BOOK OF ABSTRACTS

EQUADIFF 15

Conference on Differential Equations
and Their Applications

Brno, Czech Republic, July 11–15, 2022

(version: July 10, 2022)

Contents

Abstracts – plenary talks

Generalizations of the Poincaré – Birkhoff Theorem for Hamiltonian systems

Alessandro Fonda
University of Trieste

In 1983, Conley and Zehnder [3] proved a remarkable theorem on the periodic problem associated with a general Hamiltonian system, giving a partial answer to a conjecture by V.I. Arnold. In the same paper they also proposed a second theorem, mentioning a possible relation with the Poincaré – Birkhoff Theorem.

First conjectured by Poincaré [5] in 1912, shortly before his death, this theorem has been proved by Birkhoff in [1, 2]. Both Poincaré and Birkhoff were interested in it because of its consequences in the existence of periodic solutions for some Hamiltonian systems originating from Celestial Mechanics.

More recently, a deeper relation between the above quoted second theorem by Conley and Zehnder and the Poincaré – Birkhoff theorem has been established by the first author jointly with A.J. Ureña [4].

The main result in [4] has found so far several applications, and has been extended in different directions. Besides the usual periodic – twist assumption, the coupling with some nonresonant linear terms can now also be treated, as well as with terms involving lower and upper solutions.

I will report on some of the most recent advances in this still very fertile field.

[1] G.D. Birkhoff, *Proof of Poincaré's geometric theorem.* Trans. Amer. Math. Soc. **14** (1913), 14–22.

[2] G.D. Birkhoff, *An extension of Poincaré's last geometric theorem.* Acta Math. **47** (1925), 297–311.

[3] C.C. Conley and E.J. Zehnder, *The Birkhoff – Lewis fixed point theorem and a conjecture of V.I. Arnold.* Invent. Math. **73** (1983), 33–49.

[4] A. Fonda and A.J. Ureña, *A higher dimensional Poincaré – Birkhoff theorem for Hamiltonian flows.* Ann. Inst. H. Poincaré Anal. Non Linéaire **34** (2017), 679–698.

[5] H. Poincaré, *Sur un théorème de géométrie.* Rend. Circ. Mat. Palermo **33** (1912), 375–407.

Mathematics of nonlinear acoustics: modeling, analysis and inverse problems

Barbara Kaltenbacher
University of Klagenfurt, Austria

The importance of ultrasound is well established in the imaging of human tissue. In order to enhance image quality by exploiting nonlinear effects, recently techniques such as harmonic imaging and nonlinearity parameter tomography have been put forward. As soon as the pressure amplitude exceeds a certain bound, the classical linear wave equation loses its validity and more general nonlinear versions have to be used. Another characteristic property of ultrasound propagating in human tissue is frequency power law attenuation leading to fractional derivative damping models in time domain. In this talk we will first of all dwell on modeling of nonlinearity on one hand and of fractional damping on the other hand. Then we will give an idea on the challenges in the analysis of the resulting PDEs and discuss some parameter asymptotics. Finally, we address some relevant inverse problems in this context, in particular the above mentioned task of nonlinearity parameter imaging, which leads to a coefficient identification problem for a quasilinear wave equation.

Dissipative weak solutions of compressible fluid flows

Mária Lukáčová-Medvid'ová
Institute of Mathematics, University of Mainz

This is a joint work with E. Feireisl (Academy of Sciences, Prague, Czech Republic), H. Mizerová (Comenius University, Bratislava, Slovakia), B. She (Academy of Science, Prague, Czech Republic) and Y. Yuan (University of Mainz, Germany).

In this talk we introduce generalized solutions of compressible flows, the so-called dissipative weak solutions. We will concentrate on the inviscid flows, the Euler equations, and mention also the relevant results obtained for the viscous compressible flows, governed by the Navier-Stokes equations.

The existence of dissipative weak solutions has been shown by the convergence analysis of suitable, invariant-domain preserving finite volume schemes [1, 2, 3, 4]. In the case that the strong solution to the above equations exists, the dissipative weak solutions coincide with the strong solution on its life span [1].

Otherwise, we apply a newly developed concept of $\mathcal{K}$-convergence and prove the strong convergence of the empirical means of numerical solutions to a dissipative weak solution [5, 6]. The latter is the expected value of the dissipative measure-valued solutions and satisfies a weak formulation of the Euler equations modulo the Reynolds defect measure. In the class of dissipative weak solutions there exists a solution that is obtained as a vanishing viscosity limit of the Navier-Stokes system [7]. Theoretical results will be illustrated by a series of numerical simulations.

The present research has been partially supported by TRR 146 Multiscale simulation methods for soft matter systems, TRR 165 Waves to Weather funded by the German Science Foundation and by the Gutenberg Research College.

[1] E. Feireisl, M. Lukáčová-Medvid'ová, H. Mizerová, B. She, Numerical analysis of compressible fluid flows. Springer, 2021.

[2] E. Feireisl, M. Lukáčová-Medvid'ová, H. Mizerová, *Convergence of finite volume schemes for the Euler equations via dissipative-measure valued solutions*, Found. Comput. Math. **20** (2020), 923–966.

[3] E. Feireisl, M. Lukáčová-Medvid'ová, H. Mizerová, B. She, *Convergence of a finite volume scheme for the compressible Navier-Stokes system*, ESAIM: Math. Model. Num. **53** (2019), 1957–1979.

[4] M. Lukáčová-Medvid'ová, B. She, Y. Yuan, *Error estimate of the Godunov method for multidimensional compressible Euler equations*, J. Sci. Comput. **91** (2022), 71.

[5] E. Feireisl, M. Lukáčová-Medvid'ová, H. Mizerová, *$\mathcal{K}$-convergence as a new tool in numerical analysis*, IMA J. Numer. Anal. **40** (2020), 2227–2255.

[6] E. Feireisl, M. Lukáčová-Medvid'ová, B. She, Y. Wang, *Computing oscillatory solutions of the Euler system via $\mathcal{K}$-convergence*, Math. Math. Models Methods Appl. Sci. **31** (2021), 537–576.

[7] E. Feireisl, M. Lukáčová-Medvid'ová, S. Schneider, B. She, *Approximating viscosity solutions of the Euler system*, Math. Comp. (2022), accepted doi.org/10.1090/mcom/3738

Maxwell's equations revisited - mental imagery and mathematical symbols

Stefan Siegmund

Institute of Analysis & Center for Dynamics, Faculty of Mathematics, TU Dresden, Germany

This is joint work with Dr. Matthias Geyer, B.Sc. Jan Hausmann, M.Sc. Konrad Kitzing and B.Sc. Madlyn Senkyr (TU Dresden, Germany). Maxwell developed his famous equations

$$\operatorname{curl} E + \frac{\partial B}{\partial t} = 0, \quad \operatorname{div} D = \rho,$$
$$\operatorname{curl} H + \frac{\partial D}{\partial t} = j, \quad \operatorname{div} B = 0,$$
$$D = \varepsilon_0 E, \quad B = \mu_0 H,$$

in what Hon and Goldstein [7] call an odyssey in electromagnetics consisting of four stations:

- Station 1 (1856-1858): on Faraday's lines of force [1]
- Station 2 (1861-62): on physical lines of force [2]
- Station 3 (1865): A dynamical theory of the electromagnetic field [3]
- Station 4 (1873): A treatise on electricity and magnetism [4]

Maxwell's original work is a rich source for methodological inspiration [6, 7, 8]. In this talk we pick up some of this inspiration from his original work as well as the mental imagery which he developed and look at his ideas again. We first discuss Maxwell's imaginary fluid approach in vector calculus notation and then formulate a research hypothesis based on Maxwell's constitutive relations $D = \varepsilon_0 E$, $B = \mu_0 H$, in the language of differential forms.

[1] J.C. Maxwell, *On Faraday's lines of force*, Transactions of the Cambridge Philosophical Society **10** (1858), 27–83. Reprinted in [5] 1965, vol. 1, 155–229.

[2] J.C. Maxwell, *On physical lines of force*, Philosophical Magazine **4**, no. 21, 161–175, 281–291, 338–348. Plate. Philosophical Magazine **4**, no. 23, 12–24, 85–95. Reprinted in [5] 1965, vol. 1, 451–513.

[3] J.C. Maxwell, *A dynamical theory of the electromagnetic field*, Philosophical Transactions of the Royal Society of London **155**, 459–512. Reprinted in [5] 1965, vol. 1, 526–597.

[4] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, 2 vols. Oxford, Clarendon Press, 1873. Reprinted, Cambridge University Press, 2010.

[5] J.C. Maxwell, *The scientific papers of James Clerk Maxwell* (ed. by W.D. Niven), 2 vols. Cambridge, University Press, 1890. Reprinted, two volumes bound as one. New York, Dover, 1965.

[6] O. Darrigol, *Electrodynamics from Ampère to Einstein*, Oxford University Press, 2003.

[7] G. Hon, B.R. Goldstein, *Reflections on the Practice of Physics: James Clerk Maxwell's Methodological Odyssey in Electromagnetism*, Routledge, 2020.

[8] D.M. Siegel, *Innovation in Maxwell's electromagnetic theory: molecular vortices, displacement current, and light*, Cambridge University Press, 1991.

Stability of Sobolev inequalities and related topics

Juncheng Wei
University of British Columbia, Vancouver, Canada

Suppose $u \in \dot{H}^1(\mathbb{R}^n)$. In a seminal work Struwe (1984) proved that if $||\Delta u + u^{\frac{2n}{n-2}}||_{H^{-1}} := \Gamma(u) \to 0$ then $\delta(u) \to 0$, where $\delta(u)$ denotes the $\dot{H}^1(\mathbb{R}^n)$-distance of u from the manifold of sums of Talenti bubbles. In 2020 Figalli and Glaudo obtained the first quantitative version of Struwe's decomposition in lower dimensions, namely $\delta(u) \lesssim \Gamma(u)$ when $3 \leq n \leq 5$. In this talk, I will report the following nonlinear estimates:

$$\delta(u) \leq C \begin{cases} \Gamma(u)\,|\log \Gamma(u)|^{\frac{1}{2}} & \text{if } n = 6, \\ |\Gamma(u)|^{\frac{n+2}{2(n-2)}} & \text{if } n \geq 7. \end{cases}$$

Furthermore, we show that this inequality is optimal. Extensions to Caffarelli-Kohn-Nirenberg inequalities, Harmonic Map Inequalities and 1/2-Harmonic Maps will also be discussed.

Algorithms and applications of fractional PDEs

Barbara Wohlmuth
TUM, Garching, Germany

In this talk, we discuss several aspects of fractional differential operators. We consider theoretical existence results for a fractional in time non-linear PDE model and show the equivalence to an integer order system. From a numerical point of view, we introduce an efficient algorithm how to discretize a fractional in time operator based on a rational approximation and a sum of exponentials. We use a fractional PDE in space to generate a wide range of random structures, including two-phase and multi-phase materials and wind flow profiles. For the mean flow, a copula based approach fitting the synthetic profile to the windrose data is used, while for the turbulent part a fractional PDE operator in space with a Gaussian white noise on the right hand side is considered. All theoretical and algorithmic results are illustrated by numerical examples illustrating the large flexibility of the proposed tools

Abstracts – invited talks

Fractional periodic problems with critical growth

Vincenzo Ambrosio
Università Politecnica delle Marche, Ancona, Italy

In this talk, I will discuss the existence of 2π-periodic solutions to the following fractional critical problem:

$$\begin{cases} [(-\Delta_x + m^2)^s - m^{2s}]u = W(x)|u|^{2_s^*-2}u + f(x,u) & \text{in } (-\pi,\pi)^N \\ u(x+2\pi e_i) = u(x) & \text{for all } x \in \mathbb{R}^N, \quad i = 1,\dots,N, \end{cases}$$

where $s \in (0,1)$, $N \geq 4s$, $m \geq 0$, $2_s^* = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent, $W(x)$ is a positive continuous function, and $f(x,u)$ is a superlinear 2π-periodic (in x) continuous function with subcritical growth. When $m > 0$, the existence of a nonconstant periodic solution will be established by combining the Linking theorem and a suitable variant of the extension method in periodic setting. The case $m = 0$ will be studied through a limit procedure.

On the connection of differential, difference, delay and dynamic equations

Elena Braverman
University of Calgary, Calgary, Canada

This is a joint work with Leonid Berezansky (Ben-Gurion University of the Negev, Beer-Sheva, Israel).

Solutions of autonomous scalar nonlinear ordinary differential equations usually experience monotone behaviour, while difference models can lead to oscillation. Delay equations in some sense combine properties of the two types of equations. For some types of nonlinear difference equations $x_n = f(x_{n+1})$, eventually monotone convergence to a unique equilibrium guarantees its global attractivity for an equation with a distributed delay [1]

$$x'(t) = r(t)\left[\int_{h(t)}^{t} f(x(s))d_sR(t,s) - x(t)\right].$$

This result cannot be automatically extended to systems [2]: the notion of a strong attractor of a vector difference equation associated with a nonlinear vector differential equation is required to guarantee stability of the delay system. The stability theorem is applied to compartment-type models of population dynamics with Nicholson's blowflies growth law and to Hopfield neural networks [3]. The results extend the theorem [1] that for a one-dimensional equation with a distributed delay, delay-independent stability can be deduced from attractivity of an associated difference equation.

Difference equations can approximate solutions of differential equations and the denser the grid is, more accurate description of solutions behaviour can be given, in particular, non-oscillating character can be imitated. This also leads to an interesting question for equations on time scales: is the oscillation property monotone, for example, are there conditions such that non-oscillation of a delay equation on certain time scales leads to the same property on any finer time scale [4]?

[1] E. Braverman and S. Zhukovskiy, *Absolute and delay-dependent stability of equations with a distributed delay*, Discrete Contin. Dyn. Syst. **32** (2012), 2041–2061.

[2] E. Liz and A. Ruiz-Herrera, *Attractivity, multistability, and bifurcation in delayed Hopfield's model with non-monotonic feedback*, J. Differential Equations **255** (2013), 4244–4266.

[3] L. Berezansky and E. Braverman, *On the global attractivity of non-autonomous neural networks with a distributed delay*, Nonlinearity **34** (2021), 2381–2401.

[4] E. Braverman and B. Karpuz, *On monotonicity of nonoscillation properties of dynamic equations in time scales*, Z. Anal. Anwend. **31** (2012), 203–216.

Shape optimization in Stokes fluids

Dorin Bucur
Université Savoie Mont Blanc, Le Bourget du Lac, France

We analyse the question of minimizing the drag of an obstacle of prescribed volume in a viscous flow driven by the Stokes equation, provided that the contact between the obstacle and the fluid obeys a Navier law. Letting completely free the shape of the obstacle, we prove the existence of an optimal solution, possibly presenting lower dimensional features, and analyse its regularity. No a priori geometric constraints are imposed on the competing shapes, except a control on their total area. The analysis is carried in the framework of free discontinuity problems. A similar question for Navier-Stokes flows will be shortly commented.

Stability regions for linear fractional and delay differential equations

Jan Čermák
Brno University of Technology, Czech Republic

In this contribution, we discuss forms and properties of stability regions for various types of linear autonomous differential equations that can be considered in a joint form

$$\mathrm{D}^{\alpha} y(t) = A y(t-\tau(t)), \qquad t > 0. \tag{1}$$

Here, D^{α} means the Caputo derivative of a positive real order α, A is a constant real matrix and τ is a non-negative real function satisfying some additional properties.

Starting with the elementary case $\alpha = 1$ and τ being identically zero, we explore dependance of stability regions of (1) on a lag function τ as well as on a continuously changing derivative order α. We support these results by comments on asymptotic and oscillatory properties of (1) with a special emphasize put on similarities and dissimilarities between integer and fractional–order case. Besides, we consider (1) with several delay functions and mention some partial results also in this direction.

We intend to present these stability conditions in their optimal (i.e. non-improvable) and efficient form. Such descriptions are useful not only for theoretical reasons, but also in numerical analysis of differential equations and in some application areas. To illustrate this, we mention a simple consequence to control theory.

Finally, we are going to comment some related results of several great mathematicians connected with the history of the Czechoslovak Equadiff.

An elementary proof of existence and uniqueness for the Euler flow in uniformly localized Yudovich spaces

Gianluca Crippa
University of Basel, Switzerland

I will revisit Yudovich's well-posedness result for the 2-dimensional Euler equations. I will derive an explicit modulus of continuity for the velocity, depending on the growth in p of the (uniformly localized) L^p norms of the vorticity. If the growth is moderate at infinity, the modulus of continuity is Osgood and this allows to show uniqueness. I will also show how existence can be proved in (uniformly localized) L^p spaces for the vorticity. All the arguments are fully elementary, make no use of Sobolev spaces, Calderon-Zygmund theory, or Paley-Littlewood decompositions, and provide explicit expressions for all the objects involved. This is a joint work with Giorgio Stefani (SISSA Trieste).

New finite elements for extremely high-order FEM

Patrick Farrell
University of Oxford, UK

This is a joint work with P. D. Brubeck Martinez of the University of Oxford.

For problems with smooth solutions, high-order finite element methods offer very good convergence properties. Moreover, there exist optimal matrix-free algorithms for operator evaluation with high arithmetic intensity, arising from data locality, making them very attractive on modern parallel hardware architectures. Unfortunately, the conditioning of the stiffness matrix is severely affected by the polynomial degree p of the approximation. In order to obtain practical high-order codes, we require good preconditioners.

Pavarino proved in 1993 [1] that an additive Schwarz method with vertex patches and a low-order coarse space gives a p-robust solver for symmetric and coercive problems. However, for very high polynomial degree it is not feasible to factorize (or even assemble) the matrices for each vertex patch. In this work we introduce a new H^1-conforming finite element on tensor product cells that under certain conditions yields *sparse* stiffness matrices for the H^1 Riesz map, enabling the use of direct solvers for each patch problem. We can thus afford to assemble and factorize the matrices for the vertex-patch problems for much higher polynomial degrees than previously possible. When the conditions for sparsity do not apply, the method can be employed as a preconditioner by approximating the problem with a suitable (separable) surrogate, in a manner that is provably p-robust.

We then extend the construction to new finite elements for the entire L^2 de Rham complex [2], enabling the fast solution of the $H(\mathrm{div})$ and $H(\mathrm{curl})$ Riesz maps, which often arise as inner problems in the use of block preconditioners for coupled systems of equations [3].

We demonstrate the approach by solving the Poisson equation, a $H(\mathrm{div})$-conforming interior penalty discretization of linear elasticity, and the Riesz maps for $H(\mathrm{div})$ and $H(\mathrm{curl})$, at $p = 15$ on three-dimensional unstructured meshes.

[1] L. F. Pavarino, *Additive Schwarz methods for the p-version finite element method.* Numer. Math. **66** (1993), 493–515.

[2] D. N. Arnold, Finite element exterior calculus. SIAM, Philadelphia, 2018.

[3] K.-A. Mardal and R. Winther, *Preconditioning discretizations of systems of partial differential equations.* Numer. Linear Algebra Appl. **18** (2011), 1–40.

Numerical methods for conservation laws on networks

Ulrik Fjordholm
University of Oslo, Oslo, Norway

We review some theory of conservation laws posed on a graph, and look at some recent results on well-posedness of entropy solutions via the convergence of a finite volume method. If time permits we will look at an application to flow in porous media. This is joint work with Nils Henrik Risebro and Markus Musch (UiO).

Neutral FDEs with state–dependent delays: linearized instability principle

Jaqueline Godoy Mesquita
Universidade de Brasília, Brasíília, Brazil

In this lecture, we will discuss a linearized instability principle for neutral FDEs with state-dependent delays. Also, we will discuss some open and developing problems.

The thin film Muskat problem

Philippe Laurencot
Institut de Mathématiques de Toulouse, CNRS, France

The thin film Muskat problem describes the dynamics of the respective heights of two immiscible thin fluid layers with different densities and viscosities. It is a degenerate second-order parabolic system with a full diffusion matrix (cross-diffusion). Old and new results are presented, including the existence of weak solutions and their boundedness. For the Cauchy problem, the self-similar solutions are identified and their stability is shown. Joint works with A. Ait Hammou Oulhaj, C. Cances, C. Chainais-Hillairet, J. Escher and B.-V. Matioc.

Recent developments of computer-assisted proofs in differential equations

Jean-Philippe Lessard
McGill University, Quebec, Canada

In the study of high dimensional dynamical systems exploring the dynamics in the entire phase space is impossible. One strategy to tackle this problem is to focus on a set of special solutions that act as organizing centers. To single out these solutions computer-assisted proofs are being developed to find, for example, fixed points, periodic orbits and connecting orbits between those. Computer-assisted proofs in dynamics combine the strength of scientific computing, nonlinear analysis, numerical analysis, applied topology, functional analysis and approximation theory. While in the past decade, these techniques have primarily been applied to ODEs, we are starting to witness their applicability for infinite dimensional nonlinear dynamics generated by partial differential equations (PDEs), integral equations, delay differential equations (DDEs), and infinite dimensional maps. In this talk we will present recent advances in this direction, with a special emphasize on the dynamics of DDEs and PDEs.

Ulam stability of generalized logistic equations

Masakazu Onitsuka
Okayama University of Science, Okayama, Japan

In this talk we consider the generalized logistic equation

$$y' = y\left(1 - y^{\alpha}\right), \tag{1}$$

where $\alpha > 0$. Recently, Popa et al. [1] proposed conditional Ulam stability for (1) with $\alpha = 1$. Conditional Ulam stability is a property that guarantees the difference between the approximate solution and the exact solution to be finite. The exact definition is as follows. Let $A \subseteq (0,\infty)$ and $B \subseteq \mathbb{R}$ be nonempty sets. Define the class

$$\mathcal{C}_B := \left\{ y \in C^1\left[0, T_y\right) : y(0) \in B, T_y > 0 \text{ with } T_y = \infty \text{ or } |y(t)| \to \infty \text{ as } t \nearrow T_y \right\}.$$

Note that $[0, T_y)$ refers to the maximal existence interval of $y(t)$. The nonlinear differential equation

$$y' = F(t, y) \tag{2}$$

is *conditionally Ulam stable* on $[0, \min\{T_y, T_\eta\})$ with A in the class $\mathcal{C}_B$ if there exists $L > 0$ such that for any $\varepsilon \in A$ and any approximate solution $\eta \in \mathcal{C}_B$ that satisfy $|\eta' - F(t,\eta)| \le \varepsilon$ for $t \in [0, T_\eta)$, there exists a solution $y \in \mathcal{C}_B$ of (2) such that $|\eta(t) - y(t)| \le L\varepsilon$ for $t \in [0, \min\{T_y, T_\eta\})$. We call such an L an *Ulam constant* for (2) on $[0, \min\{T_y, T_\eta\})$. If $A = (0,\infty)$ and $B = \mathbb{R}$, then this definition is exactly the same as that for the standard Ulam stability. See [2, 3, 4, 5, 6] for previous studies on standard and conditional Ulam stabilities. The main result in this talk is as follows.

Theorem 1 ([7]). Let $A = \left(0, \alpha(\alpha+1)^{-\frac{\alpha+1}{\alpha}}\right]$ and $B = \left[(\alpha+1)^{-\frac{1}{\alpha}}, \infty\right)$. Then (1) is conditionally Ulam stable on $[0,\infty)$ with A in the class $\mathcal{C}_B$. Furthermore, $L = \max\left\{(\alpha+1)\alpha^{-1}, (\alpha+1)\alpha^{-2}\right\}$ is an Ulam constant for (1) on $[0,\infty)$.

This result is extended as a theorem applicable to more generalized logistic equations.

[1] D. Popa, I. Raşa and A. Viorel, *Approximate solutions of the logistic equation and Ulam stability*, Appl. Math. Lett. 85 (2018), 64–69.

[2] D. R. Anderson and M. Onitsuka, *Hyers–Ulam stability for differential systems with* 2×2 *constant coefficient matrix*, Results Math. 77 (2022), Paper No. 136, 23 pp.

[3] Y. W. Nam, *Hyers–Ulam stability of loxodromic Möbius difference equation*, Appl. Math. Comput. 356 (2019), 119–136.

[4] M. Onitsuka, *Conditional Ulam stability and its application to the logistic model*, Appl. Math. Lett. 122 (2021), Paper No. 107565, 7 pp.

[5] M. Onitsuka, *Conditional Ulam stability and its application to von Bertalanffy growth model*, Math. Biosci. Eng. 19 (2022), no. 3, 2819–2834.

[6] M. Onitsuka and Iz. El-Fassi, *On approximate solutions of a class of Clairaut's equations*, Appl. Math. Comput. 428 (2022), Paper No. 127205, 13 pp.

[7] M. Onitsuka, *Approximate solutions of generalized logistic equation*, submitted.

Nonlinear eigenvector problems and the simulation of Bose-Einstein condensates

Daniel Peterseim
University of Augsburg, Germany

This talk is based on joint work with P. Henning (Ruhr University Bochum, Germany), R. Altmann and T. Stykel (University of Augsburg, Germany).

Stationary states of Bose-Einstein condensates can be modelled by an eigenvalue problem for a nonlinear partial differential operator – the Gross-Pitaevskii or non-linear Schrödinger equation. It is a representative of the larger class of nonlinear eigenvector problems arising in computational physics but also in data analysis. The talk discusses the numerical solution of such nonlinear eigenvalue problems by adapting techniques from Riemannian optimization, computational PDEs and multiscale modelling and simulation.

