Pos-pretoposes and compact ordered spaces

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(joint work with Jérémie Marquès)

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Abstract.

A characterisation of the category KH of compact Hausdorff spaces and continuous maps was provided in [4] (see also [1] for a constructive approach). The key notion is that of filtrality: let us say that a (bounded, distributive) lattice L is *filtral* if it is isomorphic to the lattice of filters on the Boolean algebra of complemented elements of L. By extension, an object X in a category is *filtral* if its poset of subobjects is a filtral lattice. Filtral objects in KH are exactly the Stone spaces.

A weaker version of the main result of [4] can be stated as follows.

Theorem. *Let E be a non-trivial pretopos with terminal object $\mathbf{1}$. Suppose that:*

1. *$\mathbf{1}$ is a regular generator and, for every set S, the copower $S \cdot \mathbf{1}$ exists in E;*
2. *for every set S, the copower $S \cdot \mathbf{1}$ is filtral.*

Then E is equivalent to KH.

In a nutshell, condition 1 implies that E is equivalent to the category of algebras for the monad on Set induced by the adjunction $- \cdot \mathbf{1} \dashv \mathsf{E}(\mathbf{1}, -)$, and it follows from condition 2 that the latter is the *ultrafilter monad*. Manes' Theorem [5] then entails that E is equivalent to KH.

In this talk, I shall discuss an extension of the previous result to the category KOrd of Nachbin's compact ordered spaces and continuous monotone maps, regarded as a Pos-pretopos enriched in the category Pos of posets and monotone maps. The poset-variant of Manes' Theorem, due to Flagg [3], identifies KOrd with the category of algebras for the *prime filter monad* on Pos.

Our approach is based on the observation that a Pos-pretopos, and more generally a Pos-lex-category, can be identified with a category of internal partial orders in an ordinary lex-category. This allows us to use the internal logic to describe order-enriched variants of ordinary notions, such as filtrality. For example, in the same way that regular categories correspond to regular theories, Pos-regular categories correspond to *monotone* regular theories. Here, a theory T is monotone if for every sort X there is a binary relation symbol $\leq_X \, :: X \times X$ such that T proves that (i) $\leq_X$ is a partial order and (ii) every function symbol is monotone with respect to these orders.

If time permits, I will also discuss the Pos-enriched analogue of *extensive* categories and the connection with the study of two-dimensional exactness conditions in [2].

References

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