An intrinsic approach to kernels in general categories

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Abstract. Given a nice epimorphism $p\colon A \to B$, one would hope to be able to recover it from the data of its kernel. With that goal in mind, we consider a category $\mathbb{C}$ equipped with a coreflective subcategory $\mathbb{Z}$, so that a pullback square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & d(B) \arrow[d] \\ A \arrow[r, "p"] & B, \end{tikzcd}$$

where $d(B) \to B$ is the coreflection of B into $\mathbb{Z}$, exhibits the span $d(B) \leftarrow K \to A$ as the kernel of p. (Kernels in the classical sense are recovered when $\mathbb{Z}$ is the zero subcategory of a pointed category and $d(B) = 0$.) We require this pullback square to also be a pushout for every nice epimorphism p, which means asking that a nice epimorphism should be the cokernel of its kernel.

If we don't want the subcategory $\mathbb{Z}$ to be extra structure on top of $\mathbb{C}$, we can suppose that the smallest such subcategory $\mathbb{Z}$ (possibly satisfying some extra niceness conditions) exists, thus equipping the category $\mathbb{C}$ with a certain *intrinsic* notion of kernel. Additionally, under niceness assumptions, the data of the span $d(B) \leftarrow K \to A$ can be encoded into a pair (ρ, K), with ρ an equivalence relation (internal to $\mathbb{Z}$) on $d(A)$, and $K \rightarrowtail A$ a subobject. This is in essence similar to *star kernels* [1] which also tie together kernels as subobjects with kernels as equivalence relations.

The intuition is that we try to minimize the subcategory $\mathbb{Z}$, so that the ρ component of the pair (ρ, K) is as trivial as possible. For a semi-abelian category $\mathbb{C}$, since the minimal $\mathbb{Z}$ is the zero subcategory, the equivalence relation ρ contains no information, meaning the kernel of $A \to B$ is solely encoded into a subobject $K \rightarrowtail A$.

On the other hand, for $\mathbb{C} = \mathsf{Set}$, the minimal $\mathbb{Z}$ is Set itself. In this case the kernel of $p\colon A \to B$ is (ρ, A), with ρ the kernel pair of p and $A \rightarrowtail A$ always the largest subobject, therefore containing no information.

A motivating example that lies somewhere in-between is the category of inverse monoids, in which $\mathbb{Z}$ is not the zero subcategory, but instead the subcategory of semilattices, with $d(S) \subseteq S$ the semilattice of idempotent elements of the inverse monoid S. This approach then recovers the classical *kernel-trace* description [2, Section 5.1] of a surjective morphism of inverse monoids.

With $\mathbb{C}$ the category of unital rings, $\mathbb{Z}$ consists of the initial object, but note that $d(B) \to B$ and therefore $K \to A$ no longer need to be subobjects.

Finally we analyze this approach through the behavior of pullback squares of categories

$$\begin{tikzcd} \mathbb{K} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathcal{E} \arrow[r, "\mathsf{cod}"] & \mathbb{C}, \end{tikzcd}$$

pulling back a coreflective subcategory $\mathbb{Z}$ along the codomain fibration of nice epis.

References

[1] M. Gran, Z. Janelidze, A. Ursini, *A good theory of ideals in regular multi-pointed categories*, J. Pure Appl. Algebra 216 (2012), no. 8–9, 1905–1919.

[2] L. M. Lawson, *Inverse semigroups*, World Scientific Publishing Co., Inc., River Edge, NJ, 1998.