For the special case of the Gross-Pitaevskii equation, the numerical analysis and a series of numerical experiments demonstrate the ability of the resulting simulation methods to capture relevant physical effects of Bose-Einstein condensates such as eigenstate localization under disorder potentials and the formation of vortex lattices in fast rotating potential traps. The talk is completed by an outlook to the robust and efficient simulation of the condensate's dynamics.

[1] R. Altmann, D. Peterseim, T. Stykel, *Energy-adaptive Riemannian optimization on the Stiefel manifold.* Math. Model. Numer. Anal. (2022).

[2] R. Altmann, P. Henning, D. Peterseim, *Localization and delocalization of ground states of Bose-Einstein condensates under disorder.* SIAM J. Appl. Math. **82(1)** (2022), 330-358.

[3] R. Altmann, P. Henning and D. Peterseim, *Numerical homogenization beyond scale separation.* Acta Numer. **30(1)** (2021), 1-86.

[4] R. Altmann, D. Peterseim, *The J-method for the Gross-Pitaevskii eigenvalue problem.* Numer. Math. **148(3)** (2021), 575-610.

[5] P. Henning, D. Peterseim, *Sobolev gradient flow for the Gross-Pitaevskii eigenvalue problem: global convergence and computational efficiency.* SIAM J. Numer. Anal. **58(3)** (2020), 1744-1772.

[6] R. Altmann, P. Henning, D. Peterseim, *Quantitative Anderson localization of Schrödinger eigenstates under disorder potentials.* Math. Models Methods Appl. Sci. **30(5)** (2020), 917-955.

[7] R. Altmann, D. Peterseim, *Localized computation of eigenstates of random Schrödinger operators.* SIAM J. Sci. Comput. **41** (2019), B1211-B1227.

[8] H. Alaeian, M. Schedensack, C. Bartels, D. Peterseim, and M. Weitz, *Thermo-optical interactions in a dye-microcavity photon Bose-Einstein condensate.* New J. Phys. **19(11)** (2017), 115009.

[9] P. Henning, A. Målqvist, D. Peterseim, *Two-level discretization techniques for ground state computations of Bose-Einstein condensates.* SIAM J. Numer. Anal. **52(4)** (2014), 1525-1550.

Adaptive FEM with quasi-optimal cost for nonlinear PDEs

Dirk Praetorius
TU Wien, Austria

We consider nonlinear elliptic PDEs with strongly monotone nonlinearity. We apply an adaptive finite element method

$$\boxed{\text{SOLVE \& ESTIMATE}} \quad\longrightarrow\quad \boxed{\text{MARK}} \quad\longrightarrow\quad \boxed{\text{REFINE}}$$

which steers the linearization as well as the iterative solution of the arising linear finite element systems. We prove that the proposed algorithm guarantees *full linear convergence*, i.e., linear convergence in each step, independently of the algorithmic decision for mesh-refinement, linearization, or algebraic solver step. For sufficiently small adaptivity parameters, this allows to mathematically guarantee *optimal convergence with respect to the overall computational work*, i.e., the quasi-error decays with optimal algebraic rate when plotted versus the cumulative computational time.

The talk is based on joint work [1, 2, 3].

[1] G. Gantner, A. Haberl, D. Praetorius, S. Schimanko, *Rate optimality of adaptive finite element methods with respect to the overall computational costs.* Math. Comp., 90 (2021), 2011–2040.

[2] A. Haberl, D. Praetorius, S. Schimanko, M. Vohralik, *Convergence and quasi-optimal cost of adaptive algorithms for nonlinear operators including iterative linearization and algebraic solver.* Numer. Math., 147 (2021), 679–725.

[3] P. Heid, D. Praetorius, T. Wihler, *Energy contraction and optimal convergence of adaptive iterative linearized finite element methods.* Comput. Methods Appl. Math., 21 (2021), 407–422.

Machine learned regularization for solving inverse problems

Carola-Bibiane Schönlieb
University of Cambridge, Cambridge, UK

Inverse problems are about the reconstruction of an unknown physical quantity from indirect measurements. Most inverse problems of interest are ill-posed and require appropriate mathematical treatment for recovering meaningful solutions. Regularization is one of the main mechanisms to turn inverse problems into well-posed ones by adding prior information about the unknown quantity to the problem, often in the form of assumed regularity of solutions. Classically, such regularization approaches are handcrafted. Examples include Tikhonov regularization, the total variation and several sparsity-promoting regularizers such as the L1 norm of Wavelet coefficients of the solution. While such handcrafted approaches deliver mathematically and computationally robust solutions to inverse problems, providing a universal approach to their solution, they are also limited by our ability to model solution properties and to realise these regularization approaches computationally. Recently, a new paradigm has been introduced to the regularization of inverse problems, which derives regularization approaches for inverse problems in a data driven way. Here, regularization is not mathematically modelled in the classical sense, but modelled by highly over-parametrised models, typically deep neural networks, that are adapted to the inverse problems at hand by appropriately selected (and usually plenty of) training data. In this talk, I will present some work on unsupervised and deeply learned convex regularisers and their application to image reconstruction from tomographic and blurred measurements.

Doubly nonlinear stochastic evolution equations

Ulisse Stefanelli
Faculty of Mathematics, University of Vienna, Austria

I will review some recent results on doubly nonlinear parabolic SPDEs in Hilbert spaces [1, 2, 3], obtained in collaboration with Luca Scarpa (Politecnico di Milano, Italy).

Doubly nonlinear parabolic PDEs arise in combination with a variety of applications, including nonlinear diffusion, phase transition, and mechanics. In the deterministic case, existence, approximation, and long-time behavior results are obtained by variational methods.

The variational theory can be partly extended to the stochastic setting. I will comment on existence results based on regularization procedures. Both nonlinear viscous and degenerate rate-independent cases will be discussed.

[1] L. Scarpa, U. Stefanelli, *Doubly nonlinear stochastic evolution equations.* Math. Models Methods Appl. Sci. **30** (2020), 991–1031

[2] L. Scarpa, U. Stefanelli, *Doubly nonlinear stochastic evolution equations II.* Stoch. Partial Differ. Equ. Anal. Comput. To appear (2022). `arXiv:2009.08209`

[3] L. Scarpa, U. Stefanelli, *Rate-independent stochastic evolution equations: parametrized solutions.* Submitted (2021). `arXiv:2109.15208`

Onsager's conjecture for general conservation laws

Agnieszka Świerczewska-Gwiazda
University of Warsaw, Poland

A common feature of systems of conservation laws of continuum physics is that they are endowed with natural companion laws which are in such case most often related to the second law of thermodynamics. This observation easily generalizes to any symmetrizable system of conservation laws. They are endowed with nontrivial companion conservation laws, which are immediately satisfied by classical solutions. Not surprisingly, weak solutions may fail to satisfy companion laws, which are then often relaxed from equality to inequality and overtake a role of a physical admissibility condition for weak solutions. We want to discuss what is a critical regularity of weak solutions to a general system of conservation laws to satisfy an associated companion law as an equality. An archetypal example of such result was derived for the incompressible Euler system by Constantin et al. in the context of the seminal Onsager's conjecture. This general result can serve as a simple criterion to numerous systems of mathematical physics to prescribe the regularity of solutions needed for an appropriate companion law to be satisfied.

Periodic solutions of the Lorentz force equation

Pedro J. Torres
University of Granada, Spain

This is a joint work with Manuel Garzón (University of Granada, Spain), based on the recent papers [1, 2].

According to classical Electrodynamics, the motion of a slowly accelerated charged particle $q(t)$ in an electromagnetic field is ruled by the classical Lorentz Force equation (LFE), which is one the fundamental equations in Mathematical Physics and has its origin in the pioneering works of Poincaré and Planck. The dynamical system under study is

$$\frac{d}{dt}\left(\frac{\dot{q}(t)}{\sqrt{1-|\dot{q}(t)|^2}}\right) = E(t,q(t)) + \dot{q}(t) \times B(t,q(t)). \tag{1}$$

where $E : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ and $B : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ are the electric and magnetic field respectively, while the left-hand side denotes the relativistic acceleration of the particle, which implies the characteristic speed limitation of Special Relativity (here the speed of light is normalized to $c = 1$). The electromagnetic field (E, B) must be solution of Maxwell's equations for suitable charge and current densities.

The general aim of the talk is to consider a time-periodic dependence of the electromagnetic field (E, B) and then study sufficient conditions for the existence of periodic solutions of the LFE.

[1] M. Garzón, P.J. Torres, *Periodic solutions for the Lorentz force equation with singular potentials*, Nonlinear Analysis: Real World Applications Vol. 56 (2020), 103162

[2] M. Garzón, P.J. Torres, *Periodic dynamics in the relativistic regime of an electromagnetic field induced by a time-dependent wire*, submitted

Abstracts – keynote and minisymposia talks

Topological entropy and differential equations

Jan Andres
Palacký University, Olomouc, Czech Republic

Matching the Nielsen fixed point theory with topological entropy by means of the Ivanov inequality, we will consider impulsive differential equations on tori. More concretely, the Ivanov inequality allows us to obtain in this way a lower estimate of topological entropy of the associated Poincaré translation operators, composed with the impulsive maps, in terms of the asymptotic Nielsen numbers on tori. Because of a one to one correspondence between periodic solutions of (minimal) order k and k-periodic points of the above composition, we can therefore speak about topological entropy for impulsive differential equations. Sufficient effective conditions for a positive entropy (topological chaos) will be finally given in terms of the asymptotic Lefschetz numbers imposed just on the impulsive maps on tori. For the scalar problems (in $\mathbb{R}/\mathbb{Z}$), it is enough to assume that the topological degree of a given impulsive map on the circle is absolutely greater than 1.

Linear ordinary differential systems with generic inhomogeneous boundary conditions in Sobolev spaces

Olena Atlasiuk

Institute of Mathematics of the Czech Academy of Sciences, Czech Republic

This is a joint work with Prof Volodymyr Mikhailets (Institute of Mathematics of the Czech Academy of Sciences, Prague, Czech Republic).

We study linear systems of ordinary differential equations on a finite interval with the most general (generic) inhomogeneous boundary conditions in Sobolev spaces. These boundary problems include all known types of classical and numerous nonclassical conditions. The latter may contain derivatives of integer and (or) fractional orders, which may exceed the order of the differential equation.

We investigate the characteristic of solvability of inhomogeneous boundary-value problems, prove their Fredholm properties, and find the indices, the dimensions of the kernel, and the cokernel of these problems.

Moreover, we obtained the necessary and sufficient conditions for continuity in the parameter of solutions to the introduced boundary-value problems in the Sobolev spaces. Some applications of these results are also presented.

[1] O. M. Atlasiuk, V. A. Mikhailets, *Fredholm one-dimensional boundary-value problems in Sobolev spaces* Ukrainian Math. J. **70** (2019), 1526–1537.

[2] O. M. Atlasiuk, V. A. Mikhailets, *Fredholm one-dimensional boundary-value problems with parametr in Sobolev spaces* Ukrainian Math. J. **70** (2019), 1677–1687.

[3] O. Atlasiuk, V. Mikhailets, *Continuity in a parameter of solutions to boundary-value problems in Sobolev spaces* arXiv:2005.03494v1 (2020).

[4] O. M. Atlasiuk, *Limit theorems for solutions of multipoint boundary-value problems in Sobolev spaces* J. Math. Sci. **247** (2020), 238–247.

Qualitative properties of a multilayered elasticity system - Stokes interaction

George Avalos
University of Nebraska-Lincoln, USA

In this talk, we will discuss our recent work on a certain multilayered structure-fluid interaction (FSI) which arises in the modeling of vascular blood flow. The coupled PDE system under our consideration mathematically accounts for the fact that mammalian veins and arteries are typically composed of various layers of tissues: each layer will generally manifest its own intrinsic material properties, and will moreover be separated from the other layers by thin elastic laminae. Consequently, the resulting modeling FSI system will manifest an additional PDE, which evolves on the boundary interface so as to account for the thin elastic layer. (This is in contrast to the FSI PDE's which appear in the literature, wherein elastic dynamics are largely absent on the boundary interface.) As such, the PDE system will constitute a coupling of 3D fluid-2D elastic-3D elastic dynamics. For such multilayered FSI systems, we will discuss ongoing work concerning the wellposedness and longtime behavior of solutions.

Scalable MS-GFEM applied to composite aero-structures

Jean Bénézech
Material & Structure Research Centre, University of Bath, Bath, UK

This is a joint work with Prof Robert Scheichl (University of Heidelberg, Heidelberg, Germany).

Due to the interaction of structural meso-scale and geometric macro-scale features, composite aero-structures are inherently multi-scale in nature. In addition, complex manufacturing processes lead to the appearance of defects at multiple scales, which discourage the use of a scale separation hypothesis. Hence, a large-scale uncertainty quantification (UQ) problem must be solved. The primary challenge is the cost of the associated numerical simulation. Therefore, to tackle this problem, an efficient Multiscale Spectral Generalized Finite Element Method (MS-GFEM) [1] has been developed within the open-source software package DUNE (https://www.dune-project.org/). Hence, the GenEO [2] approximation constructed in an optimal A-harmonic subspace [3] capable of solving HPC scale problems has been implemented, providing high quality approximate solutions even for small coarse space dimensions. The approach allows for good parallel scalability across at least hundreds of processor cores. Furthermore, this multi-scale method has the advantage of being modular. Assuming a subsequent problem is identical to the previous one except for changes localized to few subdomains (such as a localized defect in an aerospace part), the approximation space for unchanged subdomains remains identical, and therefore a large quotient of the computational effort may be skipped. Thus, the MS-GFEM has been integrated into an Offline/Online framework [4] allowing a very efficient exploration of defects in of the material parameters. The proposed framework will then form a tool for composite structure design and certification.

The research is supported by the UK Engineering and Physical Sciences Research Council (EPSRC) through the Programme Grant: 'Certification of Design: Reshaping the Testing Pyramid EP/S017038/1.

[1] I. Babuška, R. Lipton, Optimal local approximation spaces for generalized finite element methods with application to multiscale problems, Multiscale Modeling & Simulation 9 (1) (2011) 373–406.

[2] Spillane, N., Dolean, V., Hauret, P., Nataf, F., Pechstein, C., Scheichl, R. (2014). Abstract robust coarse spaces for systems of PDEs via generalized eigenproblems in the overlaps. Numerische Mathematik, 126(4), 741-770.

[3] Ma, C., Scheichl, R. (2021). Error estimates for fully discrete generalized FEMs with locally optimal spectral approximations. arXiv preprint arXiv:2107.09988.

[4] Y. Efendiev, J. Galvis, T. Y. Hou, Generalized multiscale finite element methods (GMsFEM), Journal of Computational Physics 251 (2013) 116–135.

Asymptotic behavior of singular solutions of nonlinear elliptic equations

Soohyun Bae

Department of Mathematical Sciences, Hanbat National University, Republic of Korea

Nonlinear elliptic equations sometimes has an invariance of the so-called self-similarity. Naturally, solutions with self-similarity may exist for many cases. Asymptotic self-similarity describes the basic property of entire solutions and singular solutions of nonlinear elliptic equations with the invariance. We discuss the asymptotic behavior and related questions for solutions. See [1], [2] and [3] for the asymptotic behavior of radially symmetric equations.

[1] S. Bae, *Asymptotic behavior of positive solutions of inhomogeneous semilinear elliptic equations.* Nonlinear Anal. TMA **51** (2002), 1373–1403.

[2] S. Bae, *Entire solutions with asymptotic self-similarity for elliptic equations with exponential nonlinearity.* J. Math. Anal. Appl. **428** (2015), 1085–1116.

[3] S. Bae, *On the elliptic equation* $\Delta u + Ku^p = 0$ *in* $\mathbf{R}^n$. Discrete Contin. Dyn. Syst. Ser. A **33** (2013), 555–577.

An intrinsic finite element method for PDEs on surfaces

Elena Bachini
Institute of Scientific Computing, TU Dresden, Germany

From level-set based techniques [1] to the surface finite element method [2] and isogeometric analysis [3], a host of numerical approaches for surface PDEs have been proposed over the last twenty years. Many, like the surface finite element method of [2], rely on an embedding of the surface in a higher dimensional space. These methods have proven successful in applications from fluid flow to biomedical engineering and electromagnetism. We present here an alternative finite element approach based on a geometrically intrinsic formulation [4], that we call Intrinsic Surface Finite Element Method (ISFEM). By careful definition of the geometry and the transport operators, we are able to arrive at an approximation that is fully intrinsic to the surface. We consider first a scalar advection-diffusion-reaction equation defined on a surface. In this case, the numerical analysis of the scheme is also available [5], and we show numerical experiments that support theoretical results. Then, we extend the differential operators for the case of vector-valued partial differential equations. In this case the presented formulation allows the direct discretization of objects naturally defined in the tangent space, without the need of any additional projection. Finally, we extend ISFEM to consider moving surfaces via an intrinsic re-definition of the PDE that takes into account a time-dependent metric tensor. To evaluate our approach, we consider several steady and transient problems involving both diffusion and advection-dominated regimes and compare its performance to established finite element techniques.

[1] M. Bertalmio, L.-T. Cheng, S. Osher, and G. Sapiro, *Variational Problems and Partial Differential Equations on Implicit Surfaces.* J. Comput. Phys. **174(2)** (2001), 759–780.

[2] G. Dziuk and C. M. Elliott, *Finite element methods for surfaces PDEs.* Acta Numer. **22** (2013), 289–396.

[3] L. Dedé and A. Quarteroni, *Isogeometric Analysis for second order Partial Differential Equations on surfaces.* Comput. Methods Appl. Mech. Engrg. **284(1)** (2015), 807–834.

[4] E. Bachini, M. Farthing and M. Putti, *Intrinsic finite element method for advection-diffusion-reaction equations on surfaces.* J. Comput. Phys. **424** (2021).

[5] E. Bachini and M. Putti, *Convergence analysis of the intrinsic surface finite element method.* arXiv:10.48550/ARXIV.2203.07330 (2022).

Nonlinear elliptic critical problems in $\mathbb{R}^N$

Laura Baldelli
Mathematical Institute of the Polish Academy of Sciences (IMPAN), Poland

In this talk we discuss some recent results concerning elliptic problems of (p, q)-Laplacian type in all of $\mathbb{R}^N$ with a nonlinearity involving both a critical and a subcritical term, nonnegative nontrivial weights and a positive real parameter λ, namely

$$-\Delta_p u - \Delta_q u = \lambda V(x)|u|^{k-2}u + K(x)|u|^{p^*-2}u \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $1 < q < p < N$, $N \geq 3$ and $1 < k < p^*$. In particular, under suitable conditions on the exponents of the nonlinearity, we obtain multiplicity, that is existence of infinitely many solutions, and existence results of nontrivial weak solutions with negative and positive energy depending on the range of the parameter λ, overcoming the double loss of compactness due both to the presence of the critical Sobolev's exponent p^* and to the unboundedness of the domain. We analyze problem (1) also in the case of nonnegative nontrivial weights satisfying some symmetry conditions with respect to a certain group $T \subset O(N)$, where $O(N)$ is the group of orthogonal linear transformations in $\mathbb{R}^N$. Our proofs use variational methods and the concentration compactness principle.

This is a joint work with Prof Roberta Filippucci (University of Perugia, Perugia, Italy) and Dr Ylenia Brizi (University of Perugia, Perugia, Italy).

A variational approach to fluid structure interaction

Barbora Benešová
Department of mathematical analysis, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

This is a joint work with Malte Kampschulte (Charles University) and Sebastian Schwarzacher (University of Uppsala & Charles University).

In this talk we will consider the interaction of a Stokes/Navier-Stokes flow with a viscoelastic body. The elastic body is allowed to undergo large deformations (but no self-collisions). In order to handle this situation correctly, we devise a variational approximation scheme in the spirit of DeGiorgi to the combined problem. Moreover, by using a two-scale scheme, we also extend this approach to the hyperbolic regime including inertia of the solid body. These variational approaches allow us to prove proper energetic estimates while also controlling the geometric restictions posed on the solid body and, eventually, to establish existence of weak solutions.

[1] Benešová, Barbora, Malte Kampschulte, and Sebastian Schwarzacher, *A variational approach to hyperbolic evolutions and fluid-structure interactions.*, arXiv:2008.04796 (2020).

Normalized ground states of the nonlinear Schrödinger equation with at least mass critical growth

Bartosz Bieganowski
Institute of Mathematics, Polish Academy of Sciences

This is a joint work with Prof. Jarosław Mederski (Institute of Mathematics, Polish Academy of Sciences).

We propose a simple minimization method to show the existence of least energy solutions to the normalized problem

$$\begin{cases} -\Delta u + \lambda u = g(u) \quad \text{in } \mathbb{R}^N, \ N \geq 3, \\ u \in H^1(\mathbb{R}^N), \\ \int_{\mathbb{R}^N} |u|^2 \, dx = \rho > 0, \end{cases}$$

where ρ is prescribed and $(\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ is to be determined. The new approach based on the direct minimization of the energy functional on the linear combination of Nehari and Pohozaev constraints intersected with the closed ball in $L^2(\mathbb{R}^N)$ of radius ρ is demonstrated, which allows to provide general growth assumptions imposed on g. We cover the most known physical examples and nonlinearities with growth considered in the literature so far as well as we admit the mass critical growth at 0.

[1] B. Bieganowski, J. Mederski, *Normalized ground states of the nonlinear Schrödinger equation with at least mass critical growth* J. Funct. Anal. **280**, Issue 11 (2021), 108989.

Large global solutions of the parabolic-parabolic Keller-Segel system in $\mathbb{R}^d$ and blowup for related toy models

Piotr Biler
Instytut Matematyczny, Uniwersytet Wrocławski, Poland

This is a joint work with Alexandre Boritchev and Lorenzo Brandolese (Université Claude Bernard, Lyon-1, France).

Two toy models obtained after a slight modification of the nonlinearity of the usual doubly parabolic Keller–Segel system studied in [1]

$$u_t = \Delta u - \nabla \cdot (u \nabla \varphi),$$

$$\tau \varphi_t = \Delta \varphi + u,$$

are studied. For these toy models, both consisting of a system of two parabolic equations (with the same structure of steady states as is for the nonlinear heat equation $u_t = \Delta u + u^2$), we establish that for data which are, in a suitable sense, smaller than the diffusion parameter τ in the equation for the chemoattractant, we obtain global solutions, and for data larger than τ, a finite time blowup. In this way, we check that our size condition for the global existence is sharp for large τ.

[1] P. Biler, A. Boritchev, L. Brandolese, *Large global solutions of the parabolic-parabolic Keller–Segel system in higher dimensions*, arXiv: 2203.09130.

Bifurcation analysis of Rayleigh-Bénard convection using deflation

Nicolas Boullé
Mathematical Institute, University of Oxford, Oxford, United Kingdom

This is a joint work with Dr. Vassilios Dallas and Prof. Patrick Farrell (University of Oxford).

We consider Rayleigh-Bénard convection in a confined fluid heated from below and maintained at a constant temperature difference $\Delta T = T - T_0$ across a two-dimensional square domain with no-slip boundary conditions. This work employs a recent algorithm called deflation [1] to compute multiple solutions to the steady-state Rayleigh-Bénard problem

$$\begin{aligned} \mathrm{Pr}\nabla^2\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \mathrm{PrRa}T\hat{\mathbf{z}} &= 0, \\ \nabla\cdot\mathbf{u} &= 0, \\ \nabla^2 T - \mathbf{u}\cdot\nabla T &= 0, \end{aligned} \tag{1}$$

where $\mathbf{u}$ is the velocity field, p is the pressure, and T is the temperature. Deflation is based on Newton's method and consists of modifying the problem to ensure nonconvergence of the numerical solver to previously computed solutions to (1) in order to discover new solutions. In this work, we combine deflation with an initialization strategy to generate initial guesses based on perturbed solutions to the linearized equations.

We then perform bifurcation analysis of Rayleigh-Bénard convection and compute many solutions to (1) over a wide range of values of the Rayleigh number Ra [2], including several disconnected branches of the bifurcation diagram, without the need for any prior knowledge of the solutions. One of the disconnected branches we find contains an S-shaped curve with hysteresis, which is the origin of a flow pattern that may be related to the dynamics of flow reversals in the turbulent regime. Linear stability analysis is also performed to analyze the steady and unsteady regimes of the solutions in the parameter space and to characterise the type of instabilities. We finally classify the solutions based on the kinetic and potential energies as well as the Nusselt number. Our classification provides a clear view of the steady states of Rayleigh-Bénard convection up to a Rayleigh number of 10^5.

[1] P. E. Farrell, A. Birkisson, and S. W. Funke, *Deflation techniques for finding distinct solutions of nonlinear partial differential equations*, SIAM J. Sci. Comput. **37** (2015), A2026–A2045.

[2] N. Boullé, V. Dallas, and P. E. Farrell, *Bifurcation analysis of two-dimensional Rayleigh-Bénard convection using deflation*, Phys. Rev. E **105** (2022), 055106.

