monad and the profinite completion of monoids.

As the elements of the free clone over a first-order signature are the trees on that signature, we call profinite trees elements of profinite completions of free clones. After having proven that profinite trees verify a strong form of parametricity, we establish a close link with the profinite λ-calculus introduced in [3].

Theorem. For any signature, the associated clones of profinite trees and of profinite λ-terms are isomorphic.

References

[1] J.F. Kennison and D. Gildenhuys. *Equational completion, model induced triples and pro-objects.* Journal of Pure and Applied Algebra, 1:317–346, 1971.

[2] T. Leinster, *Codensity and the ultrafilter monad.* Theory and Applications of Categories, 28(13), 332-370.

[3] S.v. Gool, P.-A. Melliès, and V. Moreau. *Profinite lambda-terms and parametricity.* Electronic Notes in Theoretical Informatics and Computer Science, Volume 3 – Proceedings of MFPS XXXIX, November 2023. doi:10.46298/entics.12280.