Quasi-homeomorphisms of topological groupoids

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Abstract.
Topological spaces. In the very early days of topos theory, it was noted that two (T_0-)topological spaces X, Y have equivalent topoi of sheaves $\mathbf{Sh}(X) \simeq \mathbf{Sh}(Y)$ if and only if X and Y admit subspace inclusions $X \subseteq W \supseteq Y$ that are moreover *quasi-homeomorphisms* [5, §10]. As a result, invariants of the topos of sheaves on a space (e.g. *sheaf cohomology*) correspond to invariants of the space that are preserved by quasi-homeomorphism. In some ways, this is a vestige of point-set topology – the topos of sheaves on a space X is entirely determined by the locale of opens $\mathcal{O}(X)$.

Topological groups and groupoids. In contrast, topological groups and topological groupoids do not necessarily yield localic groups or localic groupoids, and so we cannot rely on the arguably better-behaved point-free setting (as found in [6, §7]). Indeed, the question of when two topological groups G, G' are *Morita equivalent*, i.e. their topoi of continuous actions are equivalent $\mathbf{B}G \simeq \mathbf{B}G'$, becomes more complex than the point-free variant [7] (cf. also the difference between topological and localic representation theorems [3, 4]).

Our contribution. In this talk, we discuss the case for topoi of sheaves on topological groupoids.

1. We introduce the subclass of *logical* groupoids (so named for the *logical topologies* of [1]). These generalise those T_0-topological groups whose open subgroups generate the topology, or equivalently the topological subgroups of a symmetric group $\Omega(X)$ endowed with the pointwise convergence topology. The class of logical groupoids is not restrictive since every topos with enough points is the topos of sheaves on a logical groupoid (see [2]).
2. We characterise which inclusions of logical groupoids $\mathbb{Y} \subseteq \mathbb{X}$ yield an equivalence of their sheaf topoi $\mathbf{Sh}(\mathbb{Y}) \simeq \mathbf{Sh}(\mathbb{Y})$ ([8, Theorem 2]) – the titular 'quasi-homeomorphisms' of topological groupoids.

As a consequence, we will deduce that two logical groupoids $\mathbb{X}, \mathbb{Y}$ are Morita equivalent if and only if they can be embedded into a common logical groupoid $\mathbb{X} \subseteq \mathbb{W} \supseteq \mathbb{Y}$ via 'quasi-homeomorphisms' ([8, Corollary 6.3]).

References

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