Strong stationarity for an optimal control problem for a rate independent evolution

Martin Brokate
TU München, WIAS Berlin, Germany, CTU Prague, Czech Republic

This is a joint work with Pavel Krejčí (Czech Academy of Sciences, Prague) and Constantin Christof (TU München).

We present strong stationarity conditions for the optimal control problem

$$\text{minimize} \quad J(z,u)\,, \quad \text{where } z = S(u)\,. \tag{1}$$

Here, $u \mapsto z = S(u)$ stands for the rate independent evolution given by the scalar stop operator (or, equivalently, by a corresponding scalar rate independent evolution variational inequality). While S is not smooth, it possesses directional derivatives [1]. Any local minimizer u_* of $j(u) = J(S(u),u)$ satisfies the Bouligand stationarity condition $j'(u_*;h) \geq 0$ for all variations h. We present a strong stationarity condition, that is, a system of equations and inequalities resembling a Lagrange multipliere rule which is equivalent to Bouligand stationarity [2]. This result is based on the characterization of the directional derivative $S'(u;h)$ of the stop operator by a variational inequality in [3].

[1] M. Brokate, P. Krejčí, *Weak differentiability of scalar hysteresis operators*, Discrete Contin. Dyn. Syst. **35** (2015), 2406–2421.

[2] M. Brokate, C. Christof, *Strong stationarity conditions for optimal control problems governed by a rate-independent evolution variational inequality*, arXiv 2205.01196 (2022).

[3] M. Brokate, P. Krejčí, *A variational inequality for the derivative of the scalar play operator*, J. Appl. Numer. Optim. **3** (2021), 263–283.

Convergence to equilibria for weak solutions of heat conducting non-Newtonian fluids

Miroslav Bulíček
Charles University, Prague, Czech Republic

This is a joint work with Anna Abbatiello (Sapienza Università di Roma) and Petr Kaplický (Charles University).

We consider a flow of non-Newtonian heat conducting incompressible fluid in a bounded domain subjected to the homogeneous Dirichlet boundary condition for the velocity field and the spatially inhomogeneous Dirichlet boundary condition for the temperature. The ultimate goal is to show that the fluid converges to equilibrium as time tends to infinity and also if possible to get as good as possible rate of the convergence. We discuss what is the proper metric for measuring the distance to equilibrium and show how it is affected by material parameters. Moreover, we show that formally (for sufficiently smooth solutions) we have the exponential rate of the convergence for most of the classical models. Finally, we discuss when such exponential decay can be justified also rigorously for any weak/entropy/suitable or in general “proper notion" of solution.

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Existence results for a third order three-point boundary value problem

Alberto Cabada
Universidade de Santiago de Compostela, Galicia, Spain

This is a joint work with Prof Nikolay D. Dimitrov (University of Ruse, Ruse, Bulgaria).

In this talk, we present the results, obtained in [1], of existence of solutions of the third order nonlinear differential equation

$$u'''(t) = -\lambda\, p(t)\, f(u(t)), \text{ a.e. } t \in [0,1], \tag{1}$$

coupled to the three-point boundary value conditions

$$u(0) = 0,\ u''(\eta) = \alpha\, u'(1),\ u'(1) = \beta\, u(1), \tag{2}$$

with $0 \le \alpha \le 1, 0 \le \beta < \frac{2}{2-\alpha}$ and $0 \le \eta \le \frac{1}{3}$.

Taking into account that the related Green's function is nonpositive for $0 \le s < \eta$ and nonnegative if $\eta < s \le 1$, we assume the following conditions on the nonlinear part of the equation:

(F) $\lambda > 0$ is a parameter, $p \in L^\infty(I)$ is such that $p < 0$ a.e. on $[0,\eta]$ and $p > 0$ a. e. on $[\eta,1]$ and $f : [0,\infty) \to [0,\infty)$ is a continuous function.

By defining suitable cones on $C^1([0,1])$, under additional conditions on the asymptotic behavior of function f, we deduce, for a particular set of values of the parameter λ, the existence of positive and increasing solutions on the whole interval of definition which are convex on $[0,\eta]$. The results hold by means of degree theory.

[1] A. Cabada, N. D. Dimitrov, *Third-order differential equations with three-point boundary conditions.* Open Math. **19** (2021), 1, 11–31.

Low Mach number flows and dimension reduction in fluid mechanics

Matteo Caggio
Institute of Mathematics, CAS, Prague, Czech Republic

We consider the compressible Navier–Stokes system describing the motion of a viscous fluid confined to a straight layer $\Omega_\delta = (0, \delta) \times \mathbb{R}^2$. We show that the weak solutions in the 3D domain converge strongly to the solution of the 2D incompressible Navier–Stokes equations (Euler equations) when the Mach number tends to zero as well as $\delta \to 0$ (and the viscosity goes to zero). An extension to heat-conductive fluids in presence of strong stratification will be also discussed.

[1] Caggio, Matteo; Donatelli, Donatella; Nečasová, Šárka; Sun, Yongzhong, Low Mach number limit on thin domains. Nonlinearity 33 (2020), no. 2, 840–863.

[2] Caggio, Matteo; Donatelli, Donatella; Nečasová, Šárka; Pokorný, Milan, Inviscid incompressible limits on thin domains for the Navier-Stokes-Fourier system, in preparation.

Analysis of generalized Aw-Rascle system

Nilasis Chaudhuri
Imperial College London, UK

This is a joint work with Piotr Gwiazda (IMPAN, Warsaw, Poland) and Ewelina Zatorska (Imperial College London, UK), [2].

In this talk we consider the multidimensional generalization of the Aw-Rascle system for vehicular traffic. For a large class of initial data and the periodic boundary conditions, we prove the existence of a global-in-time measure-valued solution. The idea of such solution is motivated from the works related to dissipative measure-valued solution of compressible Navier-Stokes system, [1]. Moreover, using the relative energy technique, we show that, emanating from the same initial data, a measure-valued solution coincides with the classical solution as long as the latter exists.

[1] E. Feireisl, P. Gwiazda, A. Świerczewska–Gwiazda and E. Wiedemann, *Dissipative measure-valued solutions to the compressible Navier-Stokes system*, Calc. Var. Partial Differential Equations, 55(6):Art. 141, 20, 2016.

[2] N. Chaudhuri, P. Gwiazda amd E. Zatorska *Analysis of generalized Aw-Rascle system* arXiv:2202.04130.

On numerical schemes for fractional order Mean Field Games

Indranil Chowdhury
University of Zagreb, Zagreb, Croatia

This is a joint work with Prof Espen R. Jakobsen and Olav Ersland (Norwegian University of Science and Technology, Trondheim, Norway).

Mean Field Game (MFG) equations are coupled system of PDEs consisting Hamilton–Jacobi–Bellman and Fokker–Planck equations. I will discuss the numerical approximation of a class of MFG system with nonlocal/fractional order diffusion. The problems include degenerate diffusion and our schemes are based on semi-Lagrangian approximation of the underlying control problem/ games. The prescribed schemes are monotone, stable and consistent. I will discuss the convergence results along subsequences for degenerate equations in one space dimension and also for nondegenerate equations in arbitrary dimensions.

Two solutions to a p-Laplacian supercritical Neumann problem: existence and asymptotics

Francesca Colasuonno
Alma Mater Studiorum Università di Bologna, Italy

This is a joint work with Proff. Benedetta Noris and Gianmaria Verzini (Politecnico di Milano, Italy).

For $1 < p < 2$ and q large, we prove the existence of two positive, nonconstant, radial and radially nondecreasing solutions of the supercritical equation

$$-\Delta_p u + u^{p-1} = u^{q-1}$$

under Neumann boundary conditions, in the unit ball of $\mathbb{R}^N$. We use a variational approach in an invariant cone and distinguish the two solutions upon their energy: one is a ground state inside a Nehari-type subset of the cone, the other is obtained via a mountain pass argument inside the Nehari set. As a byproduct of our proofs, we show that the constant solution 1 is a local minimizer on the Nehari set. This is a peculiarity of the case $1 < p < 2$ and is responsible for the appearance of the second (higher-energy) solution. Finally, we detect the limit profiles of the two solutions as the power in the nonlinearity goes to infinity. These results are contained in [1, 2].

[1] F. Colasuonno, B. Noris, G. Verzini, *Multiplicity of solutions on a Nehari set in an invariant cone.* Minimax Theory Appl. **7** (2022), 185–206.

[2] F. Colasuonno, B. Noris, *Asymptotics for a high-energy solution of a supercritical problem.* Submitted for publication (2022), arXiv:2203.06940.

Parametric superlinear double phase problems with singular term and critical growth on the boundary

Ángel Crespo-Blanco
Technische Universität Berlin, Germany

This is a joint work with Nikolaos S. Papageorgiou (National Technical University, Athens, Greece) and Patrick Winkert (Technische Universität Berlin, Berlin, Germany).

In this talk, we present the ideas of the article with the same name, see [1]. In that paper we study quasilinear elliptic equations driven by the double phase operator along with a reaction term, which in the right-hand side has a singular term and a parametric superlinear term, and on which we impose a nonlinear Neumann boundary condition of critical growth, namely

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)+\alpha(x)u^{p-1} &= \zeta(x)u^{-\kappa}+\lambda u^{q_1-1} && \text{in } \Omega,\\ \left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nu &= -\beta(x)u^{p_*-1} && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $0 < \kappa < 1$, $\lambda > 0$ and $\nu(x)$ is the outer unit normal of Ω at the point $x \in \partial\Omega$. In our setting,

$$1 < p < N, \qquad p < q < p^*, \qquad 0 \leq \mu \in L^\infty(\Omega), \tag{2}$$

and further assumptions are imposed on q_1, α, β and ζ.

Based on a new equivalent norm for Musielak–Orlicz Sobolev spaces and the Nehari manifold along with the fibering method, we prove the existence of at least two weak solutions with positive and negative energy, provided the parameter is sufficiently small.

[1] Á. Crespo-Blanco, N.S. Papageorgiou, P. Winkert, *Parametric superlinear double phase problems with singular term and critical growth on the boundary.* Math. Methods Appl. Sci. **45** (2022), 2276– 2298.

Incompressible limit for a two-species tumour growth model

Tomasz Debiec
Laboratoire Jacques-Louis Lions, Sorbonne Université, France

We study a two-species advection-reaction system, a well-known model with applications in modelling tumour growth. The cell densities are advected by the gradient of a chemical potential which satisfies the so-called Brinkman law, while the growth rate of each population is governed by a function of the joint population pressure.

We present a rigorous argument on connecting the coupled PDE system to a more geometric formulation, wherein the total population density is limited to a critical value and the pressure vanishes on unsaturated regions.

Joint work with B.Perthame, M.Schmidtchen and N.Vauchelet.

A bifurcation phenomenon for the critical Laplace and p-Laplace equation in the ball

Francesca Dalbono
University of Palermo, Italy

This is a joint work with Prof. Matteo Franca (University of Bologna, Italy), and Prof. Andrea Sfecci (University of Trieste, Italy).

We focus on positive radial solutions to the Dirichlet problem associated with the eigenvalue equation

$$\begin{cases} \Delta_p u + \lambda\, K(|x|) u^{q-1} = 0, & x \in B_1 \\ u(x) = 0 & |x| = 1, \end{cases} \tag{1}$$

where $\Delta_p u = \mathrm{div}(\nabla u |\nabla u|^{p-2})$ denotes the p-Laplace operator, B_1 is the ball of radius 1 in $\mathbb{R}^n$, $2n/(2+n) \leq p \leq 2,\ 1 < p < n$, and q is the Sobolev critical exponent

$$q = p^* = \frac{np}{n-p}.$$

The function K is assumed to be C^1, positive and to satisfy the ℓ-flatness condition in a neighborhood of zero. We show that the existence of positive solutions depends on the slope ℓ of K at zero, and on choice of the positive parameter λ. In particular, we prove that the order of flatness at zero presents a critical value, $\ell_p^* := \frac{n-p}{p-1}$, which is the supremum of the slopes for which problem (1) admits radial solutions independently of the choice of λ.

An additional strictly monotonicity assumption on K allows us to describe more in details the supercritical case $\ell > \ell_p^*$, ensuring the existence of a bifurcation number λ_0 such that problem (1) has at least two positive radial solutions for $\lambda > \lambda_0$, but no solutions exist whenever $\lambda < \lambda_0$.

Our main purpose is to improve and extend the result in [1] to the p-Laplacian case. Our approach, based on Fowler transformation, invariant manifold theory, phase plane analysis, and energy estimates, offers a new geometrical perspective and exploits the construction of suitable barrier sets for the solutions.

[1] C.S. Lin and S.S. Lin, *Positive radial solutions for* $\Delta u + K u^{\frac{n+2}{n-2}} = 0$ *in* $\mathbb{R}^n$ *and related topics.* Appl. Anal. **38** (1990), 121–159.

Nonparametric Bayesian inference of discretely observed diffusions

Masoumeh Dashti
University of Sussex, UK

This is a joint work with Jean-Charles Croix, Stylianos Katsarakis (University of Sussex, Brighton) and Tanja Zerenner (University of Bristol, Bristol).

We consider the inverse problem of recovering the diffusion and drift functions of a stochastic differential equation from discrete measurements of its solution. We give conditions for the well-posed approximations of the posterior measure allowing for priors with unbounded support. We then consider the case where the diffusion coefficient is small. We use random perturbation methods to approximate the solution of the stochastic differential equation, and study the resulting approximated posterior measure.

The existence of a solution for nonlinear fractional differential equations where nonlinear term depends on the fractional and first order derivative of an unknown function

Sladjana Dimitrijevic
University of Kragujevac, Faculty of Science, Serbia

This is a joint work with Prof. Alberto Cabada (University of Santiago de Compostela, Spain), and Prof. Suzana Aleksic and Prof. Tatjana Tomovic Mladenovic (both from University of Kragujevac, Serbia).

In this paper, we consider the existence of solutions for the nonlinear fractional differential equation boundary-value problem

$$ {}^{C}D^{\alpha}u(t) = f(t, u(t), u'(t), {}^{C}D^{\beta}u(t)), \quad 0 < t < 1,\ 1 < \alpha < 2,\ 0 < \beta \leqslant 1, \tag{1} $$

$$ u(0) = A, \quad u(1) = Bu(\eta), \tag{2} $$

where $0 < \eta < 1$, $A \geqslant 0$, $B\eta > 1$ and ${}^{C}D^{\alpha}$, ${}^{C}D^{\beta}$ are the Caputo's differential operators of order α, β, respectively, and f is a function in $C^1([0,1] \times R \times R \times R)$. Existence results and conditions for a positive solution are obtained.

[1] E. Pourhadi, R. Saadati and S. K. Ntouyas, *Application of fixed-point theory for a nonlinear fractional three-point boundary-value problem*, Mathematics **7**(6) (2019), Article ID 526.

[2] S. Xinwei and L. Landong, *Existence of solution for boundary value problem of nonlinear fractional differential equation*, Appl. Math. J. Chinese Univ. Ser. B **22**(3) (2007), 291–298.

[3] J. R. L. Webb, *Initial value problems for Caputo fractional equations with singular nonlinearities*, Electron. J. Differential Equations (2019), Paper No. 117, 32 pp.

[4] J. R. L. Webb, *Compactness of nonlinear integral operators with discontinuous and with singular kernels*, J. Math. Anal. Appl. **509** (2022), no. 2, Paper No. 126000, 17 pp.

Application of the Karamata theory in the study of asymptotic properties of the half-linear q-difference equation

Katarina Djordjević
Faculty of Sciences and Mathematics, University of Niš, Serbia

This is a joint work with Prof Jelena Manojlović (Faculty of Sciences and Mathematics, University of Niš, Serbia).

In this talk, the asymptotic behavior of positive solutions of the second-order half-linear q-difference equation

$$D_q(p(t)\Phi(D_q(x(t)))) + r(t)\Phi(x(qt)) = 0,\ t \in q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\},$$

with $q > 1$, where $\Phi(x) = |x|^\alpha \mathrm{sgn} x$, $\alpha > 0$, $p : q^{\mathbb{N}_0} \to (0, \infty)$, $r : q^{\mathbb{N}_0} \to \mathbb{R}$, will be analyzed in the framework of the Karamata theory. More precisely, under the assumption that coefficient p is a q-regularly varying function, necessary and sufficient conditions for the existence of q-regularly varying solutions will be presented. Moreover, the asymptotic formulas of the existing q-regularly varying solutions, under certain assumptions, will be established.

On stability of delay differential equations

Alexander Domoshnitsky
Ariel University, Ariel, Israel

The following first order delay differential equation

$$x'(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [0, \infty) \tag{1}$$

with $\omega - periodic$ coefficient $p(t)$ and delay $\tau(t)$ satisfying the conditions $t - \tau(t) \geq 0$, $p(t) \geq 0$, $\int\limits_0^\omega p(t)dt > 0$, is considered.

A version of the Floquet theory for delay equations is proposed and new results on stability are obtained on this basis. One of them is the following:

Theorem. *There exist infinite number of intervals* (α_n, β_n), *where* $\alpha_n \to \infty$ *for* $n \to \infty$, *that the equation is exponentially stable in the case of* $\omega \in \bigcup\limits_{n=1}^{\infty} (\alpha_n, \beta_n)$.

Note that in the classical results by A.D.Myshkis a corresponding smallness of the integral $\int\limits_{t-\tau(t)}^{t} p(s)ds < \frac{3}{2}$ is assumed.

In our results there is no such assumption.

Shadowing for nonautonomous differential equations

Davor Dragičević
Faculty of Mathematics, University of Rijeka, Croatia

This is a joint work with L. Backes (Departamento de Matemática, Universidade Federal do Rio Grande do Sul, Porto Alegre, Brazil), M. Pituk (Department of Mathematics, University of Pannonia, Veszprém, Hungary) and L. Singh (Department of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia)

In this talk we plan to survey several recent results dealing with the shadowing property of nonautonomous and nonlinear differential equations. More precisely, let us consider a nonautonomous semilinear differential equation

$$x' = A(t)x + f(t,x) \quad t \in \mathbb{R}, \tag{1}$$

acting on an arbitrary Banach space X. Roughly speaking, we say that (1) exhibits shadowing property if in a vicinity of each approximate solution of (1), we can construct its exact solution.

Our goal is to describe sufficient conditions under which (1) has shadowing property. Besides considering ordinary differential equations [1, 2, 4], we will also discuss the case of delay differential equations [3].

[1] L. Backes and D. Dragičević, *Shadowing for infinite dimensional dynamics and exponential trichotomies*, Proc. Roy. Soc. Edinburgh Sect. A **151** (2021), 863–884.

[2] L. Backes and D. Dragičević, *A general approach to nonautonomous shadowing for nonlinear dynamics*, Bull. Sci. Math. **170** (2021), 30pp.

[3] L. Backes, D. Dragičević, M. Pituk and L. Singh, *Weighted shadowing for delay differential equations*, submitted.

[4] L. Backes, D. Dragičević and L. Singh, *Shadowing for nonautonomous and nonlinear dynamics with impulses*, Monatsh. Math., in press.

On a structure-preserving variational scheme for nonlinear diffusion equations

Bertram Düring
University of Warwick, UK

Many nonlinear diffusion equations can be interpreted as gradient flows whose dynamics are driven by internal energies and given external potentials. Examples include the heat equation, the porous medium equation, the thin film equation and the fourth-order Derrida-Lebowitz-Speer-Spohn equation. When solving these equations numerically, schemes that respect the equations' special structure are of particular interest. In this talk we present a structure-preserving Lagrangian scheme for nonlinear diffusion equations. For discretisation of the Lagrangian map, we use a finite subspace of linear maps in space and a variational form of the implicit Euler method in time. We present numerical experiments for the porous medium equation in two space dimensions.

Dynamical system with the non-Markovian process: stability and optimization

Irada Dzhalladova
Kyiv National Economic Univesity, Ukraine, VUT, Brno, Czech Republic

This is a joint work with doc. Veronika Novotna (VUT, Brno, Czech Republic).

In [1] investigation of a dynamical system with Markovian parameters is given. In [2] postulated problems with non-Markovian processes and also models are considered. The non-Markovian process is a rule but the Markovian process is an idealization.

In this and future papers we research stability and optimization problems of a dynamical system with non-Markovian parameters.

On probability space (Ω, F, P) we consider a system of differential equations (1) with initial condition (2), where ω is a random variable. The solution to this problem is denoted by $\Phi(t)$, but, in fact, any solution depends on the random variable ω. The F on the ride side of the system (1) is equal to zero for the trivial solution. The F depends on $\xi(t)$, which is the non-Markov process.

Here is some equation

$$\frac{d\Phi(\omega,t)}{dt} = F(t, \xi(t)), \Phi(\omega,t)) \tag{1}$$

$$F(0, \xi(t)), \Phi(0)) = 0 \tag{2}$$

where $\xi(t)$ is a Non- Markovian process.

Our results concern the $L-2$ stability, so it is good to clarify this term. If the integral which depends on mathematical expectation $\Phi(t)$ with bounded initial value converges for any solution of (1),(2) then the trivial solution of the differential systems (1),(2) is said to be $L-2$-stable.

Our purpose is to establish conditions for the stability of the zero solution of the considered system. An effective method for investigating the stability of linear or nonlinear differential systems is the Lyapunov functions method. But its use is complicated for the investigation of the differential systems with random parameters. This can be explained by the fact that the concept of the Lyapunov stability makes the Lyapunov functions inconvenient to use. Therefore, we use so-called $L-2$-stability instead of Lyapunov stability in our investigation.

Moreover, we create a new method of construction of Lyapunov functions.

[1] Dzhalladova Irada and Ruzickova Miroslava. A dynamical system with a random structure and its applications. Cambridge Scientific Publishers, 2020. ISBN 978-1-908106-66-7.

[2] Dzhalladova I., Ruzickova M., Dacjuk M. Non Markovian process: postulates and model problems. Modeling and information systems in economics, No. 101, 2022, p. 122-127. ISSN 1511-1100. | https://doi.org/10.33111/mise.101.5

[3] Gardiner C.W., Handbook of Stochastic Methods.-Springer, Berlin 1983.

Boundary behaviour for viscous and inviscid Hamilton-Jacobi equations

Carlos Esteve Yagüe
University of Cambridge, UK

In this talk, I will discuss the boundary condition for different viscous and inviscid Hamilton-Jacobi equations arising in optimal control theory. It is well-known that, under suitable hypotheses, the value function associated to an optimal control problem satisfies a Hamilton-Jacobi equation, and that the boundary condition is given by the final pay-off of the associated optimal control problem. However, in many situations, the value function develops discontinuities at the boundary, and hence, the boundary condition has to be understood in the viscosity sense. First, I will discuss this phenomenon for a viscous Hamilton-Jacobi equation in a bounded domain, for which the loss of the boundary condition appears after the so-called gradient blow-up [1]; then, for a degenerate parabolic equation arising in game theory [2]; and finally, for inviscid Hamilton-Jacobi equations with prescribed final condition [3].

[1] C. Esteve, *Single-point Gradient Blow-up on the Boundary for Diffusive Hamilton-Jacobi Equation in domains with non-constant curvature*, Journal de Mathématiques Pures et Appliquées, 137 (2020), pp. 143–177.

[2] C. Esteve, P. Blanc and J. D. Rossi. *A game theoretical approach for the evolution problem associated with eigenvalues of the Hessian*, J. London Math. Soc. (2) 00 (2020), 1-25.

[3] C. Esteve and E. Zuazua. *The inverse problem for Hamilton–Jacobi equations and semiconcave envelopes*, SIAM J. Math. Anal., 52(6), 5627–5657.

Some results for nonautonomous linear Hamiltonian systems

Roberta Fabbri
Università di Firenze, Italy

The investigation and properties of the rotation number for a family of linear nonautonomous Hamiltonian systems and its relation with the exponential dichotomy concept are fundamental for the role they have in the study of qualitative behavior of various dynamical systems. Applications to spectral and control theory are given. In particular, the problem of the solvability of a minimization problem of infinite horizon type, formulated in terms of the property of a corresponding linear Hamiltonian system ([2]), and the presence of exponential dichotomy and its relation with the corresponding Floquet coefficent for a perturbed Hamiltonian family which does not verify the classical Atkinson condition are considered.

[1] R. Johnson, R. Obaya, S. Novo, C. Núñez , R. Fabbri, Nonautonomous Linear Hamiltonian Systems: Oscillation, Spectral Theory and Control. Springer, 2016.

[2] R. Fabbri, C. Núñez , *On the solvability of the Yakubovich linear-quadratic infinite horizon minimization problem* Ann. Mat. Pura Appl. ser. IV **199** (2020), 1713-1735.

Permanence and stability for a Nicholson's equation with mixed monotonicities

Teresa Faria
Departamento de Matemática and CMAFCIO, Faculdade de Ciências,
Universidade de Lisboa, Portugal

This is a joint work with Henrique Prates (University of Lisbon, Portugal) [2].

We consider a Nicholson's equation with multiple pairs of time-varying delays where each nonlinear term incorporates two delays and is given by a mixed monotone function:

$$x'(t) = \beta(t)\Big(\sum_{j=1}^{m} p_j x(t-\tau_j(t))e^{-a_j x(t-\sigma_j(t))} - \delta x(t)\Big), \quad t \geq t_0, \tag{1}$$

where $p_j, a, \delta \in (0,\infty)$ and $\beta(t), \sigma_j(t), \tau_j(t)$ are continuous, non-negative and bounded, $\beta(t) \geq \beta^- > 0$. Equation (1) is shown to be permanent if $\sum_j p_j > \delta$. In this situation there exists a positive equilibrium K, and additional sufficient conditions for the global attractivity of K are established. Our criteria depend on the size of some delays and enhance results in recent literature [1].

[1] H.A. El-Morshedy and A. Ruiz-Herrera, *Global convergence to equilibria in non-monotone delay differential equations*, Proc. Am. Math. Soc. **147** (2019), 2095–105.

[2] T. Faria and H.C. Prates *Global attractivity for a nonautonomous Nicholson's equation with mixed monotonicities*, Nonlinearity **35** (2022), 589–607.

Oscillatory behaviour of a nonlinear Becker-Döring type model for prion dynamics

Fellner Klemens
University of Graz, Austria

This is a joint work with Prof. Marie Doumic (CNRS, Ecole Polytechnique), Mathieu Mezache (CNRS), Human Rezaei (INRA), Juan L. Velazquez (University of Bonn).

Prions are able to self-propagate biological information through the transfer of structural information from a misfolded/infectious protein in a prion state to a protein in a non-prion state. Prions cause diseases like Creuzfeldt-Jakob. Prion-like mechanisms are associated to Alzheimer, Parkinson ans Huntington diseases. We present a fundamental bi-monomeric, nonlinear Becker-Döring type model, which aims to explain experiments in the lab of Human Rezaei showing sustained oscillatory behaviour over multiple hours. We exemplify a mechanism of oscillatory behaviour and show numerical simulations, [1].

[1] M. Doumic, K. Fellner, M. Mezache, H. Rezaei, A bi-monomeric nonlinear Becker-Döring type system to capture oscillatory aggregation kinetics in prion dynamics, Journal of Theoretical Biology, 480 (2019) 241–261.

Periodic solutions to a perturbed relativistic Kepler problem

Guglielmo Feltrin
University of Udine, Italy

The motion of a relativistic particle in a Kepler potential can be described by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\dot{x}}{\sqrt{1-|\dot{x}|^2/c^2}}\right) = -\alpha\,\frac{x}{|x|^3}, \quad x \in \mathbb{R}^2 \setminus \{0\},$$

where $m > 0$ is the mass of the particle, c is the speed of light, and $\alpha > 0$ is a constant. Firstly, we illustrate the Hamiltonian formulation of the problem and we focus our attention on the description of the periodic and quasi-periodic solutions. Secondly, we deal with the perturbed equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\dot{x}}{\sqrt{1-|\dot{x}|^2/c^2}}\right) = -\alpha\,\frac{x}{|x|^3} + \varepsilon\,\nabla_x U(t,x), \quad x \in \mathbb{R}^2 \setminus \{0\},$$

where $U(t,x)$ is T-periodic in the first variable and $\varepsilon \in \mathbb{R}$. The analysis of the action-angle coordinates and an application of an higher dimensional version of the Poincaré–Birkhoff fixed point theorem allow to prove that, for ε small enough, the perturbed problem admits T-periodic solutions with prescribed winding number, bifurcating from invariant tori of the unperturbed problem. The talk is based on the paper [1] written in collaboration with Prof. Alberto Boscaggin (University of Torino, Italy) and Prof. Walter Dambrosio (University of Torino, Italy).

[1] A. Boscaggin, W. Dambrosio, G. Feltrin, Periodic solutions to a perturbed relativistic Kepler problem, SIAM J. Math. Anal. 53 (2021), no. 5, 5813–5834.

Ordering properties for radial solutions of p-Laplace equations

Matteo Franca
Universita' di Bologna, Italy

This is a joint work with Prof R. Colucci (Politechnic University of Marche, Ancona, Italy). In this talk we discuss the ordering properties of positive radial solutions of

$$\Delta_p u + k(|x|)u^{q-1} = 0$$

where $x \in \mathbb{R}^n$, $n > p > 1$, $q > p$ and k is a positive function. We are interested both in regular ground states, defined and positive in the whole of $\mathbb{R}^n$, and in singular ground states v, defined and positive in $\mathbb{R}^n \setminus \{0\}$ and such that $\lim_{|x|\to 0} v(x) = +\infty$. A key role in this analysis is played by two bifurcation parameters $p_{jl} < p^{jl}$ which are the natural generalization of the classical Joseph–Lundgren exponent [3] and its dual.

This kind of results, besides the intrinsic mathematical interest, in the $p = 2$ case play a key role in determining the stability properties of the associated parabolic problem. More precisely, when the solutions are ordered they gain some stability properties, in suitable weighted spaces, while when this ordering property is lost these solutions determine the threshold between blowing up and fading solutions [2].

[1] R. Colucci, M. Franca, *Ordering properties of radial ground states and singular ground states of quasilinear elliptic equations* NoDEA Nonlinear Differential Equations Appl. **27**(6) (2020), 54.

[2] M. Hoshino, E. Yanagida *Convergence rate to singular steady states in a semilinear parabolic equation* Nonlinear Anal., **131** (2016), 98–111.

[3] D.D. Joseph, T.S. Lundgren, *Quasilinear elliptic problems driven by positive sources* Arch. Ration. Mech. Anal., **49** (1972/1973), 241–269.

Super-localized orthogonal decomposition for convection-dominated diffusion problems

Philip Freese
Augsburg University, Germany

This is a joint work with Francesca Bonizzoni (Augsburg University, Germany) and Daniel Peterseim (Augsburg University, Germany).

We present a multi-scale method for the convection-dominated diffusion equation

$$\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with singular perturbation parameter $0 < \varepsilon \ll 1$ and incompressible velocity field b. The application of the solution operator to piecewise constant right-hand sides on some arbitrary coarse mesh defines a finite-dimensional coarse ansatz space with favorable approximation properties. For some relevant error measures, including the L^2-norm, the Galerkin projection onto this generalized finite element space even yields ε-independent error bounds.

We construct an approximate local basis that turns the approach into a novel multi-scale method in the spirit of the Super-Localized Orthogonal Decomposition (SLOD). The error caused by basis localization, which we conjecture to decay super-exponentially, can be estimated in an a posteriori way. Numerical experiments indicate ε-independent convergence without preasymptotic effects, even in the under-resolved regime of large mesh Péclet numbers.

Numerical implementation of constitutive models of materials with strongly coupled dissipative processes

Miroslav Frost
Institute of Thermomechanics of the CAS, Prague, Czech Republic

Many interesting material systems exhibit simultaneously activated, strongly coupled dissipative processes, e.g. NiTi shape memory alloys [1]. A very flexible framework for formulation and mathematical analysis of constitutive models of such a type is the Generalized Standard Materials/Models (GSM) concept developed in [2]. After parametrization of dissipative processes by so-called *internal variables*, the model is completed by specifying two scalar constitutive functions: one for energy conservation, the other for energy dissipation. For rate-independent processes, the variational formulation of the corresponding evolutionary boundary value problem can be recast (under some conditions) into a sequence of *time-incremental minimization problems*, each of them in a generic symbolic form

$$\Delta E + \Delta D + \Delta\Omega \to \min_{v,\alpha} \quad \text{subject to relevant constraints, initial and boundary conditions.} \tag{1}$$

Here ΔE, ΔD and $\Delta\Omega$ denote global incremental stored energy, global incremental dissipation (pseudo-potential) and the increment of potential energy of the conservative loads, respectively. The minimum is searched with respect to both the set of state variables, v, and the set of internal variables, α. Let us note that a mathematically rigorous theory covering this formulation is summarized in [3].

Common methods for numerical solution of the incremental minimization problem (1) are based on alternative minimization strategy, which relies on the mutual independence of state-update procedure within independent material points. Often, "check and treat" strategy is employed to resolve the state-update: activity of respective dissipative processes is checked first; then, based on the findings, the evolution is resolved considering only the set of active processes. These methods were developed and streamlined for conventional plasticity models, where they have proved robustness and efficiency. However, searching for the set of activated processes turns into a complex and elaborate task for models with many inelastic processes and internal constraints; moreover, periodic oscillations of the active set may result into instabilities of iterative solution procedures. Fortunately, the variational character of the global problem provides several alternative ways how to deal with (1). Some of them will be discussed with relation to the constitutive model coupling various inelastic dissipative processes in NiTi shape memory alloys [1].

This is a joint work with J. Valdman, M. Kružík (Institute of Information Theory and Automation of the CAS) and P. Sedlák, B. Benešová (Institute of Thermomechanics of the CAS).

[1] M. Frost, B. Benešová, P. Sedlák, *A microscopically motivated constitutive model for shape memory alloys: formulation, analysis and computations,* Math. Mech. Solids **21** (2016), 358–382.

[2] B. Halphen, Q.S. Nguyen, *Sur les matériaux standard généralisés,* J. Mecanique **14** (1975), 39–63.

[3] A. Mielke, T. Roubíček, Rate-Independent Systems: Theory and Application. Springer, New York, 2015.

Singular solutions of ordinary differential equations with $p(t)$-Laplacian

Kōdai FUJIMOTO

Osaka Metropolitan University, Japan

This talk is based on the collaboration with Professor Miroslav Bartušek (Masaryk University, Brno, Czech Republic) [1].

In this talk, we consider the second order nonlinear differential equation with $p(t)$-Laplacian

$$(a(t)|x'|^{p(t)-2}x')' = b(t)|x|^{q(t)-2}x, \quad t \geq 0, \tag{1}$$

where $a(t) > 0$, $b(t)$, $p(t) > 1$, and $q(t) > 1$ are continuous functions defined on $\mathbb{R}_+ := [0, \infty)$.

A function $x(t)$ is said to be a *solution* of equation (1) defined on $[t_0, \tau) \subset \mathbb{R}_+$, if $x(t)$ and its quasiderivative $x^{[1]}(t) := a(t)|x'(t)|^{p(t)-2}x'(t)$ are continuously differentiable, and $x(t)$ satisfies equation (1) on $[t_0, \tau)$. We discuss solutions of equation (1) which are defined on $[t_0, \tau)$; if $\tau < \infty$ then we suppose that $x(t)$ is nonextendable to the right, i.e.,

$$\limsup_{t \to \tau-} (|x(t)| + |x'(t)|) = \infty.$$

A nontrivial solution $x(t)$ of equation (1) on $[t_0, \infty)$ is said to be a *singular solution of the first kind* if there exists a positive constant $T_x > t_0$ such that $x(t) \equiv 0$ for $t \geq T_x$. It is said to be a *singular solution of the second kind* if $\tau < \infty$. It is said to be a *proper* solution if $x(t)$ is nonsingular.

When we investigate the asymptotic behavior of solutions of nonlinear differential equations, we often assume the existence of proper solutions. However, sometimes the considered equation possesses singular solutions only. In this sense, the problem of existence and nonexistence of proper and singular solutions has the great importance.

The purpose of this study is to give sufficient conditions for the existence and nonexistence of singular solutions of the first (second) kind. In addition, we deal with the existence of proper solutions and the coexistence of proper solutions and singular solutions of the second kind. The proofs are based on the results of T. A. Chanturiya [2], [3], and [4]. In addition, we present some applications and examples.

[1] M. Bartušek and K. Fujimoto, *Singular solutions of nonlinear differential equations with $p(t)$-Laplacian*, J. Differential Equations **269** (2020), 11646–11666.

[2] T. A. Chanturiya, *A Kneser-type problem for a system of ordinary differential equations*, Mat. Zametki **15** (1974), 897–906 (in Russian), English version: Math. Notes **15** (1974), 536–541.

[3] T. A. Chanturia, *On singular solutions of strongly nonlinear systems of ordinary differential equations*, in: Function Theoretic Methods in Differential Equations, in: Res. Notes in Math., vol. 8, Pitman, London, 1976, 196–204.

[4] T. A. Čanturia, *On singular solutions of nonlinear systems of ordinary differential equations*, in: Differential equations, Keszthely (Hungary), in: Coll. Math. Soc. J. Bolyai, 1977, 107–119.

Convergence of a regularized finite element discretization of the two–dimensional Monge–Ampère equation

Dietmar Gallistl
Friedrich-Schiller-Universität Jena, Germany

This is a joint work with Dr. Ngoc Tien Tran (Universität Jena, Germany).

This contributions proposes a regularization of the Monge–Ampère equation

$$\det D^2u = f \text{ in } \Omega \quad \text{and} \quad u = g \text{ on } \partial\Omega.$$

in a planar convex domain Ω through uniformly elliptic Hamilton–Jacobi–Bellman equations. The regularized problem possesses a unique strong solution u_ε and is accessible to the discretization with finite elements. This work establishes locally uniform convergence of u_ε to the convex Alexandrov solution u to the Monge–Ampère equation as the regularization parameter ε approaches 0. A mixed finite element method for the approximation of u_ε is proposed, and the regularized finite element scheme is shown to be locally uniformly convergent. Numerical experiments provide empirical evidence for the efficient approximation of singular solutions u.

[1] D. Gallistl and N. T. Tran, *Convergence of a regularized finite element discretization of the two–dimensional Monge–Ampère equation.* Arxiv e-prints **2112.10711**, `https://arxiv.org/abs/2112.10711`.

Sharp oscillation criteria for first order linear differential equations with variable delays

Ábel Garab
University of Klagenfurt, Austria

This is a joint work with Professor Mihály Pituk (University of Pannonia, Veszprém, Hungary) and Professor Ioannis P. Stavroulakis (University of Ioannina, Ioannina, Greece).

We consider linear differential equations with a variable delay of the form

$$x'(t) + p(t)x(t - \tau(t)) = 0, \qquad t \geq t_0, \tag{1}$$

where $p\colon [t_0, \infty) \to [0, \infty)$ and $\tau\colon [t_0, \infty) \to (0, \infty)$ are continuous functions, such that $t - \tau(t) \to \infty$ (as $t \to \infty$). It is well-known that, for the oscillation of all solutions, it is necessary that

$$B := \limsup_{t\to\infty} A(t) \geq \frac{1}{e} \quad \text{holds, where} \quad A(t) := \int_{t-\tau(t)}^{t} p(s)\,ds.$$

We show that if the function A is slowly varying at infinity, then under mild additional assumptions on p and τ, condition $B > 1/e$ implies that all solutions of (1) are oscillatory.

We also obtain several generalizations of the above result for the equation with multiple delays:

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(t - \tau_i(t)) = 0, \qquad t \geq t_0,$$

and for its discrete-time analogue:

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0.$$

Absence of small solutions and existence of Morse decomposition for a cyclic system of delay differential equations

Ábel Garab
University of Klagenfurt, Austria

We consider the unidirectional cyclic system of nonautonomous delay differential equations

$$\dot{x}^i(t) = g^i(x^i(t), x^{i+1}(t-\tau^i), t), \qquad 0 \le i \le N,$$

where the indexes are taken modulo $N+1$, with $N \in \mathbb{N}_0$, $\tau := \sum_{i=0}^{N} \tau^i > 0$, and for all $0 \le i \le N$, the feedback functions $g^i(u,v,t)$ are continuous in $t \in \mathbb{R}$ and C^1 in $(u,v) \in \mathbb{R}^2$, and each of them satisfies either a positive or a negative feedback condition in the delayed term.

In the theory of infinite dimensional dynamical systems, the question of existence of small solutions (i.e. nonzero solutions that converge to an equilibrium faster than any exponential function) have a crucial role, since only in the absence of such solutions can one describe asymptotically the solutions from the stable manifold by the associated linear equation.

We show that all components of a small solution have infinitely many sign-changes on any interval of length τ. As a corollary we obtain that if a pullback attractor exists, then it does not contain any small solutions. In the autonomous case we also prove that the global attractor possesses a Morse decomposition that is based on an integer valued Lyapunov function. This generalizes former results by Mallet-Paret [1] and Polner [2] in which the scalar ($N = 0$) case was studied.

[1] J. Mallet-Paret, *Morse decompositions for delay-differential equations.* J. Differential Equations **72** (1988), 270–315.

[2] M. Polner, *Morse decomposition for delay-differential equations with positive feedback.*, Nonlinear Anal. **48** (2002), 377–397.

Nonlinear diffusions and wave fronts: some recent results

Maurizio Garrione
Dipartimento di Matematica, Politecnico di Milano, Italy

We present some recent results taken from [1, 2, 3], regarding the existence and the asymptotic shape of monotone traveling fronts in parameter-dependent reaction-diffusion equations with nonlinear diffusion. In particular, we consider diffusive terms of mean curvature type, both in the Euclidean and in the Lorentz-Minkowski space, and different kinds of reactions (monostable, bistable and of combustion type). We discuss the appearance of some new phenomena which are peculiar of the considered diffusions, illustrating them by means of some numerical experiments.

[1] M. Garrione, *Vanishing diffusion limits for planar fronts in bistable models with saturation*, Trans. Amer. Math. Soc. **374** (2021), 3999–4021.

[2] M. Garrione, *Asymptotic study of critical wave fronts for parameter-dependent Born-Infeld models: physically predicted behaviors and new phenomena*, preprint, 2022

[3] M. Garrione and L. Sanchez, *Monotone traveling waves for reaction-diffusion equations involving the curvature operator*, Bound. Value Probl. **2015** (2015), 2015:45, 31 pp.

Numerical computations for a novel description of rate-type inelastic responses in solids

Pablo Alexei Gazca-Orozco
Charles University, Faculty of Mathematics and Physics, Mathematical Institute, Czech Republic

This work focuses on a novel description of classical elastic-perfectly plastic behaviour, exploiting the concept of implicit rate-type constitutive relations. In the one-dimensional setting, originally introduced by [1], the constitutive relation takes the form:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = E\left[1 - H\left(\sigma\frac{\mathrm{d}\epsilon}{\mathrm{d}t}\right) H(|\sigma| - \sigma_y)\right]\frac{\mathrm{d}\epsilon}{\mathrm{d}t}, \tag{1}$$

where σ denotes the stress, ϵ is the relative deformation, σ_y is the yield stress, E denotes the Young modulus and H denotes the Heaviside step function. A generalisation to the higher-dimensional finite deformation setting was carried out in [2], where a complete thermodynamical basis was also established. We discuss the numerical approximation of solutions to these systems using the finite element method. One of the advantages with respect to traditional approaches is that it is not necessary to employ additional concepts such as the plastic strain; the formulation employed also does not require the use of variational inequalities. Moreover, since the system is derived in a manner that is consistent with thermodynamics, we are able to compute the evolution of the temperature field without additional complication.

The concept of implicit rate-type constitutive relations can be used in the description of more complex inelastic phenomena such as the Mullins effect, and our work explores the feasibility of finite element based computations of systems with such constitutive relations.

[1] K.R. Rajagopal, A.R. Srinivasa, *Inelastic response of solids described by implicit constitutive relations with nonlinear small strain elastic response.* Int. J. Plast. 2015, 71, pp. 1-9.

[2] D. Cichra, V. Průša, *A thermodynamic basis for implicit rate-type constitutive relations describing the inelastic response of solids undergoing finite deformation.* Math. Mech. Solids 25(12), pp. 2222-2230.

On impact of disturbance in the deployment problem of multi-agent system

Ewa Girejko
Bialystok University of Technology, Poland

This is a joint work with Katarzyna Topolewicz (Bialystok University of Technology Poland), prof. Sorin Olaru (CentraleSupelec, Univesity Paris-Saclay France) and prof. Carlos E. T. Dórea (Universidade Federal do Rio Grande do Norte, UFRN-CT-DCA, 59078-900 Natal, RN, Brazil).

The class of Multi-Agent System (MAS) covers a generic family of dynamics composed of multiple interacting subsystems called agents. One of the main tasks while dealing with MAS is to design the control strategies for a group of agents in view of covering a known, predetermined target area in order to obtain a static configuration so that the region of coverage is maximized. This problem is known as the *deployment problem* or *coverage problem.* There are different approaches to the problem of deploying agents according the local or global information exchanged and the knowledge of the environment.

The present work is considering the decentralized deployment using a dynamic Voronoi partition. This method is built on the agents' current position and induces a control policy, which is nonlinear due to the agents arrangement. This means that at each time instant, a bounded convex polyhedron, which is the working environment, is partitioned using a Voronoi algorithm. In these schemes, the polytopic target environment is partitioned into a finite collection of polytopic Voronoi cells as there are agents. Moreover, it is necessary to designate internal target points where agents can reach a static configuration. For this purpose, we consider the Chebyshev centers which can be expressed geometrically the corresponding Voronoi cell.

Within this framework recent results show (see [1]) that nominal closed-loop dynamics are stable and converge to a consensus-like equilibrium. Our aim is to go beyond the state of the art and analyse the impact of additive disturbances on the multi-agent behaviour and the overall coverage problem. Such a robustness analysis is particularly useful in practice given the uncertainties available on the sensing and actuation channels that can be modelled in terms of additive uncertainties.

This work considers a one-dimensional case $\mathbb{R}$, i.e. the agents' work environment trivializes to the interval, on the discrete-time framework. The main purpose of the paper is to analyze the behaviour of agents for the system in the presence of disturbances. Since disturbance gains have a significant impact on the behaviour of the agents such as switching of their positions, the main focus is to establish a robust invariant set around the equilibrium of the nominal dynamics for a bounded additive uncertainty. Our consideration was conducted in two ways: as a theoretical analysis and a numerical one based on numerous simulations.

[1] K. Topolewicz, E. Girejko, and S. Olaru. *Decentralized control for de- ployment of multi-agent dynamical systems.* In 23rd International Conference on Process Control (PC), IEEE, (2021), 25–30.

Stochastic models of epidemics

Dan Goreac
University of Shandong, Weihai, China

This is based on joint works with Juan Li (Shandong Univ. Weihai) and Yi Wang (Shandong Univ. Weihai).

We present some recent advances on SIR-like models with random features and state-constraints. The randomness is present either by jump or by mean-field features.

We identify the feasible and the safe zones and give some insight into variational approaches to optimality.

Homogenization of the Navier-Stokes equations in perforated domains in the inviscid limit

Richard Höfer
IMJ-PRG, Université Paris Cité, France

We consider the Navier-Stokes equations with viscosity ε^γ in the whole space $\mathbb{R}^3$ perforated by small holes . The holes are assumed to be identical, with centers on the lattice $(\varepsilon\mathbb{Z})^3$, and of size ε^α. We study the homogenization limit $\varepsilon \to 0$, in different regimes $\gamma > 0, \alpha > 1$.

If the local Reynolds number on the length-scale of the particles is small, one expects that the effective influence of the holes is governed by the asymptotic validity of a linear friction law (Stokes law). This reasoning can be made rigorous. In particular regimes, this leads to an effective equation of Euler-Brinkman type with momentum equation

$$\partial_t u + u \cdot \nabla u + \mathcal{R}u + \nabla p = f,$$

where the effective resistance tensor $\mathcal{R} \in \mathbb{R}^{3\times 3}$ depends on the shape of the holes.

Periodic, permanent, and extinct solutions in population models

Robert Hakl

Institute of Mathematics, Czech Academy of Sciences, Czech Republic

This is a joint work with Dr. José Oyarce (Universidad del Bío-Bío, Concepción, Chile).

The existence of a threshold $\lambda_c > 0$ is proven for some population models, that splits the set of parameters into two parts where the existence, resp. nonexistence, of a positive periodic solution is guaranteed. Moreover, it is shown that in a quite wide class of population models, all the positive solutions are permanent, resp. extinct ones, provided there exists, resp. does not exist, a positive periodic solution. The results are based on a theoretical research dealing with a boundary value problem for functional differential equation with a real parameter

$$u'(t) = \ell(u)(t) + \lambda F(u)(t) \quad \text{for a. e. } t \in [a,b], \qquad h(u) = 0,$$

where ℓ and $F : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ are, respectively, linear and nonlinear operators, $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ is a linear functional, and $\lambda \in \mathbb{R}$ is a real parameter.

Differentiability of solutions with respect to parameters in a class of neutral differential equations with state-dependent delays

Ferenc Hartung
University of Pannonia, Hungary

In this talk we discuss the problem of differentiability of the solutions with respect to parameters in several classes of differential equations with state-dependent delays. After reviewing related earlier results, we consider nonlinear neutral functional differential equations with state-dependent delays of the form

$$\dot{x}(t) = f\Big(t, x_t, x(t - \tau(t, x_t, \xi)), \dot{x}(t - \mu(t)), \dot{x}(t - \rho(t, x_t, \lambda)), \theta\Big), \qquad \text{m.m. } t \geq 0$$

and the associated initial condition

$$x(t) = \phi(t), \qquad t \in [-r, 0].$$

Here x_t denotes the solution segment function defined by $x_t(\zeta) = x(t + \zeta)$ for $\zeta \in [-r, 0]$, where $r > 0$ is a fixed finite constant. $\xi \in \Xi$, $\lambda \in \Lambda$ and $\theta \in \Theta$ represent parameters in the formula of τ, ρ and f, repsectively. The parameter spaces Ξ, Λ and Θ are finite or infinite dimensional normed linear spaces. We show differentiability of the maps $(\phi, \xi, \lambda, \theta) \mapsto x(t)$ and $(\phi, \xi, \lambda, \theta) \mapsto x_t$ for fixed t under certain conditions and using appropriate norm on the domain of the parameters.

[1] F. Hartung, *Differentiability of Solutions with respect to Parameters in a Class of Neutral Differential Equations with State-Dependent Delays.* Electron. J. Qual. Theory Differ. Equ. **2021** No. 56 (2021), 1–41.

Surrogate models for computational fluid dynamics simulations using convolutional autoencoder neural networks and physical constraints

Alexander Heinlein

Delft Institute of Applied Mathematics, Delft University of Technology, The Netherlands

This is a joint work with Viktor Grimm and Axel Klawonn (Department of Mathematics and Computer Science, University of Cologne, Germany).

Simulations of fluid flow are generally very costly because high grid resolutions are not only required to obtain quantitatively accurate results, but too low grid resolutions may also lead to qualitatively incorrect results. In applications, however, one is often not interested in accurate approximations of the complete flow field but only in the qualitative behavior of the flow or in individual quantities (e.g., maximum velocity, pressure drop within a section of a pipe, or wall shear stresses at certain locations).

In this talk, the use of convolutional autoencoder neural networks to construct efficient reduced order surrogate models for high fidelity computational fluid dynamics (CFD) simulations is discussed; cf. [1, 2]. Using this approach, which is inspired by [3], it is possible to build surrogate models for varying geometries. In particular, the geometry is the input of the neural network, and the flow and pressure fields are the output. In order to construct accurate surrogate models, U-Net [4] type convolutional neural networks, which are very successful in image recognition and segmentation tasks, are employed and the architecture and hyper parameters are optimized to this application. As a first step, a fully supervised approach, which requires the availability of simulation results as the training data, is presented. After that, a novel approach is introduced, which does not require CFD simulation results but is based on introducing physical constraints via the loss function.

As a test-bed for the surrogate models, flow around obstacles with varying shape and size within a channel is considered. Moreover, results for the application to geometries of arteries with aneurysms are presented. The results show that the surrogate models provide good predictions of the flow and pressure fields while being computationally much cheaper compared to classical CFD codes.

[1] M. Eichinger, A. Heinlein, and A. Klawonn. *Stationary flow predictions using convolutional neural networks*, In F. J. Vermolen and C. Vuik, editors, Numerical Mathematics and Advanced Applications ENUMATH 2019, pages 541–549, Cham, 2021. Springer International Publishing.

[2] M. Eichinger, A. Heinlein, and A. Klawonn. *Surrogate convolutional neural network models for steady computational fluid dynamics simulations*, Electron. Trans. Numer. Anal., **56** (2022), 235–255.

[3] X. Guo, W. Li, and F. Iorio. *Convolutional neural networks for steady flow approximation*, In Proceedings of the 22nd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, KDD '16, page 481–490, New York, NY, USA, 2016. Association for Computing Machinery.

[4] O. Ronneberger, P. Fischer, and T. Brox. U-net: *Convolutional networks for biomedical image segmentation*, In N. Navab, J. Hornegger, W. M. Wells, and A. F. Frangi, editors, Medical Image Computing and Computer-Assisted Intervention – MICCAI 2015, pages 234–241, Cham, 2015. Springer International Publishing.

Multiplicity and concentration results for some nonlinear Schrödinger equations with the fractional p-Laplacian

Teresa Isernia
Università Politecnica delle Marche, Ancona, Italy

This is a joint work with Prof. V. Ambrosio (Università Politecnica delle Marche, Ancona (Italy)) and Prof. G.M. Figueiredo (Universidade de Brasilia, Brasilia (Brazil)).

We consider a class of parametric Schrödinger equations driven by the fractional p-Laplacian operator and involving continuous positive potentials and nonlinearities with subcritical or critical growth. By using variational methods and Ljusternik-Schnirelmann theory, we study the existence, multiplicity and concentration of positive solutions for small values of the parameter.

Global asymptotic stability and periodic solutions in scalar differential delay equations

Anatoli Ivanov
Department of Mathematics, Pennsylvania State University, USA

Simple form scalar differential delay equations are considered. The first equation was recently proposed as a mathematical model of the megakaryopoiesis [2]:

$$x'(t) = -\mu x(t) + f(x(t))g(x(t-\tau)). \tag{1}$$

The second equation serves as a model of dynamics in certain economics markets [1]:

$$x'(t) = G(x(t-\tau)) - F(x(t)). \tag{2}$$

Sufficient condition for the global asymptotic stability of the unique equilibrium in both equations are derived. The stability criteria are given in terms of induced interval maps, one set being delay independent conditions and another one involving the delay [5]. The existence of periodic solutions slowly oscillating about the equilibrium is also established. The periodic oscillations in both models always exist when the positive equilibrium is linearly unstable. The proof of existence uses the established ejective fixed point techniques [3, 4] with necessary modifications due to the specific forms of the equations.

Part of this talk on periodic solutions represents a joint work with Prof. Dr. Bernhard Lani-Wayda of Giessen University, Germany [6].

[1] J. Bélair and M.C. Mackey, *Consumer memory and price fluctuations in commodity markets: an integrodifferential model.* J. Dynam. Differential Equations **1** (1989), 299–325.

[2] L. Boullu, M. Adimy, F. Crauste, and L. Pujo-Menjouet, *Oscillation and asymptotic convergence for a delay differential equation modeling platelet production.* Discrete and Continuous Dynamical Systems, Ser. B **24**, no. 6 (2019), 2417–2442.

[3] O. Diekmann, S. van Gils, S. Verdyn Lunel, and H.O. Walther, Delay Equations: Complex, Functional, and Nonlinear Analysis. Springer-Verlag, New York, 1995.

[4] J.K. Hale and S.M. Verduyn Lunel, Introduction to Functional Differential Equations. Springer Applied Mathematical Sciences, vol. 99, 1993.

[5] A.F. Ivanov, *Global asymptotic stability in a differential delay equation modeling megakaryopoiesis.* Functional Differential Equations **28**, no. 3-4 (2021), 103–116.

[6] A. Ivanov and B. Lani-Wayda, *Periodic solutions in a differential delay equation modelling megakaryopoiesis.* Proceedings of ISAAC-2021, Springer-Verlag, 10 pp. (submitted)

Enhancing a multigrid solver for the Navier-Stokes equations with deep learning

Robert Jendersie
Otto-von-Guericke-University, Magdeburg, Germany

This is joint work with Nils Margenberg (Helmut Schmidt University, Hamburg, Germany), Christian Lessig (Otto-von-Guericke University, Magdeburg, Germany) and Thomas Richter (Otto-von-Guericke-University, Magdeburg, Germany).

Our work further develops the Deep Neural Network Multigrid Solver (DNN-MG) introduced in [1] for the incompressible Navier-Stokes equations by improving the utilized neural networks. DNN-MG is a novel method for machine learning enhanced simulations, which improves computational efficiency by combining a geometric multigrid solver and a neural network. The multigrid solver is used in DNN-MG for the coarse levels while the neural network corrects interpolated solutions on fine levels, avoiding the high computational costs there. DNN-MG thereby corrects small patches of the mesh domain using the neural network. This local approach greatly facilitates generalizability and allows us to use a network trained on one mesh domain also on different ones. The locality also results in a compact neural network with a small number of parameters which facilitates fast training with limited example simulations and allows for fast evaluation of the network. The efficiency and generalizability of DNN-MG was demonstrated in [1] for variations of the 2D laminar flow around an obstacle. DNN-MG improved the solutions as well as lift and drag functionals while requiring less than half of the computation time of a full multigrid solution [2].

In this talk, we show that through the use of different neural network architectures and regularization techniques, one can greatly improve the consistency of DNN-MG. Furthermore, by combining multiple neural networks to an ensemble we gain insights into the confidence of the networks' predictions on different flow scenarios. This can be used to further increase the robustness.

Acknowlegdement. Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 422037413 – TRR 287.

[1] N. Margenberg, D. Hartmann, C. Lessig, T. Richter. A neural network multigrid solver for the Navier-Stokes equations, J. Comput. Phys. (2022): 110983

[2] N. Margenberg, R. Jendersie, C. Lessig, T. Richter. Deep neural networks for geometric multigrid methods. arXiv preprint arXiv:2106.07687, (2021).

Discretising boundary conditions of fully nonlinear equations

Max Jensen
University of Sussex, UK

Boundary conditions of fully nonlinear boundary value problems are more diverse compared to the standard PDE theory of semi-linear equations. For example, Dirichlet boundary conditions can be posed in multiple non-equivalent forms: they model different physical/real-world effects and the well-posedness of the BVP may rely on their particular form. Boundary conditions of Neumann- and Robin-type are frequently fully-nonlinear themselves and interact with the PDE operator in intricate ways.

In this talk, we review the taxonomy of common boundary conditions, highlight common pitfalls and misconceptions, and discuss recent methodologies to discretise them. Particular emphasis will be paid to Hamilton-Jacobi-Bellman and Isaacs equations and their approximation by finite element methods.

This is a joint work with Dr Jaroszkowski (University of Sussex) [1, 2] and Dr Smears (University College London) [3].

[1] Jaroszkowski, Bartosz and Jensen, Max (2022) *Finite element methods for isotropic Isaacs equations with viscosity and strong Dirichlet boundary conditions.* Applied Mathematics and Optimization, 85. 1-32. https://doi.org/10.1007/s00245-022-09860-5

[2] Jaroszkowski, Bartosz and Jensen, Max (2021) *Finite Element Approximation of Hamilton-Jacobi-Bellman equations with nonlinear mixed boundary conditions*, https://arxiv.org/abs/2105.09585

[3] Jensen, Max and Smears, Iain (2018) *On the notion of boundary conditions in comparison principles for viscosity solutions.* Radon Series on Computational and Applied Mathematics. http://dx.doi.org/10.1515/9783110543599-007

Ordinary differential equations derived from cluster integrals

Hiroshi Kajimoto
Professor Emeritus at Nagasaki University, Japan

In his study of virial coefficients of an imperfect gas whose molecules interract each others by the square well potential, Katsura [3] and others [1] calculated several cluster integrals of the type:

$$W^{\lambda}_{\mu_1,\mu_2,\mu_3,\mu_4}(a_1,a_2,a_3,a_4) := \int_0^{\infty} J_{\mu_1}(a_1t)J_{\mu_2}(a_2t)J_{\mu_3}(a_3t)J_{\mu_4}(a_4t)t^{-\lambda}dt \tag{1}$$

where $J_\mu(t)$ are Bessel functions.

In this talk we give a way of making ordinary differential equations for the integrals:

$$W^{\lambda}_{\mu,\ \nu^{n-1}}(\rho, 1^{n-1}) := \int_0^{\infty} J_\mu(\rho t)J_\nu(t)^{n-1}t^{-\lambda}dt \tag{2}$$

which come from higher order virial coefficients for the hard sphere potential. Here n is the number of molecules and the cluster is of a line graph A_n. We take a complex integral which is a constant multiple of the integral (2):

$$W_n(\rho) := \int_C J_\mu(\rho t)J_\nu(t)^{n-1}t^{-\lambda}dt = \int_C \Phi(\rho)\varphi_n \tag{3}$$

where C is a proper contour in the complex t-plane, we put $\Phi = \Phi(\rho) = J_\mu(\rho t)$ and $\varphi_n = J_\nu(t)^{n-1}t^{-\lambda}dt$. Let $\theta = \rho\dfrac{d}{d\rho}$ and $\Theta = \Phi^{-1}\theta\circ\Phi$. Bessel ODE is then $(\theta_t^2 + t^2 - \nu^2)J_\nu(t) = 0$. Then we know

$$\theta W_n(\rho) = \int_C \theta(\Phi\varphi_n) = \int_C \Phi\Theta\varphi_n \qquad \therefore \qquad L(\rho,\theta)W_n(\rho) = \int_C \Phi L(\rho,\Theta)\varphi_n \tag{4}$$

where Θ and $L(\rho,\Theta)$ are operators acting on differential 1-forms. We sought the operator $L(\rho,\Theta)$ that vanishes φ_n, and got the following recurrence formulae. Let $L = \Theta - \lambda + 1$ and $K = \nu^2 + \rho^{-2}(\Theta^2 - \mu^2)$. Fix $n \geq 1$, for $1 \leq k \leq n$, put

$$L^{(k,n)} = LL^{(k-1,n)} - (k-1)(n-k+1)KL^{(k-2,n)}, \qquad L^{(0,0)} = I, \quad L^{(-1,0)} = 0. \tag{5}$$

Then we have

$$L^{(k,n)}\varphi_n \equiv (-1)^k(n-1)(n-2)\cdots(n-k)\left(\frac{\theta J_\nu}{J_\nu}\right)^k \varphi_n \mod B^1 \tag{6}$$

where B^1 is the linear space of 1-forms η such that $\int_C \Phi\eta = 0$. So we know $L^{(n,n)}\varphi_n \equiv 0$. Hence by (4) we get $L^{(n,n)}(\rho,\theta)W_n(\rho) = \int_C \Phi L^{(n,n)}\varphi_n = 0$.

For the hard sphere potential:$(\lambda,\mu,\nu) = \left(\frac{3}{2}(n-2), \frac{1}{2}, \frac{3}{2}\right)$, we list $L^{(n)} := L^{(n,n)}(\rho,\theta)$ ($n = 1,2,3$): $L^{(1)} = \theta + \frac{5}{2}$, $\rho^2L^{(2)} = (\theta+\frac{1}{2})(\rho^2-1)(\theta-\frac{1}{2})$, $\rho^2L^{(3)} = (\theta+\frac{1}{2})((\rho^2-4)\theta - \frac{1}{2}(7\rho^2-12))(\theta-\frac{1}{2})$.

[1] D. Ruelle, Statistical Mechanics. Benjamin, 1969.

[2] H. Kajimoto, *On several virial coefficients for the hard sphere potential.* International J. of Pure & Engg. Math. **vol. 2** no. III (2014), 109–119.

[3] S. Katsura, *Fourh virial coefficients for the square well potential.* Physical Review. **115** (1959), 1418–1426.

Decreasing solutions of cyclic systems of second-order Emden-Fowler type difference equations

Aleksandra Kapešić
University of Niš, Faculty of Science and Mathematics, Niš, Serbia

This is a joint work with Prof Jelena Manojlović (University of Niš, Faculty of Science and Mathematics, Niš, Serbia).

The cyclic system of second-order difference equations

$$\Delta(p_i(n)|\Delta x_i(n)|^{\alpha_i-1}\Delta x_i(n)) = q_i(n)|x_{i+1}(n+1)|^{\beta_i-1}x_{i+1}(n+1),$$

for $i = \overline{1,N}$ where $x_{N+1} = x_1$, is analyzed. Under the assumption that α_i and β_i are positive constants such that $\alpha_1\alpha_2\cdots\alpha_N > \beta_1\beta_2\cdots\beta_N$ and p_i and q_i are positive sequences, it is shown that the situation in which this system possesses primitive decreasing solutions can be completely characterized. In addition, if coefficients p_i and q_i are regularly varying sequences, necessary and sufficient conditions for the existence of strongly decreasing solutions are obtained. Besides, precise information can be acquired about the asymptotic behavior at infinity of these solutions.

Global weak and renormalized solutions to a class of energy-reaction-diffusion systems

Michael Kniely
WIAS Berlin, Germany

The existence of global solutions to reaction-diffusion equations has been widely studied in the literature assuming an isothermal setting. The more realistic situation of temperature-dependent models is, however, much less understood. In this talk, we present a global existence result for thermodynamically consistent reaction-diffusion systems coupled to an equation for the internal energy describing heat transfer. Using time discretization, suitable entropy estimates, and several regularization procedures, we construct global weak and renormalized solutions in the presence (respectively absence) of growth restrictions on the reactions.

Thermodynamic consistency ensures that fundamental laws of physics are satisfied, such as the production of entropy and various conservation laws. A characteristic feature of non-isothermal models is the presence of cross-diffusion-type phenomena like the Soret and the Dufour effect. The former one causes a concentration flux due to a nonvanishing temperature gradient, while the latter induces a heat flux due to a nonvanishing concentration gradient. This is a joint work with Julian Fischer, Katharina Hopf, and Alexander Mielke.

On solution of initial boundary value problems in hypoplasticity

Victor A. Kovtunenko
Karl-Franzens University of Graz, Graz, Austria, Siberian Branch of the Russian Academy of Sciences, Novosibirsk, Russia

The hypoplasticity is motivated by engineering application to granular materials and soils in geomechanics. The reference hypoplastic constitutive equations of Kolymbas type are fully nonlinear and rate-independent, they are a practice example of implicit theories as suggested by Truesdell and Rajagopal. Mathematically, our study is faced to ill-posed dynamic PDE problems. Moreover, for a cohesionless granular material the constraint of non-positive principal stresses preserving compression of grains should be satisfied within the solution. The well-posedness analysis results known in the literature are restricted for hypoplastic models simplified either to a semi-discrete nonlinear Cauchy problem following Chambon, or to quasi-static nonlinear rate problems. For the particular models of Bauer, von Wolffersdorff, Toll, and others, we aim at the dynamic behavior of the nonlinear ODE systems under proportional and cyclic loading, which leads to hysteresis and ratcheting phenomena, especially important for applications.

The author thanks for support the OeAD Scientific and Technological Cooperation (WTZ 18/2020) financed by the Austrian Federal Ministry of Science, Research and Economy (BMWFW) and by the Czech Ministry of Education, Youth and Sports (MSMT).

[1] E. Bauer, V.A. Kovtunenko, P. Krejčí, N. Krenn, L. Siváková, A.V. Zubkova, *On proportional deformation paths in hypoplasticity.* Acta Mechanica **231** (2020), 1603–1619.

[2] V.A. Kovtunenko, E. Bauer, J. Eliaš, P. Krejčí, G.A. Monteiro, L. Straková (Siváková), *Cyclic behavior of simple models in hypoplasticity and plasticity with nonlinear kinematic hardening.* J Sib Fed Univ - Math Phys. **14** (2021), 1–12.

[3] V.A. Kovtunenko, P. Krejčí, E. Bauer, L. Siváková, A.V. Zubkova, *On Lyapunov stability in hypoplasticity.* In: Proceedings Equadiff 2017 Conference, K. Mikula, D. Ševčovič, J. Urbán J (eds.), 107–116. Slovak University of Technology, Bratislava, 2017.

[4] V.A. Kovtunenko, P. Krejčí, N. Krenn, E. Bauer, L. Siváková, A.V. Zubkova, *On feasibility of rate-independent stress paths under proportional deformations within hypoplastic constitutive model for granular materials.* Mathematical Models in Engineering **5** (2019), 119–126.

Periodic and connecting orbits for Mackey–Glass type equations

Tibor Krisztin
University of Szeged, Hungary

This is a joint work with Ferenc Bartha, Gábor Benedek, Ngoc Bach Pham Le, Mónika Polner, Alexandra Vigh (University of Szeged, Hungary).

In this talk we consider Mackey–Glass type equations of the form

$$y'(t) = -ay(t) + bf_{k,n}(y(t-1)) \tag{1}$$

with real parameters $b > a > 0$, $k \geq 1$, $n \in \mathbb{N}$, and nonlinearity

$$f_{k,n}(\xi) = \frac{\xi^k}{1+\xi^n} \qquad (\xi \geq 0).$$

The aim is to present some results indicating the richness of the dynamics generated by the solutions of equation (1).

A limiting version, as $n \to \infty$, of (1) is the equation with discontinuous right hand side

$$x'(t) = -ax(t) + bf_k(x(t-1)) \tag{2}$$

where $f_k(\xi) = \xi^k$ for $\xi \in [0,1]$, and $f_k(\xi) = 0$ for $\xi > 1$.

First, for some parameter values a, b, k, we construct stable periodic orbits, connecting orbits between periodic orbits ($k > 1$), and homoclinic orbits ($k > 1$) of the limiting equation (2). In the next step it is shown that the same type of orbits exist for equation (1), as well, provided n is sufficiently large.

Traveling waves close to the Couette flow

Daniel Lear
Charles University, Czech Republic

In this talk we shall study the existence of smooth traveling waves close to the Couette flow for the 2D incompressible Euler equation for an ideal fluid. It is well known that this kind of solution does not exist arbitrarily close to the Couette flow if the distance is measured in H^s with $s > 3/2$, at the level of the vorticity. In this presentation we will deal with the case $s < 3/2$.

Unidirectional flocks in Collective Dynamics

Daniel Lear
Charles University, Czech Republic

In this talk we will focus on the so-called Cucker-Smale model, which encode one of the simplest communication protocols that lead to emergence of two fundamental phenomena of collective action: alignment and flocking. Such systems arise in a variety of applications including biological, social and technological contexts. Kinetic and hydrodynamic models will be presented and the problems of global well-possendess, long time behavior, and stability of flocks on the macroscopic level will be addressed. Finally, we discuss unidirectional flocks, which have been recently introduced and used to obtain some results in higher dimensions.

A note on asymptotically exact a posteriori error estimates for mixed Laplace eigenvalue problems

Philip Lederer
Aalto University, Finland and TU Wien, Austria

We derive optimal and asymptotically exact a posteriori error estimates for the approximation of the Laplace eigenvalue problem. To do so, we combine two results from the literature. First, we use the hypercircle techniques developed for mixed eigenvalue approximations with Raviart-Thomas Finite elements, [1]. In addition, we use the post-processings introduced for the eigenvalue and eigenfunction based on mixed approximations with the Brezzi-Douglas-Marini Finite element, [2]. To combine these approaches, we define a novel additional local post-processing for the fluxes that appropriately modifies the divergence. Consequently, the new flux can be used to derive upper bounds and still shows good approximation properties.

[1] F. Bertrand, D. Boffi, R. Stenberg, *Asymptotically exact a posteriori error analysis for the mixed Laplace eigenvalue problem* Comput. Methods Appl. Math. **20** (2020), 215–225.

[2] R. Stenberg, *Postprocessing schemes for some mixed finite elements* RAIRO. **25** (1991), 151–167.

Overconfidence in numerical Bayesian inference

Han Cheng Lie
Universität Potsdam, Germany

This is joint work with Martin Stahn (Freie Universität Berlin, Berlin, Germany), T. J. Sullivan (University of Warwick, Coventry, United Kingdom), and Aretha Teckentrup (University of Edinburgh, Edinburgh, United Kingdom).

Many physics-based models of natural phenomena are expressed in terms of differential equations. These models are often subject to uncertainty because they depend on parameters that are not known precisely, such as initial conditions, boundary conditions, or coefficients in the differential equation. To make predictions about the natural phenomenon of interest, one must first perform inference for the unknown parameters using noisy, incomplete observations of the solution to the differential equation. One then applies the inferred parameter values to the model.

Bayesian approaches to solving such inference problems involve encoding all available knowledge about the unknown parameter into the 'prior' probability measure μ_{pri}, and then assimilating the information in a new observation y into the 'posterior' probability measure μ^y_{pos} by using Bayes' formula

$$\mu^y_{\mathrm{pos}}(\mathrm{d}x) = f(x;y)\mu_{\mathrm{pri}}(\mathrm{d}x),$$

where $f(x;y)$ is the likelihood of the observation y given that the unknown parameter has the value x.

In general, one must draw samples from the posterior in order to quantitatively describe the uncertainty in the model. For differential equation-based models, drawing a single sample from the posterior requires computing the solution of the differential equation. In practice, the differential equation can be solved only numerically. If the numerical solution contains an approximation error, then the resulting approximate posterior can be overconfident, in the sense that the bias between the posterior mean and the true parameter is large relative to the variance.

In this talk, we shall present an analytical example that illustrates how overconfidence can arise due to the approximation error of a numerical solver, and how randomisation of the numerical solver can help to mitigate the problem of overconfidence. We shall also present theoretical results from from [1] concerning error bounds for randomised approximate numerical solvers for initial value problems defined on Gelfand triples. In addition, we shall describe how the latter error bounds can be combined with the error bounds from [2] for posterior measures corresponding to randomised approximate likelihoods.

[1] H. C. Lie, M. Stahn, T. J. Sullivan, *Randomised one-step time integration methods for deterministic operator differential equations.* Calcolo **59** (2022), No. 13.

[2] H. C. Lie, T. J. Sullivan, A. L. Teckentrup, *Random forward models and log-likelihoods in Bayesian inverse problems.* SIAM/ASA J. Uncertain. Quantif. **6** (2018), No. 4, 1600-1629

On rate-type viscoelastic fluids with stress diffusion and their large-data analysis

Josef Málek
Charles University, Faculty of Mathematics and Physics, Prague, Czech Republic

This presentation is based on joint works with Michal Bathory (University of Vienna, Austria), Miroslav Bulíček (Charles University, Prague) and Casey Rodrigues (University of North Carolina, USA).

We present the result concerning the large-data and long-time existence of a weak solution to an initial- and boundary-value problem associated with a PDE system governing unsteady flows of a robust class of rate-type viscoelastic fluid with stress diffusion in two and three dimensions. The fluid is described by the incompressible Navier-Stokes equations for the velocity $\mathbf{v}$, coupled with a diffusive variant of a combination of the Oldroyd-B and the Giesekus models for a tensor $\mathbb{B}$. By a proper choice of the constitutive relations for the Helmholtz free energy (which, however, is non-standard in the current literature, despite the fact that this choice is well motivated from the point of view of physics) and for the energy dissipation, we are able to proves that $\mathbb{B}$ enjoys the same regularity as $\mathbf{v}$ in the classical three-dimensional Navier-Stokes equations. This enables us to handle any kind of objective derivative of $\mathbb{B}$, thus obtaining existence results for the class of diffusive Johnson-Segalman models as well. Moreover, using a suitable approximation scheme, we are able to show that $\mathbb{B}$ remains positive definite if the initial datum was a positive definite matrix (in a pointwise sense). This talk is based on the results published in [1] and [2].

[1] Bathory, Michal and Bulíček, Miroslav and Málek, Josef *Large data existence theory for three-dimensional unsteady flows of rate-type viscoelastic fluids with stress diffusion.* Adv. Nonlinear Anal. **10** (2021) 501–521.

[2] Bulíček, Miroslav and Málek, Josef and Rodriguez, Casey *Global well-posedness for two-dimensional flows of viscoelastic rate-type fluids with stress diffusion.* J. Math. Fluid Mech. **24** (2022) Paper No. 61.

Diffusion-convection reaction equations with sign-changing diffusivity

Luisa Malaguti
University of Modena and Reggio Emilia, Italy

This is a joint work with Diego Berti (University of Florence, Florence, Italy) and Andrea Corli (University of Ferrara, Ferrara, Italy)

We consider a diffusion-convection reaction equation

$$\rho_t + f(\rho)_x = \left(D(\rho)\rho_x\right)_x + g(\rho), \qquad t \geq 0,\, x \in \mathbb{R}. \tag{1}$$

in one space variable. The diffusivity D changes sign once or even more than once; then, we deal with a forward-backward parabolic equation. We investigate the existence of globally defined traveling wave solutions (i.e. wavefront solutions) $\rho(t,x) = \varphi(x - ct)$, which connect two equilibria and cross both regions where the diffusivity is positive and regions where it is negative. The real constant c is the wave speed and the continuous function φ is the wave profile. We show the appearance of sharp behaviors at the points where the diffusivity degenerates. The presence of the convection f reveals several new features of wavefront solutions especially when the reaction term g is bistable. In this case, indeed, profiles can occur either for a single value of the speed or for a bounded interval of such values, uniqueness (up to shifts) is lost and plateaus of arbitrary length can appear. The results are then applied for studying the dynamics of a population formed by isolated and grouped individuals. We assume that the movement occurs in a random way, in one spatial dimension and it is biased. By passing to the limit, we obtain a forward-backword-forward parabolic equation as (1) which includes, in particular, a convective term and we are able to perform the study of its wavefront solutions.

[1] D. Berti, A. Corli, L. Malaguti, *Wavefronts for degenerate diffusion-convection reaction equations with sign-changing diffusivity.* Discrete Contin. Dyn. Syst., **41** (2021)6023–6046.

[2] D. Berti, A. Corli, L. Malaguti, *Diffusion-convection reaction equations with sign-changing diffusivity and bistable reaction term.* Nonlinear Anal. Real World Appl. **67** (2022), Paper No. 103579, 29 pp.

[3] D. Berti, A. Corli, L. Malaguti, *Wavefronts in forward-backward parabolic equations and applications to biased movements.* Proceedings of the Conference ISAAC 2021, to appear.

Stability of multi-agents systems under DoS attacks

Agnieszka B. Malinowska
Faculty of Computer Science, Bialystok University of Technology, Poland

This is a joint work with: R. Almeida (Center for Research and Development in Mathematics ad Applications, Department of Mathematics, University of Aveiro, Portugal), E. Girejko (Faculty of Computer Science, Bialystok University of Technology, Poland), L. Machado (Institute of Systems and Robotics – University of Coimbra and Department of Mathematics – University of Trás-os-Montes e Alto Douro (UTAD), Portugal) and N. Martins (Center for Research and Development in Mathematics ad Applications, Department of Mathematics, University of Aveiro, Portugal).

We investigate multi-agent systems under Denial-of-Service (DoS) attacks. In the presence of DoS attacks, information can neither be received nor sent between the attacked nodes, and adversaries can attack partial or all channels at any time. Therefore, in multi-agent systems, these attacks may lead to a loss of stability. Our goal is to provide sufficient conditions for the stability of multi-agent systems under DoS attacks. Two types of systems are considered:

- systems with memory
- systems defined on hybrid time domains

A.B. Malinowska was supported by the Bialystok University of Technology Grant (WZ/WI-IIT/1/2020), financed from a subsidy provided by the Minister of Education and Science.

Strongly monotone solutions of systems of nonlinear differential equations with rapidly varying cofficients

Jelena Manojlović
University of Niš, Faculty of Science and Mathematics, Serbia

This is a joint work with Jelena Milošević (University of Niš, Faculty of Science and Mathematics, Serbia).

The two-dimensional systems of first order nonlinear differential equations

$$(S1)\ :\ \begin{cases} x' = p(t)y^{\alpha}, \\ y' = q(t)x^{\beta}, \end{cases} \qquad (S2)\ :\ \begin{cases} x' + p(t)y^{\alpha} = 0, \\ y' + q(t)x^{\beta} = 0, \end{cases} \qquad \alpha, \beta > 0$$

are analyzed using the theory of rapid variation. We show that all strongly increasing solutions of the system $(S1)$ as well as all strongly decreasing solutions of the system $(S2)$ are rapidly varying under the assumption that p and q are rapidly varying functions. Next, we introduce some asymptotic equivalence relations in the set of rapidly varying functions and determine the asymptotic behavior of strongly monotone solutions of these two systems in terms of these relations.

On the role of arbitrary pollution effects on the stability of swirling free-surface flows

Laurent Martin Witkowski
Sorbonne Université, Faculté des Sciences et Ingénierie, Paris, France
& LISN-CNRS, Université Paris-Saclay, Orsay, France

This is a joint work with Antoine FAUGARET, Yohann DUGUET and Yann FRAIGNEAU (LISN-CNRS, Université Paris-Saclay, Orsay, France).

The boundary condition at the air-water interface plays a major role in the stability of a rotating flow with a free surface. We consider here a generic configuration to investigate such effects both experimentally and numerically. For the flow driven by a rotating disc in a fixed cylindrical vessel partially filled with water, the standard free-slip condition in numerical simulations does not predict the instability threshold found experimentally. The unavoidable surface contamination changes the stresses at the interface and has a strong impact on the velocity field, at least when H, the fluid height, is small compared to R, the disc radius. The possible effect of unidentified pollutants at the interface can be modelled using an advection-diffusion equation and a closure equation linking the surface tension to their concentration. This modelling has been proposed in [1]

A even simpler numerical model without superficial transport of the surfactants, adaptable into any code for single-phase flows has been proposed in [2]. The model does not possess any free parameter and is independent on the closure model for surfactants. For the stationary axisymmetric base flow, the radial velocity at the interface is set to zero whereas the usual stress-free boundary conditions are retained for the perturbations. For a geometrical aspect ratio $G = H/R$ equal to $1/4$, known to display ambiguous behaviour regarding stability thresholds, the modal selection as well as a nonlinear stability island found in the experiments are well reproduced by the model, both qualitatively and quantitatively. We present experimental and numerical results in a systematic study for G ranging from 0.1 to 2. By varying the aspect ratio, we describe the modal selection of the primary instability. We show that the simple model is robust over the entire range studied.

[1] A. Faugaret, Y. Duguet, Y. Fraigneau, and L. Martin Witkowski *Influence of interface pollution on the linear stability of a rotating fluid* J. Fluid Mech., **900** (2020), A42.

[2] A. Faugaret, Y. Duguet, Y. Fraigneau, and L. Martin Witkowski *A simple model for rotating free-surface flow* J. Fluid Mech., **935** (2022), A2.

Stability switches in linear delay differential equations

Hideaki Matsunaga
Department of Mathematics, Osaka Metropolitan University, Japan

We are concerned with stability properties of linear delay differential equations with delayed and non-delayed terms. It is well known that for the autonomous case, the zero solution being asymptotically stable is equivalent to all solutions having limit zero as $t \to \infty$ which in turn is true if and only if all roots of an associated characteristic equation have negative real parts. Clarifying the dependence of some parameters on asymptotic stability is important but not so easy for differential equations with multiple delayed terms. In this case, the zero solution may change finite times from stability to instability to stability as the delay parameter increases monotonously. Such phenomena for increasing delay are often referred to as *stability switches*; see, e.g., [1, 2, 3, 4, 5, 6].

In this talk we summarize some recent results on stability switches in linear delay differential equations.

[1] J. Čermák, T. Kisela, *Delay-dependent stability switches in fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul. **79** (2019), 104888, 19 pp.

[2] K. L. Cooke, Z. Grossman, *Discrete delay, distributed delay and stability switches*, J. Math. Anal. Appl., **86** (1982), 592–627.

[3] H. I. Freedman, Y. Kuang, *Stability switches in linear scalar neutral delay equations*, Funkcial. Ekvac., **34** (1991), 187–209.

[4] Y. Hata, H. Matsunaga, *Stability switches in a linear differential equation with two delays*, submitted.

[5] H. Matsunaga, *Stability switches in a system of linear differential equations with diagonal delay*, Appl. Math. Comput. **212** (2009), 145–152.

[6] X. Yan, J. Shi, *Stability switches in a logistic population model with mixed instantaneous and delayed density dependence*, J. Dynam. Differential Equations, **29** (2017), 113–130.

Existence and multiplicity of decaying solutions to BVP on the half-line for nonlinear equations with p-Laplacian

Serena Matucci
Department of Mathematics and Computer Science, University of Florence, Italy

Based on joint works with Prof Zuzana Došlá (Masaryk University of Brno, CZ), and, partially, with Prof Gennaro Infante (Università della Calabria, Cosenza, Italy) and Prof Mauro Marini (University of Florence, Italy).

Some recent results are presented about existence of positive decaying radial solutions, defined outside a fixed ball, for nonlinear equations with p-Laplacian operator, both for the super-homogeneous and for the sub-homogeneous case, in presence of a changing sign nonlinearity describing diffusion/absorption phenomena. The problem reduces to a BVP problem for a nonlinear second order ODE on a noncompact interval, with initial and asymptotic conditions. A fixed point approach in some Frechét space is used to solve this problem. A key role is played by the properties of solutions of some auxiliary half-linear equations, by the notion of disconjugacy and of principal solution, and by some comparison results. Sufficient conditions are proved for the existence of solutions both in case of negative initial derivative and in case of zero initial derivative, see [1] and [2].

We also show how these results can be used to study existence and multeplicity of positive solutions on the half-line for second order ordinary differential equations, possibly subject to functional boundary conditions acting on a compact interval, by means of a combination with a fixed point approach for operators on compact intervals [3], [4].

[1] Z. Došlá and S. Matucci, *Ground state solutions to nonlinear equations with p-Laplacian.* Nonlinear Anal. **184** (2019), 1–16.

[2] Z. Došlá, M. Marini and S. Matucci, *Global Kneser solutions to nonlinear equations with indefinite weight.* DCDS-B **23** (2018), 3297–3308.

[3] Z. Došlá, M. Marini and S. Matucci, *A Dirichlet problem on the half-line for nonlinear equations with indefinite weight.* Ann. Mat. Pura Appl. **196** (2017), no. 1, 51—64.

[4] G. Infante and S. Matucci, *Positive solutions of BVPs on the half-line involving functional BCs.* AIMS Math. **6** (2021), no. 5, 4860–4872.

On impulsive functional coupled systems of differential equations

Feliz Minhós
Department of Mathematics, School of Science and Technology, Research Center in Mathematics and Applications (CIMA), University of Évora, Portugal

This is a joint work with Dr. Rui Carapinha (Research Center in Mathematics and Applications (CIMA),University of Évora, Portugal).

In this paper we consider a first-order coupled impulsive system of equations with functional boundary conditions, subject to the generalized impulsive effects. It is pointed out that this problem generalizes the classical boundary assumptions, allowing two-point or multipoint conditions, nonlocal and integro-differential ones or global arguments, as maxima or minima, among others. Our method is based on lower and upper solutions technique together with the fixed point theory.

The main theorem is applied to a SIRS model where, to the best of our knowledge, for the first time it includes impulsive effects combined with global, local, and the asymptotic behavior of the unknown functions.

[1] F. Minhós, R. Carapinha, *Functional coupled systems with generalized impulsive conditions with application to a SIRS model.* J. Funct. Spaces. **Article ID 3758274** (2021), 10 pages.

A-posteriori-steered h- and p-robust multigrid solvers

Ani Miraçi
TU Wien, Austria

This is a joint work with Jan Papež (Institute of Mathematics, Czech Academy of Sciences, Czech Republic), Dirk Praetorius (TU Wien, Austria), Julian Streitberger (TU Wien, Austria), and Martin Vohralík (Inria Paris, France).

We study a symmetric second-order linear elliptic PDE discretized by piecewise polynomials of arbitrary degree $p \geq 1$. To treat the arising linear system, we propose a geometric multigrid method with zero pre- and one post-smoothing by an overlapping Schwarz (block Jacobi) method [1]. Introducing optimal step sizes which minimize the algebraic error in the level-wise error correction step of multigrid, one obtains an explicit Pythagorean formula for the algebraic error. Importantly, this inherently induces a fully computable a posteriori estimator for the energy norm of the algebraic error. We show the two following results and their equivalence: 1) the solver contracts the algebraic error independently of the polynomial degree p; 2) the estimator represents a two-sided p-robust bound on the algebraic error. The p-robustness results are obtained by carefully applying the results of [2] for one mesh, combined with a multilevel stable decomposition for piecewise affine polynomials of [3]. Moreover, recent developments in [4] allow to prove that a local variant of the solver is robust also with respect to the number of mesh levels used for rate-optimal adaptive finite element method. Finally, we present a variety of numerical tests to confirm the theoretical results and to illustrate the advantages of our approach.

[1] A. Miraçi, J. Papež, and M. Vohralík: A-posteriori-steered p-robust multigrid with optimal step-sizes and adaptive number of smoothing steps. SIAM J. Sci. Comput. 43, 5 (2021), S117–S145.

[2] J. Schöberl, J. M. Melenk, C. Pechstein, and S. Zaglmayr: Additive Schwarz preconditioning for p-version triangular and tetrahedral finite elements. IMA J. Numer. Anal., 28 (2008), pp. 1–24.

[3] L. Chen, R. H. Nochetto, and J. Xu: Optimal multilevel methods for graded bisection grids. Numer. Math., 120(1):1–34, 2012

[4] A. Miraçi, D. Praetorius, S. Schimanko, and J. Streitberger: Optimal local p-robust multigrid for FEM on graded bisection grids. In preparation.

Numerical simulations of fractional consensus approach in attitude dynamics modeling

Dorota Mozyrska
Bialystok University of Technology, Białystok, Poland

This is a joint work with Professors: **Jerzy Baranowski** (Department of Automatic Control and Robotics, AGH University of Science & Technology, Kraków, Poland), **Waldemar Bauer** (Department of Automatic Control and Robotics, AGH University of Science & Technology, Kraków, Poland), **Karolina Dukała** (SWPS University of Social Sciences and Humanities, Katowice, Poland) and **Małgorzata** Wyrwas (Faculty of Computer Science, Bialystok University of Technology, Białystok, Poland). The work of M. Wyrwas and D. Mozyrska was supported by Bialystok University of Technology grant No. WZ/WI-IIT/1/2020 and funded by the resources for research by Ministry of Science and Higher Education.

Research on attitude and opinion change has a long history in psychology and has important practical ramifications. What it lacks, are mathematical modelling tools, allowing for analysis or predictions regarding both individual agents and their group. Promising methodology is based on the methods of consensus modelling. Use of classical models introduced by Krause in [1] (or sometimes referred to as the Hegselmann–Krause model in [2]) is however not adequate, as it does not capture the memory based influences on human behavior. Since fractional models have infinite memory, fractional calculus can be used in attitude dynamics modelling. Recently, fractional calculus in both continuous-time and discrete-time cases gained considerable development as an interesting extension multi-agent modelling methodology, see for instance [3]. The main difference of fractional models from integer order ones is the infinite memory horizon of the models. Fractional differential and difference equation solutions are not generated by semigroups and because of that cannot ignore the long memory effects. Such effects are however obviously present in modelling of human opinions, as these opinions depend of humans' experiences.

We propose consensus model using fractional calculus, which is an emerging topic in multi-agent modelling. Fractional models have infinite memory, and can be understood as relatively simple extension of traditional calculus. We propose a model structure and parametrization motivating it by psychological research. For this new interdisciplinary model for attitude dynamics we present the examples of agent networks of different complexity. Numerical simulations shows how the new interdisciplinary model represents attitude dynamics and provides psychological context.

[1] U. Krause, A discrete nonlinear and non-autonomous model of consensus formation, Proc. Commun. Difference Equations (2000) 227–236.

[2] V. D. Blondel, J. M. Hendrickx, F. N. Tsitsiklis, On Krause's multi-agent consensus model with state-dependent connectivity, IEEE Transactions on Automatic Control 5 (11) (2009) 2586–2597.

[3] G. Ren, Y. Yu, C. Xu, X. Hai, Consensus of fractional multi-agent systems by distributed event-triggered strategy, Nonlinear Dynamics 96 (2019) 541–555.

Nodal solutions in a class of Sturm-Liouville BVP's with nonnegative degenerate weights

Eduardo Muñoz-Hernández
Complutense University of Madrid, Spain

This is a joint work with Prof Julián López-Gómez (Complutense University of Madrid, Spain) and Prof. Fabio Zanolin (Udine University, Italy).

In this talk, we will analyze the existence and multiplicity of nodal solutions for a class of Sturm–Liouville boundary value problems associated to second order nonlinear ODE's of the type

$$-(\phi(u'))' = \lambda u + a(t)g(u) \quad \text{in } (0, L), \tag{1}$$

where $\lambda \in \mathbb{R}$, ϕ an increasing homeomorphism, with some minimal regularity constraints, g a continuous function such that $g(u)u > 0$, and $a \gneq 0$ is a piecewise constant function. More precisely, we assume that $[0, L]$ splits out into finitely many intervals where, alternately, either a is a positive constant or $a \equiv 0$. The motivation for such a choice for $a(t)$ goes back to the pioneering work of Moore and Nehari [2].

Then, denoting $h := \phi^{-1}$, the study of (1) will be carried out by analyzing the dynamical system associated to the general Sturm–Liouville boundary problem

$$\begin{cases} x' = h(y), \\ y' = -\lambda x - a(t)g(x), \\ x(0) \in r_0 \quad x(L) \in r_L, \end{cases} \tag{2}$$

where r_0 and r_L are curves satisfying a suitable crossing condition. First, we will perform a phase plane analysis in the intervals where $a > 0$ finding out appropriate annular regions of periodic orbits around the origin where different twist conditions are imposed. Then, in the regions where $a \equiv 0$, we will impose some stretching properties that are rather natural when $\lambda \leq 0$, but a bit more delicate if $\lambda > 0$. Once performed this preliminary analysis, by composing appropriately the Poincaré maps associated to each of the subintervals of $[0, L]$, and according to the sign of λ, we will deliver a series of multiplicity results for the system (2) together with a rather precise description of the nodal structure of the solutions.

As an application of the abstract results we will focus attention into the simpler, but paradigmatic, Dirichlet problem

$$\begin{cases} -u'' = \lambda u + a(t)|u|^{p-1}u \qquad \text{in } (0, L), \\ u(0) = u(L) = 0, \end{cases} \tag{3}$$

where the global bifurcation diagrams of nodal solutions in terms of the parameter λ will be also ascertained for both the superlinear case $p > 1$ and the sublinear case $0 < p < 1$ when $a \gneq 0$ is a piecewise constant function.

[1] J. López-Gómez, E. Muñoz-Hernández and F. Zanolin, *Examples of rich dynamics in planar systems with nonnegative degenerate weights*, preprint (2022).

[2] R. A. Moore and Z. Nehari, *Nonoscillation theorems for a class of nonlinear differential equations* Trans. Amer. Math. Soc. **93** (1959), 30–52.

A generalized DeTurck trick for anisotropic curve shortening flow

Robert Nürnberg
University of Trento, Italy

This is a joint work with Klaus Deckelnick (University of Magdeburg, Germany).

We extend the DeTurck trick from the classical isotropic curve shortening flow to the anisotropic setting. The evolution law that we consider arises as a natural gradient flow for the anisotropic energy

$$\mathcal{E}(\Gamma) = \int_\Gamma \gamma(z,\nu)\, a(z)\ \mathrm{d}\mathcal{H}^1(z) = \int_\Gamma \gamma(\cdot,\nu)\, a\ \mathrm{d}\mathcal{H}^1 \tag{1}$$

for a closed curve Γ, with unit normal ν, that is contained in the domain $\Omega \subset \mathbb{R}^2$. In the above, $\gamma : \Omega \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ denotes the anisotropy function and $a : \Omega \to \mathbb{R}_{>0}$ is a positive weight function. In the spatially homogeneous case, i.e.

$$\gamma(z,p) = \gamma_0(p) \quad \text{and} \quad a(z) = 1 \qquad \forall\ z \in \Omega = \mathbb{R}^2,$$

the corresponding functional $\mathcal{E}$ frequently occurs as an interfacial energy, e.g. in models of crystal growth. Since the anisotropic energy density in (1) may depend on space, it also allows an interpretation in the context of Finsler metrics, giving rise to e.g. geodesic curvature flow in Riemannian manifolds.

The natural gradient flow for the energy $\mathcal{E}$ evolves a family of curves $\Gamma(t) \subset \Omega$ according to the law

$$\mathcal{V}_\gamma = \varkappa_\gamma, \tag{2}$$

where $\mathcal{V}_\gamma$ and $\varkappa_\gamma$ are the anisotropic normal velocity and the anisotropic curvature, respectively. On considering a parametric formulation of the geometric evolution equation (2), we propose a novel and strictly parabolic system of PDEs for the position vector. The system can be viewed as an anisotropic analogue of the DeTurck trick applied to the classical isotropic curve shortening flow [1]. The obtained PDE is discretized by linear finite elements and optimal H^1–error bounds are proved in the semidiscrete case. In addition, we consider some fully practical fully discrete schemes and prove their unconditional stability. Finally, we present several numerical simulations, including some convergence experiments that confirm the derived error bound, as well as applications to crystalline curvature flow and geodesic curvature flow.

[1] K. Deckelnick and G. Dziuk, *On the approximation of the curve shortening flow*, in Calculus of Variations, Applications and Computations (Pont-à-Mousson, 1994), C. Bandle, J. Bemelmans, M. Chipot, J. S. J. Paulin, and I. Shafrir, eds., vol. 326 of Pitman Res. Notes Math. Ser., Longman Sci. Tech., Harlow, (1995), 100–108.

[2] K. Deckelnick and R. Nürnberg, *A novel finite element approximation of anisotropic curve shortening flow*, arXiv:2110.04605, (2021).

Existence of a weak solution to the problem of an interaction of compressible fluid with elastic structure

Šárka Nečasová

Institute of Mathematics of the Academy of Sciences of the Czech Republic, Czech Republic

This is a joint work with Václav Mácha (Institute of Mathematics of the Academy of Sciences of the Czech Republic), Boris Muha (University of Zagreb, Croatia), Arnab Roy (Basque Center for Applied Mathematics, Spain), Srdjan Trifunovic (University of Novi Sad, Serbia), Martin Kalousek (Institute of Mathematics of the Academy of Sciences of the Czech Republic), Sourav Mitra (Institute of Mathematics of the Academy of Sciences of the Czech Republic) .

We will focus on the study a nonlinear interaction problem between a shell and a compressible fluid. Firstly, the shell is governed by linear thermoelasticity equations and encompasses a time-dependent domain which is filled with a fluid governed by the full Navier-Stokes-Fourier system. The fluid and the shell are fully coupled, giving rise to a novel nonlinear moving boundary fluid-structure interaction problem involving heat exchange. Secondly, the shell is governed by a shell of Koiter type and a fluid is governed by noninteracting compressible fluids. The existence of a weak solution is obtained by combining three approximation techniques – decoupling, penalization and domain extension. In particular, the penalization and the domain extension allow us to use the methods already developed for compressible fluids on moving domains. In such a way, the proof is more elegant and the analysis is drastically simplified. Let us stress that this is the first time the heat exchange in the context of fluid-structure interaction problems is considered.Moreover, the first time when the noninteraction compressible fluids with elastic structure is studied.

[1] Martin Kalousek, Šárka Nečasová, Sourav Mitra, *Existence of weak solution for a compressible multicomponent fluid structure interaction problem*, preprint.

[2] Václav Mácha, Boris Muha, Šárka Nečasová, Arnab Roy, Srdjan Trifunovic, *Existence of a weak solution to a nonlinear fluid-structure interaction problem with heat exchange* Communications in Partial Differential Equations, https://www.tandfonline.com/doi/full/10.1080/03605302.2022.2068425.

Critical delay and stability for linear delay differential equations

Junya Nishiguchi
Tohoku University, Japan

For a linear delay differential equation (DDE), it is well-known that the zero solution of the linear DDE is exponentially stable if all the roots of its characteristic equation have negative real parts (ref. [1]). In this case, the characteristic equation is often said to be *stable*. There are various method to determine the stability of the characteristic equation of a linear DDE (ref. [2, Section 3 in Chapter 3]). Here we do not pursue a general method that is applicable to any linear DDE. Instead, we focus on an important class of linear DDEs, differential equations with a delay parameter $\tau > 0$. Then the characteristic equation also contains the delay parameter $\tau > 0$, and this should be a distinguished feature of the characteristic equation to study its stability. Such a consideration dates back to [3]. For this purpose, we mainly study a linear DDE

$$\dot{x}(t) = \alpha x(t) + Bx(t - \tau), \tag{1}$$

where $\alpha \in \mathbb{R}$, B is a 2×2 real matrix, and $\tau > 0$ is the delay parameter. We call $\tau = \tau(\alpha, B) > 0$ the *critical delay* of DDE (1) if for any $\tau \in (0, \tau(\alpha, B))$, the characteristic equation

$$\det(zI - [\alpha I + \mathrm{e}^{-\tau z} B]) = 0 \tag{2}$$

is stable. In this talk, we will make clear that the method by using the critical delay gives another proof of Hayes's and Sakata's results ([4], [5]). We will also obtain the stability condition on α, B, τ for which Eq. (2) is stable as an extension of the stability condition obtained by Hara and Sugie [6].

[1] J. K. Hale and S. M. Verduyn Lunel, *Introduction to functional differential equations*, Appl. Math. Sci., Vol. 99. Springer-Verlag, New York, 1993.

[2] L. E. Èl'sgol'ts and S. B. Norkin, *Introduction to the theory and application of differential equations with deviating arguments*, Translated from the Russian by John L. Casti. Mathematics in Science and Engineering, Vol. 105. Academic Press [A Subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1973.

[3] M. S. Lee and C. S. Hsu, *On the τ-decomposition method of stability analysis for retarded dynamical systems*, SIAM J. Control **7** (1969), 242–259.

[4] N. D. Hayes, *Roots of the transcendental equation associated with a certain difference-differential equation*, J. London Math. Soc. **25** (1950), 226–232.

[5] S. Sakata, *Asymptotic stability for a linear system of differential-difference equations*, Funkcial. Ekvac. **41** (1998), 435–449.

[6] T. Hara and J. Sugie, *Stability region for systems of differential-difference equations*, Funkcial. Ekvac. **39** (1996), no. 1, 69–86.

Heterogeneous gradient flows with applications to collective dynamics

Jan Peszek
University of Warsaw, Poland

In 2001 F. Otto [1] discovered a (nowadays well-known) relationship between the continuity equation

$$\partial_t \rho + \operatorname{div}(u\rho) = 0, \qquad \rho = \rho(t,x), \quad u = u(t,x)$$

and gradient flows with respect to the 2-Wasserstein metric. This connection provides a convenient description of many new and classical models and PDEs including Keller-Segel and Fokker-Planck as well as models of first-order collective dynamics.

I am going to present a recent work [2] (joint with David Poyato), wherein we introduce the so-called *fibered 2-Wasserstein metric* (which admits only transportation along fibers controlled by a prescribed probabilistic distribution $\nu = \nu(\omega)$) and explore its applicability in gradient flows. Based on such a metric, we develop the notion of heterogeneous gradient flows, and prove that they are equivalent to solutions of the parameterized continuity equation

$$\partial_t \rho + \operatorname{div}_x(u\rho) = 0, \qquad \rho = \rho(t,x,\omega), \quad u = u(t,x,\omega).$$

Note that the divergence above is taken only with respect to x, while ω controls the fibers and is prescribed though the distribution ν. Lastly, I will present a collection of applications ranging from mixtures of fluids, to multispecies models of collective dynamics, and to (the essential) applications in alignment models.

[1] F. Otto, *The geometry of dissipative evolution equations: the porous medium equation*, Comm. Part. Differ. Equat. **26** (2001), no. 1–2, 101–174

[2] J. Peszek and D. Poyato, *Heterogeneous gradient flows in the topology of fibered optimal transport*, arXiv:2203.08104 (2022).

Explicit values of the oscillation bounds for linear delay differential equations

Mihály Pituk
University of Pannonia, Veszprém, Hungary

This is a joint work with Prof. Ioannis P. Stavroulakis (University of Ioannina, Ioannina, Greece) and John Ioannis Stavroulakis (National Technical University of Athens, Athens, Greece).

We are interested in the oscillation of the linear delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \tag{1}$$

where p is a nonnegative, locally integrable function on $[0, \infty)$ and τ is a nondecreasing function on $[0, \infty)$ such that $\tau(t) \leq t$ for $t \geq 0$ and $\tau(t) \to \infty$ as $t \to \infty$. It is well-known (see, e.g., [1] and [2]) that if

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > \frac{1}{e} \qquad \text{or} \qquad \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > 1,$$

then all solutions of (1) are oscillatory. For $a \in [0, 1/e]$, the corresponding *oscillation bound* is the smallest number $\kappa = \kappa(a)$ with the property that

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds = a \qquad \text{and} \qquad \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > \kappa$$

imply that all solutions of (1) are oscillatory. The problem of finding the values of $\kappa(a)$ for $a \in [0, 1/e]$ has been in the focus of the oscillation theory for a long time. Although numerous estimates for the oscillation bounds are available in the literature, their explicit values were not known. In this talk, for every $a \in [0, 1/e]$, we give the value of $\kappa(a)$ explicitly in terms of the real branches of the Lambert W function.

[1] R.G. Koplatadze and T.A. Chanturia, *On the oscillatory and monotonic solutions of first order differential equations with deviating arguments,* Differ. Uravn. **18** (1982), 1463–1465. (in Russian)

[2] G. Ladas, V. Laskhmikantham, and J.S. Papadakis, *Oscillations of higher-order retarded differential equations generated by retarded arguments,* in: Delay and Functional Differential Equations and Their Applications, Academic Press, New York, 1972, pp. 219–231.

[3] M. Pituk, I.P. Stavroulakis, and J.I. Stavroulakis, *Explicit values of the oscillation bounds for linear delay differential equations with monotone argument,* Commun. Contemp. Math. (2021), 2150087, (35 pages)

Two-phase compressible/incompressible Navier–Stokes system with inflow-outflow boundary conditions

Milan Pokorný
Charles University, Prague, Czech Republic

This is a joint work with Aneta Wróblewska-Kamińska from IMPAN, Warsaw and Ewelina Zatorska from Imperial College, London.

We prove the existence of a weak solution to the compressible Navier–Stokes system with singular pressure that explodes when density achieves its congestion level. This is a quantity whose initial value evolves according to the transport equation. We then prove that the “stiff pressure" limit gives rise to the two-phase compressible/incompressible system with congestion constraint describing the free interface. We prescribe the velocity at the boundary and the value of density at the inflow part of the boundary of a general bounded C^2 domain. For the positive velocity flux, there are no restrictions on the size of the boundary conditions, while for the zero flux, a certain smallness is required for the last limit passage.

Nonlinear Poincaré–Perron theorem

Pavel Řehák
Brno University of Technology, Czech Republic

We establish a direct half-linear extension of the classical (second order) Poincarré–Perron theorem which says that the solutions y of certain perturbations of autonomous differential equations have the property $\lim_{t\to\infty} y'(t)/y(t) = \lambda$, λ being a root of the associated algebraic equation. In addition, we provide asymptotic formulae for these solutions, which are new even in the linear case. An important role in our considerations is played by the theory of regular variation.

A coupled phase-field and crystal plasticity model for deformation twinning and plastic slip

Mohsen Rezaee-Hajidehi
Institute of Fundamental Technological Research, Polish Academy of Sciences (IPPT PAN), Warsaw, Poland

Deformation twinning is a prevalent inelastic deformation mechanism in some specific metals and alloys. Due to the critical differences in the underlying mechanism and the characteristics of deformation twinning with respect to plastic slip, modeling the phenomena of deformation twinning poses additional challenges to be dealt with in comparison with plastic slip. In this study, a model of coupled deformation twinning and plastic slip is developed by combining the phase-field method and crystal plasticity [1]. Phase-field method is a powerful computational approach that allows to simulate complex microstructure patterns. The essence of the phase-field method is that it employs a diffuse-interface description by using an order parameter that distinguishes between the untwinned and twinned regions and varies smoothly across the interface.

The proposed model is formulated in the finite-deformation setting. A variational structure of the model is established by casting the model into the incremental energy minimization framework. The resulting minimization problem is non-smooth due to the rate-independent dissipation terms and the bound constraints enforced on the order parameter describing the diffuse twin interfaces. To circumvent this issue, a global-to-local regularization technique (called micromorphic regularization) is used, which largely facilitates the computer implementation of the model, in particular by shifting the complexities (in this context, non-smoothness) to the local level (e.g., Gauss points).

Two-dimensional simulations are carried out for the magnesium of the HCP crystal structure, in which two conjugate twinning systems and three effective slip systems (one basal and two pyramidal slip systems) are considered. The features of the model are illustrated by studying various problems, including twin evolution, twin transmission across grain boundaries, and the overall response of a unit-cell containing several grains, see Figure 1.

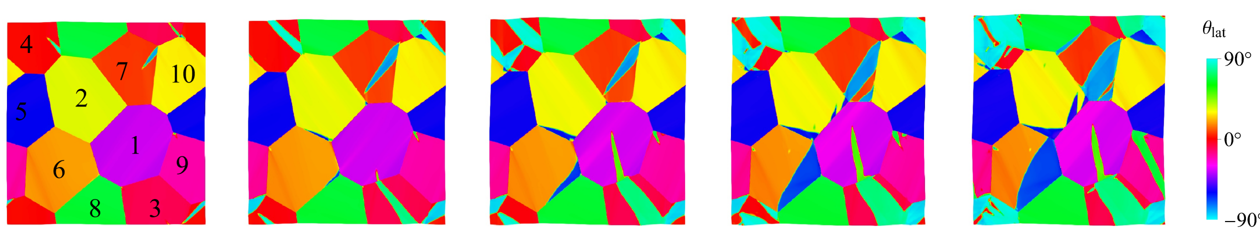

Figure 1: Snapshots of the twin microstructure evolution (represented by the lattice orientation angle θ_{lat}) in a periodic 10-grain unit cell under isochoric tension.

[1] M. Rezaee-Hajidehi, P. Sadowski, S. Stupkiewicz, *Deformation twinning as a displacive transformation: Finite-strain phase-field model of coupled twinning and crystal plasticity.* J. Mech. Phys. Solids **163**, 2022, 104855.

On elastic strain-limiting special Cosserat rods

Casey Rodriguez
University of North Carolina at Chapel Hill, USA

This talk is based on joint work with Prof. K. R. Rajagopal (Texas A&M University, College Station, TX, USA).

The theory of *implicit constitutive relations* in continuum mechanics, introduced by Rajagopal in the early 2000's, provides a simple and elegant framework for developing a broad spectrum of material models. One type of these relations is a *strain-limiting relation* where, as stresses applied to a body increase in magnitude, the body's strains converge to *finite* limiting values (either at finite stresses or asymptotically). A simple example of such a relation is given by

$$\boldsymbol{E} = a(1 + b|\bar{\boldsymbol{S}}|^p)^{-1/p}\bar{\boldsymbol{S}} \tag{1}$$

where $a, b, p > 0$, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$ is the Green-St. Venant strain tensor, and $\bar{\boldsymbol{S}}$ is the second Piola-Kirchhoff stress tensor. Over the past 15 years there has been intense research in the mathematical properties of strain-limiting relations and in developing concrete models for gum metal, brittle materials, biological fibers and many others.

All of the past research in strain-limiting relations have been within three-dimensional solid and fluid mechanics. In this talk we will discuss recent work towards developing the theory of strain-limiting relations for slender bodies modeled by intrinsically one-dimensional continua deforming in space; *perfectly flexible strings* and *special Cosserat rods.* In these theories, the constitutive relations are between nonlinear, geometrically exact measures of strain (flexural, twisting, shearing, or stretching) and components of the rod's contact couple and contact force vector fields (bending or twisting couples, shearing or tensile/compressive forces). We will give an overview of recent results as well as open problems related to strain-limited catenaries [2], to dynamic longitudinal motion of strain-limited strings [3], and to an intrinsic analog of (1) for extensible and shearable special Cosserat rods, including shearing bifurcations and Poynting effects [1].

[1] K. R. Rajagopal and C. Rodriguez, *On an elastic strain-limiting special Cosserat rod model.* arXiv: 2205.10875 (2022), 23 pages.

[2] C. Rodriguez, *On stretch-limited elastic strings.* Proc. R. Soc. Lond. A, **477**(2249):Paper No. 20210181, 15 pp, 2021.

[3] C. Rodriguez, *Longitudinal shock waves in a class of semi-infinite stretch-limited elastic strings.* Math. Mech. Solids, **27** (2022), 474–490.

Nonlinear Fokker-Planck equation on unbounded domain and featuring generalized reflecting boundary conditions

Eduard Rotenstein
"Alexandru Ioan Cuza" University of Iasi, Romania

This is a joint work with Prof. Dan Goreac (Laboratoire d'Analyse et de Mathématiques Appliquées, Université Gustave Eiffel, Paris, France) and Prof. Ioana Ciotir (Laboratoire de Mathématiques de l'INSA Rouen Normandie, France).

We study the existence and uniqueness of the solution to a nonlinear Fokker-Planck equation featuring oblique reflection at the boundary. The solution we are interested on is of mild or distributional type. We consider the following Neumann problem:

$$\begin{cases} \dfrac{\partial}{\partial t}\rho\left(t,x\right)-\dfrac{1}{2}\Delta\beta\left(\rho\left(t,x\right)\right)-\operatorname{div}\left(b\left(x\right)\rho\left(t,x\right)\right)=0, & \left(t,x\right)\in\left(0,T\right)\times\mathcal{O}, \\ \dfrac{1}{2}\dfrac{\partial}{\partial n}\beta\left(\rho\left(t,x\right)\right)+\left(b\left(x\right)\cdot H\left(t,x\right)n\left(x\right)\right)\rho\left(t,x\right)=0, & \left(t,x\right)\in\left(0,T\right)\times\partial\mathcal{O}, \\ \rho\left(0,x\right)=\rho_0\left(x\right), & x\in\mathcal{O}, \end{cases}$$

where $\mathcal{O}\in\mathbb{R}^d$, $d\geq 1$, is assumed to be an open unbounded convex domain with smooth boundary $\partial\mathcal{O}$. Denote by n the outward normal to $\partial\mathcal{O}$ and by $\partial/\partial n$ the outward normal derivative. Also, let denote $K:=\overline{\mathcal{O}}$. If we choose the solution ρ as a probability density functions on $\mathcal{O}$, then it can be seen as the probability density of the law $\mathcal{L}_X$, where X is the weak solution (in the stochastic sense) to the following stochastic differential equation with generalized reflection

$$\begin{cases} dX_t+b\left(X_t\right)dt+H\left(t,X_t\right)\partial I_K\left(X_t\right)\left(dt\right)\ni\sqrt{\dfrac{\beta\left(\rho\left(t,X_t\right)\right)}{\rho\left(t,X_t\right)}}dW_t, \\ X\left(0\right)=X_0, \end{cases}$$

considered in a filtred probability space $\left(\Omega,\mathcal{F},\mathbb{F}:=\left(\mathcal{F}_t\right)_t,\mathbb{P}\right)$, $\mathbb{F}$ being the natural filtration of the Wiener process W. Recall that X_0 is a random variable with the probability density ρ_0. By ∂I_K we denote the subdifferential operator of the convexity indicator function $I_K:\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$, $I_K\left(x\right)=0$ if $x\in K$ and $I_K\left(x\right)=+\infty$ otherwise. The term $H\partial I_K$ will provide oblique subgradients at the frontier of the domain.

Asymptotic stability of solutions of differential equations with delay

Paola Rubbioni

University of Peruggia, Italy

We discuss existence and asymptotic stability of solutions of the parametric differential equation arising from population dynamics models

$$\frac{\partial u}{\partial t}(t,x) = -b(t,x)u(t,x) + g\left(t, u(t,x), \int_{-\tau}^{0} u(t+\theta,x)d\theta\right),\ t \geq t_0,\ x \in [0,1]. \qquad (1)$$

The term $\int_{-\tau}^{0} u(t+\theta,x)d\theta$ means that at every time t the system has memory of the evolution of the state up to that moment for a past of fixed amplitude $\tau > 0$.

The study is carried out in the context of semilinear differential equations in Banach spaces. We see equation (1) as a particular case of the semilinear differential equation with functional delay

$$y'(t) = A(t)y(t) + f\left(t, y(t), y_t\right),\ t \geq t_0, \qquad (2)$$

where y_t stands for the function $y_t(\theta) = y(t+\theta)$, $\theta \in [-\tau, 0]$, $t \geq t_0$. The results are achieved on (2) by combining iterative methods and fixed point theorems for condensing maps, and then made to fall back on (1).

We also show results when systems driven by (1) are subject to impulsive forces $\mathcal{I}_k$ at times t_k, $\{t_k\}_{k\in\mathbb{N}}$ increasing diverging sequence, or to feedback controls $\omega(t,x) \in W(u(t,x))$, W set-valued map. Once these conditions are reread in the abstract setting, in the first case we get to work with functions I_k, $k \in \mathbb{N}$, defined in a suitable functions space. In the second, however, we are naturally faced with the semilinear differential inclusion $y'(t) \in A(t)y(t) + F\left(t, y(t), y_t\right)$, which requires the use of multivalued analysis tools.

Qualitative theory for monotone and sublinear non-autonomous finite-delay FDEs for an exponential ordering

Ana M. Sanz
University of Valladolid, Spain

This is a joint work with Prof Sylvia Novo and Rafael Obaya (University of Valladolid, Spain) and Prof Víctor M. Villarragut (Politecnical University of Madrid, Spain).

Monotonicity methods in dynamical systems have been extensively applied since the initial works by M.W. Hirsch in the 80's. Lately, also the field of monotone non-autonomous dynamical systems has received great attention, with applications to a class of non-autonomous differential equations. In this talk we pay attention to finite-delay equations where the monotonicity conditions for the usual ordering fail to hold. Then, one can still look for different cones of positive vectors, so that a new partial ordering can be defined and solutions generate a monotone dynamical system under not so restrictive conditions.

More precisely, we give a description of the long-term behaviour of the trajectories of the skew-product semiflow induced by a family of delay FDEs, which is monotone and sublinear for an exponential ordering (see [3]). As an application, we will consider an almost periodic Nicholson compartmental system which is uniformly persistent (see [2]), for which we obtain the existence of a unique positive almost periodic solution asymptotically attracting every other positive solution, provided that some easy-to-check conditions hold, which make the induced semiflow monotone for an exponential ordering.

[1] S. Novo, R. Obaya, A.M. Sanz, V.M. Villarragut, *The exponential ordering for non-autonomous delay systems with applications to compartmental Nicholson systems*, submitted for publication, 2022.

[2] R. Obaya, A.M. Sanz, *Is uniform persistence a robust property in almost periodic models? A well-behaved family: almost-periodic Nicholson systems*, Nonlinearity **31** (2018), 388–413.

[3] H.L. Smith, H.R. Thieme, *Strongly order preserving semiflows generated by functional differential equations*, J. Differential Equations **93** (1991), 332–363.

Bayesian calibration and comparison of models of tumour cell dynamics

Laura Scarabosio
Radboud University, The Netherlands

This presentation is based on joint work [1] with M.Sc. Sabrina Schönfeld and Prof. Dr. Christina Kuttler (Technical University of Munich, Munich, Germany), and with Dr. Alican Ozkan (Harvard University, Boston, United States) and Prof. Dr. Marissa Nichole Rylander (The University of Texas, Austin, United States).

Survival of living tumor cells underlies many influences such as nutrient saturation, oxygen level, drug concentrations or mechanical forces. However, considering numerous environmental factors when modeling tumour cell evolution can get challenging. To overcome this, we present an approach to model the separate influences of each environmental quantity on the cells in a collective manner by introducing the "environmental stress level". This is an immeasurable auxiliary variable, which quantifies to what extent viable cells would get in a stressed state, if exposed to certain conditions.

We consider the nutrient saturation as environmental variable, and we use time resolved measurements of *in vitro* populations of liver cancer cells to compare our new model using the environmental stress level with another model for tumour cell evolution where the influence of the nutrient is incorporated directly without using the stress level. These are both systems of ordinary differential equations. For both of them, we calibrate the parameters using sequential Monte Carlo based on filtering. Model comparison is performed by computing the Bayes factor.

While predictions of both models show good agreement with the data, there is indication that the model considering the stress level yields a better fitting. The proposed modeling approach offers a flexible and extendable framework for considering systems with additional environmental factors affecting the cell dynamics.

[1] Sabrina Schönfeld, Alican Ozkan, Laura Scarabosio, Marissa Nichole Rylander, Christina Kuttler. *Environmental stress level to model tumor cell growth and survival*. Mathematical Biosciences and Engineering **19(6)** (2022), 5509–5545.

Multiscale spectral generalised finite methods

Robert Scheichl
Institute of Applied Mathematics, Heidelberg University, Germany

This is a joint work with Chupeng Ma (Great Bay University, Guangdong, China) and Christian Alber (Heidelberg University, Germany).

In this talk I will present error estimates of the fully discrete generalized finite element method with optimal, multiscale spectral, local approximation spaces (MS-GFEM) for solving elliptic problems with heterogeneous coefficients [4]. The local approximation spaces are constructed using eigenvectors of local eigenvalue problems solved by the finite element method on some sufficiently fine mesh with mesh size h. The error bound of the discrete MS-GFEM approximation is proved to converge as $h \to 0$ towards that of the continuous MS-GFEM approximation, which was shown to decay nearly exponentially in previous works [1, 3]. Moreover, even for fixed mesh size h, a nearly exponential rate of convergence of the local approximation errors with respect to the dimension of the local spaces is established. An efficient and accurate method for solving the discrete eigenvalue problems is proposed by incorporating the discrete -harmonic constraint directly into the eigenproblem via a Lagrange multiplier approach. Numerical experiments are carried out to confirm the theoretical results and to demonstrate the effectiveness of the method.

Finally, I will also show how this MS-GFEM approach extends beyond elliptic problems, considering in particular a high-frequency heterogeneous Helmholtz problem [2]. Assuming that the size of the subdomains is $\mathcal{O}(1/k)$ (where k is the wavenumber) a quasi-optimal and nearly exponential (wavenumber explicit) global convergence of the method can be established even in that case.

[1] I. Babuska and R. Lipton, *Optimal local approximation spaces for generalized finite element methods with application to multiscale problems.* Multiscale Model. Simul. **9** (2011), 373—406.

[2] C. Ma, C. Alber, and R. Scheichl, *Wavenumber explicit convergence of a multiscale GFEM for heterogeneous Helmholtz problems.* Preprint arXiv:2112.10544 (2021), 1–35.

[3] C. Ma, R. Scheichl, and T. Dodwell, *Novel design and analysis of generalized finite element methods based on locally optimal spectral approximations.* SIAM J. Numer. Anal. **60** (2022), 244–273.

[4] C. Ma and R. Scheichl, *Error estimates for fully discrete generalized FEMs with locally optimal spectral approximations.* Math. Comput., to appear (2022), 1–29.

New comparison theorems for solutions of Riccati matrix differential equations without controllability condition

Peter Šepitka
Masaryk University, Czech Republic

In this talk we present some new results regarding the qualitative theory of Riccati matrix differential equations. These equations are associated with linear Hamiltonian differential systems, which naturally arise in the nonlinear variational theory (see [3]). Let $n \in \mathbb{N}$ be a given dimension and $\mathcal{I} := [a, \infty)$ be a given unbounded real interval. For the pair of linear Hamiltonian systems

$$y' = \mathcal{J}\mathcal{H}(t)\, y, \quad \hat{y}' = \mathcal{J}\hat{\mathcal{H}}(t)\, \hat{y}, \quad \mathcal{J} := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad t \in \mathcal{I}, \tag{1}$$

where $\mathcal{H}, \hat{\mathcal{H}} : [a, \infty) \to \mathbb{R}^{2n \times 2n}$ are symmetric and piecewise continuous matrix-valued functions, we consider the corresponding Riccati matrix differential equations

$$Q' + \begin{pmatrix} I_n & Q \end{pmatrix} \mathcal{H}(t) \begin{pmatrix} I_n \\ Q \end{pmatrix} = 0, \quad \hat{Q}' + \begin{pmatrix} I_n & \hat{Q} \end{pmatrix} \hat{\mathcal{H}}(t) \begin{pmatrix} I_n \\ \hat{Q} \end{pmatrix} = 0, \quad t \in \mathcal{I}. \tag{2}$$

It is well-known (see [1]) that if the systems in (1) are *completely controllable* and nonoscillatory on the interval $\mathcal{I}$ and if they satisfy the conditions

$$\mathcal{H}(t) \geq \hat{\mathcal{H}}(t) \quad \text{and} \quad \begin{pmatrix} 0 & I_n \end{pmatrix} \hat{\mathcal{H}}(t) \begin{pmatrix} 0 \\ I_n \end{pmatrix} \geq 0 \quad \text{for all } t \in \mathcal{I}, \tag{3}$$

then the distinguished (or principal) solutions Q_∞ and $\hat{Q}_\infty$ of the Riccati equations in (2) at infinity are related by the inequality $Q_\infty(t) \geq \hat{Q}_\infty(t)$ for all large $t \in \mathcal{I}$. In [2] Fabbri, Johnson, Novo, and Núñez showed that the same result holds also for the case of the *weak controllable* systems in (1). Based on the recent studies on Riccati matrix differential equations in [4] and on the singular Sturmian theory for linear Hamiltonian systems in [5] we provide a generalization of the above classical result for the nonoscillatory *uncontrollable* systems in (1) satisfying the conditions in (3).

[1] W. A. Coppel, Disconjugacy. Lecture Notes in Mathematics, Vol. 220, Springer-Verlag, Berlin - New York, 1971.

[2] R. Fabbri, R. Johnson, S. Novo, C. Núñez, *Some remarks concerning weakly disconjugate linear Hamiltonian systems.* J. Math. Anal. Appl. **380** (2011), no. 2, 853–864.

[3] W. Kratz, Quadratic Functionals in Variational Analysis and Control Theory. Akademie Verlag, Berlin, 1995.

[4] P. Šepitka, *Riccati equations for linear Hamiltonian systems without controllability condition.* Discrete Contin. Dyn. Syst. **39** (2019), no. 4, 1685–1730.

[5] P. Šepitka, R. Šimon Hilscher, *Singular Sturmian comparison theorems for linear Hamiltonian systems.* J. Differential Equations. **269** (2020), no. 4, 2920–2955.

Contactless rebound of elastic bodies in a viscous incompressible fluid

Schwarzacher Sebastian
Uppsala Univ., Sweden & Charles Univ., Czech Republic

This is a joint work with G. Gravian (Temple Univ.), O. Soucek and K. Tuma (both Charles Univ.).

In this talk, I present our resent results on the phenomenon of particle rebound in a viscous incompressible fluid environment. We focus on the important case of no-slip boundary conditions, for which it is by now classical that, under certain assumptions, collisions cannot occur in finite time. Motivated by the desire to understand this fascinating yet counterintuitive fluid-structure interaction, we introduce a reduced model which we study both analytically and numerically. In this simplified framework, we provide conditions which allow to prove that rebound is possible even in the absence of a topological contact. Our results lead to conjecture that a qualitative change in the shape of the solid is necessary and sufficient for obtaining a physically meaningful rebound. We support the conjecture by comparing numerical simulations performed for the reduced model with the finite element solutions obtained for the corresponding well-established PDE system.

Spatial maxima, unimodality, and asymptotic behavior of solutions to discrete diffusion equations

Antonín Slavík
Charles University, Prague, Czech Republic

We study the discrete diffusion (heat) equation

$$u(x,t+1)-u(x,t)=a(u(x+1,t)-2u(x,t)+u(x-1,t)),\quad x\in\mathbb{Z},\quad t\in\mathbb{N}_0,\tag{1}$$

and focus on two basic problems:

- *Properties of the fundamental solution as a function of x, for a fixed time t.* In particular, we are interested in the location of spatial maxima and unimodality.
- *Asymptotic behavior of solutions for $t\to\infty$.* We consider initial-value problems with bounded initial data, and prove that under suitable assumptions, the solutions of Eq. (1) converge to the spatial average of the initial values (provided it exists). The corresponding result for the classical diffusion equation is well known. The semidiscrete case was recently studied in [1], and the purely discrete case can be treated similarly as soon as we establish the basic properties of the fundamental solution, including unimodality; see [2].

In fact, our results apply to a much more general class of partial difference equations having the form

$$u(x,t+1)-u(x,t)=\sum_{i=-m}^{m}a_i u(x+i,t),\quad x\in\mathbb{Z},\quad t\in\mathbb{N}_0,\tag{2}$$

where $m\in\mathbb{N}$ and $a_{-m},\dots,a_m\in\mathbb{R}$. Under certain assumptions, Eq. (2) describes a random walk on $\mathbb{Z}$, which makes it possible to utilize some powerful theorems from probability theory (the local limit theorem for random variables with lattice distribution, and results on convolutions of unimodal distributions).

[1] A. Slavík, *Asymptotic behavior of solutions to the semidiscrete diffusion equation.* Appl. Math. Lett. **106** (2020), 106392.

[2] A. Slavík, *Spatial maxima, unimodality, and asymptotic behaviour of solutions to discrete diffusion-type equations.* J. Difference Equ. Appl. **28** (2022), 126–140.

Adaptive methods for fully nonlinear PDE

Iain Smears
University College London, UK

Hamilton–Jacobi–Bellman and Isaacs equations are important classes of fully nonlinear PDE with applications from stochastic optimal control and two player stochastic differential games. In this talk, we present our recent proof in [1] of the convergence of a broad family of adaptive nonconforming DG and C^0-interior penalty methods for the class of these equations that satisfy the Cordes condition in two or three space dimensions. The adaptive mesh refinement is driven by reliable and efficient a posteriori error estimators, and convergence is proven in H^2-type norms without higher regularity assumptions of the solution. A foundational ingredient in the proof of convergence is the concept of the limit space used to describe the limiting behaviour of the finite element spaces under the adaptive mesh refinement algorithm. We develop a novel approach to the construction and analysis of these nonstandard function spaces via intrinsic characterizations in terms of the distributional derivatives of functions of bounded variation. We provide a detailed theory for the limit spaces, and also some original auxiliary function spaces, that resolves some foundational challenges and that is of independent interest to adaptive nonconforming methods for more general problems. These include Poincaré and trace inequalities, a proof of the density of functions with nonvanishing jumps on only finitely many faces of the limit skeleton, symmetry of the Hessians, approximation results by finite element functions and weak convergence results.

[1] E. L. Kawecki and I. Smears, *Convergence of adaptive discontinuous Galerkin and C^0-interior penalty finite element methods for Hamilton–Jacobi–Bellman and Isaacs equations,* Found. Comput. Math. **22** (2022), no. 2, 315–364.

Randomized multiscale methods for nonlinear partial differential equations

Kathrin Smetana
Stevens Institute of Technology, United States

This is a joint work with Dr. Tommaso Taddei (Inria, Bordeaux, France).

Heterogeneous problems that take place at multiple scales are ubiquitous in science and engineering. Examples are wind turbines made from composites or groundwater flow relevant e.g., for the design of flood prevention measures. However, finite element or finite volume methods require an often prohibitively large amount of computational time for such tasks. Multiscale methods that are based on ansatz functions which incorporate the local behavior of the (numerical) solution of the partial differential equations (PDEs) have been developed to tackle these heterogeneous problems. Localizable multiscale methods that allow controlling the error due to localization and the (global) approximation error at a (quasi-optimal) rate and do not rely on structural assumptions such as scale separation or periodicity have only been developed within the last decade. Here, localizable multiscale methods allow the efficient construction of the basis functions by solving the PDE (in parallel) on several small subdomains at low cost.

While there has been a significant progress in recent years for these types of multiscale methods for linear PDEs, very few results have been obtained so far for nonlinear PDEs. In this talk, we will show how randomized methods and their probabilistic numerical analysis can be exploited for the construction and numerical analysis of such types of multiscale methods for nonlinear PDEs.

Diffusive Hamilton-Jacobi equations and their singularities

Philippe Souplet
LAGA, Université Sorbonne Paris Nord, France

We consider the diffusive Hamilton-Jacobi equation $u_t - \Delta u = |\nabla u|^p$ with homogeneous Dirichlet boundary conditions, which plays an important role in stochastic optimal control theory and in certain models of surface growth (KPZ). Despite its simplicity, in the superquadratic case $p > 2$, it displays a variety of interesting and surprising behaviors.

We will discuss two classes of phenomena:

- Gradient blow-up (GBU) on the boundary: time rate, single-point GBU, space and time-space profiles, Liouville type theorems and their applications;
- Continuation after GBU as a global viscosity solution with loss and recovery of boundary conditions. In particular, we will present the recently obtained, complete classification of solutions in one space dimension, which describes the losses and recoveries of boundary conditions at multiple times, as well as all the possible GBU and recovery rates.

This talk is based on a series of joint works in collaboration with A. Attouchi, R. Filippucci, Y. Li, N. Mizoguchi, A. Porretta, P. Pucci, Q. Zhang.

Reactive-convective Perona-Malik equations: regular vs. nonregular wavefronts

Elisa Sovrano
University of Modena and Reggio Emilia, Italy

The Perona-Malik operator provides a paradigmatic example of flux-saturated diffusion in image processing where the diffusion term is driven by a nonlinear, bounded, and non-monotone function (of the gradient) that approaches zero at infinity. In this framework, we investigate nonlinear reaction-convection-diffusion equations. When the reaction term is monostable, the goal is to give a complete picture of regular monotone wavefronts between two steady states in terms of their wave speed. Wavefronts' existence results are proved by showing that the set of admissible wave speeds is an unbounded interval and providing estimates for the threshold speed. Finally, we discuss the appearance of nonregular wavefronts. This talk is based on joint works with A. Corli (University of Ferrara, Italy) and L. Malaguti (University of Modena and Reggio Emilia).

Bifurcations in Nagumo equations on graphs

Petr Stehlík
University of West Bohemia, Czech Republic

This is a joint work with Vladimír Švígler and Jonáš Volek (Department of Mathematics, Faculty of Applied Sciences, University of West Bohemia).

We study the Nagumo equation on a graph $G = (V, E)$

$$\dot{u}_i(t) = d \sum_{j \in \mathcal{N}(i)} (u_j(t) - u_i(t)) + f(u_i(t); a), \quad i = 1, 2, \dots, |V|,$$

in which the reaction is described by the cubic nonlinearity

$$f(s; a) = s(1-s)(s-a), \quad a \in [0,1],$$

and $\mathcal{N}(i)$ is the set of all neighbours of the vertex $i \in V$. This dynamical system has an exponential number of stationary solutions for sufficiently small diffusion rate $d > 0$, [3]. On the other hand, for sufficiently large diffusion rate d there are only three spatially homogeneous stationary solutions. We present a result on how the graph structure affects the bifurcations of the the first spatially heterogeneous stationary solutions. Namely, we investigate the influence of the algebraic connectivity (the second eigenvalue of the discrete Laplacian) and its corresponding eigenvector, the Fiedler vector. This enables us to describe three qualitatively distinct configurations which can be observed even for small graphs.

We are motivated by the fact that the full understanding of bifurcation phenomena for small graphs can improve the theory bichromatic and multichromatic waves, [1, 2] .

The talk is based on [4].

[1] H. J. Hupkes, L. Morelli and P. Stehlík, *Bichromatic Travelling Waves for Lattice Nagumo Equations.* SIAM Journal on Applied Dynamical Systems **18**(2019), no. 2, 973–1014.

[2] H. J. Hupkes, L. Morelli, P. Stehlík and V. Švígler, Multichromatic travelling waves for lattice Nagumo equations. Applied Mathematics and Computation **361**(2019), 430-452.

[3] P. Stehlík, *Exponential number of stationary solutions for Nagumo equations on graphs* J. Math. Anal. Appl., **455**(2017), 1749–1764.

[4] P. Stehlík, V. Švígler and J. Volek , *Bifurcations in Nagumo Equations on Graphs and Fiedler Vectors* J. Dynam. Differential Equations, to appear.

Propagation reversal for bistable differential equations on trees

Vladimír Švígler
Department of Mathematics & NTIS, Faculty of Applied Sciences, University of West Bohemia, Czech Republic

This is a joint work with Hermen Jan Hupkes, Mia Jukić (Mathematisch Instituut, Universiteit Leiden) and Petr Stehlík (Department of Mathematics, Faculty of Applied Sciences, University of West Bohemia).

We study traveling wave solutions to the bistable differential equations on infinite k-ary trees in the form

$$\begin{aligned}\dot{u}_i &= d(ku_{i+1} - (k+1)u_i + u_{i-1}) + g(u_i; a),\\ &= d(u_{i+1} - 2u_i + u_{i-1}) + d(k-1)(u_{i+1} - u_i) + g(u_i; a),\end{aligned}$$

in which $i \in \mathbb{Z}$, $d > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a bistable nonlinearity of the Nagumo type, e.g.,

$$g(s; a) = s(1-s)(s-a), \quad a \in (0,1).$$

In the talk, we discuss how comparison principles and construction of explicit lower and upper solution can be used to obtain information about the dependence of the wave speed $c \in \mathbb{R}$ on the parameters a, d, k.

We show that wave-solutions are pinned provided the diffusion parameter d is small. Upon increasing the diffusion d, the wave starts to travel with non-zero speed $c \neq 0$, in a direction that depends on the detuning parameter a. However, once the diffusion is sufficiently strong, the wave propagates in a single direction up the tree irrespective of the detuning parameter a.

As a consequence, we show that for certain range of the detuning parameter a the changes to the diffusion parameter d lead to a reversal of the propagation direction.

Differential equations involving homeomorphism with nonlinear boundary conditions

Katarzyna Szymańska-Dębowska
Institute of Mathematics, Lodz University of Technology, Poland

This is a joint work with Prof Mirosława Zima.

We establish existence results for differential equations of the form:

$$(\varphi(u'))' = f(t, u, u'),$$

where f is continuous and φ is a homeomorphism, with nonlinear boundary conditions:

$$g(u(0), G(u')) = h(u'(1), H(u)) = 0; \;\; g(u'(0), G(u)) = h(u(1), H(u')) = 0;$$

$$g(u(0), G(u)) = h(u'(1), H(u')) = 0 \;\text{ or }\; g(u'(0), G(u')) = h(u(1), H(u)) = 0,$$

where g, h are continuous and G, H are continuous functionals. Our methods of proofs are based on the extension of Mawhin's continuation theorem for quasi–linear operators.

Multiple solutions of certain Dirichlet problems in billiard spaces

Jan Tomeček
Faculty of Science, Palacký University Olomouc, Czech Republic

This is a joint work with Prof. Grzegorz Gabor (Nicolaus Copernicus University in Toruń, Poland).

We investigate a Dirichlet problem for an ODE of the second order with state-dependent impulses in the form

$$\begin{aligned} &\ddot{x}(t) = f(t, x(t)) \quad \text{if } t \in [0, T],\ x(t) \in \operatorname{int} K, \\ &\dot{x}(s+) = \dot{x}(s) + I(x(s), \dot{x}(s)), \quad \text{if } x(s) \in \partial K, \\ &x(0) = A \in \operatorname{int} K, \quad x(T) = B \in \operatorname{int} K. \end{aligned}$$

where K is an interval in $\mathbb{R}^n$, $f : [0, T] \times K \to \mathbb{R}^n$ satisfies Carathéodory conditions, the impulse function I corresponds with the absolutely elastic impact of the ball at the boundary of the "billiard table" K, $T > 0$. We give existence and multiplicity result for solutions with prescribed number of impacts.

Time-periodic solutions to a compressible fluid and beam interaction problem

Srdan Trifunović
Faculty of Sciences, University of Novi Sad, Serbia

In this lecture, I will talk about the problem of interaction between a compressible fluid and a viscoelastic beam under the influence of time-periodic external forces in 2D. For this problem, at least one weak solution is constructed which is periodic in time and perserves the mass which is a given constant. Approximate solutions are obtained by solving a decoupling problem which is in a finite base for space and time. Weak solution is then obtained as a limit of the approximate solutions.

This is a joint work with Ondřej Kreml, Václav Mácha and Šarka Nečasová from the Institute of Mathematics of Czech Academy of Sciences.

Phase-field model for evolution of martensitic microstructure in shape memory alloys: large-scale finite element simulations

Karel Tůma
Mathematical Institute of Charles University, Faculty of Mathematics and Physics, Charles University, Prague, Czechia

Shape memory alloys such as CuAlNi are interesting for exhibiting two important phenomena, the shape-memory effect and pseudoelasticity. They originate from the reversible martensitic transformation between the parent phase (austenite) and the product phase (martensite), which is characterized by the microstructure evolution. Because austenite cannot form a compatible interface with a single variant of martensite, the martensitic transformation is realized via the formation of a complex twinning microstructure.

A recently developed phase-field model for multivariant martensitic transformation [1] is used to study the effect of crystal lattice orientation and material anisotropy on the indentation-induced microstructure evolution in CuAlNi shape memory alloy. To capture a detailed microstructure (see Figure 2) it is needed to solve computationally very large problems reaching up to 150 million degrees of freedom [2]. The model is implemented with Firedrake finite-element code that provides automatic differentiation and is tightly connected with PETSc solver library, which enables to use effective iterative solvers. The non-linear problem is solved with the Newton method. Consequent linear sub-problems are solved with GMRES with geometric multigrid used as preconditioner where on each level of discretization the point-block Jacobi iteration is used as a smoother.

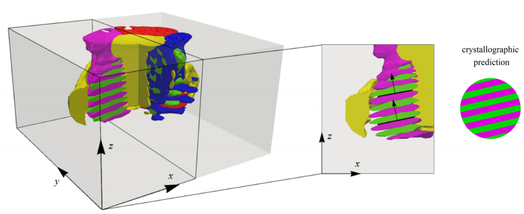

Figure 2: Twinning and saw-tooth morphologies obtained during the martensitic transformation.

[1] M. Rezaee-Hajidehi, S. Stupkiewicz, *Phase-field modeling of multivariant martensitic mi- crostructure and size effects in nano-indentation.* Mech. Mat. **141**, 2020, 103267.

[2] K. Tůma, M. Rezaee-Hajidehi, J. Hron, P.E. Farrell, S. Stupkiewicz, *Phase-field modeling of multivariant martensitic transformation at finite-strain: Computational aspects and large- scale finite-element simulations.* Comput. Meth. Appl. Mech. Eng. **377**, 2021, 113705.

Duality for Stieltjes integral equations

Milan Tvrdý
Institute of Mathematics, Czech Academy of Sciences, Czech Republic

Let BV be the space of n-vector valued functions with bounded variation on $[0,1]$ and G_L be the space of n-vector valued functions regulated on $[0,1]$ and left continuous on $(0,1]$.

First, we will recall the following older result from [1]:

Assume that A is an $n\times n$-matrix valued function of bounded variation on $[0,1]$ left-continuous on $(0,1]$, right-continuous at 0, and such that $\det[I+\Delta^+A(t)]\neq 0$ on $[0,l)$; M is a constant $n\times n$-matrix; K is an $n\times n$-matrix valued function of bounded variation on $[0,1]$. Put

$$(\mathcal{L}\,x)(t)=\begin{pmatrix} x(t)-x(0)-\displaystyle\int_0^t [dA]\,x \\ M\,x(0)+\displaystyle\int_0^1 K\,[dx] \end{pmatrix} \quad \text{for } x\in G_L \text{ and } t\in[0,1], \tag{1}$$

where the integrals are the Kurzweil-Stieltjes ones, cf. [3]. $\mathcal{L}$ is a linear bounded mapping of G_L into $Z:=G_L\times\mathbb{R}^n$. Moreover, under the above assumptions $\mathcal{L}$ has a closed range in Z and its proper dual $\mathcal{L}^*: BV\times\mathbb{R}^n\times\mathbb{R}^n\to$ is given by

$$\begin{cases} (\mathcal{L}^*((y^*,\gamma^*,\delta^*))(t)=\Big(y^*(t)+\delta^*\,K(t)-\displaystyle\int_t^1 y^*(s)\,[dA(s+)],\delta^*M-\int_0^1 y^*[dA]\Big) \\ \text{for } y\in BV,\gamma,\delta\in\mathbb{R}^n \text{ and } t\in[0,1], \end{cases} \tag{2}$$

where z^* stands for the transposition of the vector z.

Then, for a properly modified scalar problem we will present related results without assuming left-continuity of A. Second part of the contribution is based on a joint work with Ignacio Márquez Albés (University of Santiago de Compostela, Spain) and Antonín Slavík (Charles University, Prague, Czechia).

[1] M. Tvrdý, *Generalized differential equations in the space of regulated functions (Boundary value problems and controllability).* Math. Bohem. **116** (1991), 225–244.

[2] I. Márquez Albés, A. Slavík, M. Tvrdý, *Duality for Stieltjes differential and integral equations.* Submitted.

[3] G. A. Monteiro, A. Slavík, M. Tvrdý, Kurzweil-Stieltjes Integral. Theory and Applications. World Scientific, 2019.

A posteriori error bounds for eigenfunctions

Tomáš Vejchodský
Institute of Mathematics, Czech Academy of Sciences, Czech Republic

This is a joint work with Xuefeng Liu (Niigata University, Japan).

We propose two types of fully computable a posteriori error bounds for approximate eigenfunctions of compact self-adjoint operators in Hilbert spaces [1]. Special attention is paid to the case of tight clusters and multiple eigenvalues because corresponding eigenfunctions are then sensitive to small changes of the problem.

The first type of bounds is based on the Rayleigh quotient and the min-max principle equivalently characterizing the eigenvalue problem. These bounds are easy and quick to compute, they have optimal rates of convergence for single and multiple eigenvalues, and they apply well to situations where the exact problem is uncertain. On the other hand, their accuracy is limited by the positive cluster width and the accumulation of errors from previous clusters.

The second type of bounds is motivated by the Davis–Kahan $\sin\theta$ theorem, which estimates the error by a norm of the residual. We generalize this approach to weakly formulated eigenvalue problems and bound the suitable (usually dual and hence not computable) norm of the residual by the Prager–Synge technique using $H(\mathrm{div})$ conforming flux reconstructions of gradients of approximate eigenfunctions. Compared with the first type bounds, the second type requires more computational work to find the flux reconstructions, but it provides bounds with considerably higher accuracy. Note that bounds with similarly favourable properties were recently derived in [2].

[1] X. Liu, T. Vejchodský, *Fully computable a posteriori error bounds for eigenfunctions,* preprint arXiv:1904.07903.

[2] E. Cancès, G. Dusson, Y. Maday, B. Stamm, and M. Vohralík, *Guaranteed a posteriori bounds for eigenvalues and eigenvectors: multiplicities and clusters,* Math. Comp. **89** (2020), no. 326, 2563–2611.

Optimal experimental design in the deterministic and probabilistic settings

Karen Veroy-Grepl
Eindhoven University of Technology, Netherlands

This is joint work with H. Bansal, N. Cvetković, H. C. Lie, F. Silva, and N. Aretz.

Models of physical processes often depend on parameters, such as material properties or source terms, that are only known with some uncertainty. Measurement data can be used to estimate these parameters and thereby improve the model's credibility. When measurements become expensive, it is important to choose the most informative data. This task becomes even more challenging when the model configurations vary and the data noise is correlated.

In this talk, we consider the problem of optimal experimental design (OED) in the context of PDE-constrained inverse problems. We start with an iterative (greedy) sensor placement strategy based on a stability analysis of a deterministic inverse problem. In this approach, the measurement functionals are chosen to improve an observability coefficient related to the stability of the estimation problem.

We connect this work to Bayesian inversion, particularly to more standard Bayesian OED techniques based on A-, D, and E-optimality criteria. Here, we briefly discuss an optimal sensor placement algorithm for hyper-parameterized Bayesian inverse problems, where the hyper-parameter characterizes nonlinear flexibilities in the forward model, and is considered for a range of possible values. This model variability needs to be taken into account for the experimental design to guarantee that the Bayesian inverse solution is uniformly informative. We briefly describe an algorithm that iteratively chooses the sensor locations to improve the observability coefficient and thereby decrease the eigenvalues of the posterior covariance matrix. This algorithm exploits the structure of the solution manifold in the hyperparameter domain via a reduced basis surrogate solution for computational efficiency. The algorithms are suitable for correlated noise models as well as large-scale forward models, achieving computational efficiency through model order reduction. Finally, we reflect on the implications on OED, and on data assimilation in general, of modelling errors and biases.

Acknowledgments: This work was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (ERC Consolidator Grant agreement No. 818473) and by the German Research Foundation (DFG) through Grants GSC 111 and 33849990/GRK2379 (IRTG Modern Inverse Problems).

Semigroup generation for Biot-Stokes interactions

Justin Webster
University of Maryland, Baltimore County, USA

The coupling of poro-elastic dynamics (given by the inertial Biot equations) to free-flow (incompressible Stokes equations) is an interesting model, known as the filtration system. It is motivated by both geosciences and biological systems, where free flows may run adjacent to a saturated poro-elastic solid and the Beavers-Joseph-Saffman (slip) condition is in force. Although the system has been treated abundantly on the numerical side of the literature, very few papers address the theory of solutions for this particular coupling. In the simplified setting of a 3D periodic box, we demonstrate semigroup generation (providing strong and generalized solutions) for the coupled system. The definition of the generator is particularly delicate, which plays out in the proof of maximality. Following the elimination of pressure (using the approach of Avalos and Triggiani), we obtain a mixed variational formulation which is then solved via LBB, yielding semigroup generation. Time-permitting we discuss the construction of weak solutions, including the degenerate case of incompressible constituents. This work is joint with E. Gurvich and G. Avalos.

Lagrangian schemes for Wasserstein Gradient Flows

Marie-Therese Wolfram
University of Warwick, Coventry, UK

Multiscale analysis, low Mach number limit: from compressible to incompressible system

Aneta Wróblewska-Kamińska
Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland

We will discuss asymptotic analysis for hydrodynamic system, as Navier–Stokes–Fourier system, as a useful tool in in the situation when certain parameters in the system – called characteristic numbers – vanish or become infinite. The choice of proper scaling, namely proper system of reference units, the parameters determining the behaviour of the system under consideration allow to eliminate unwanted or unimportant for particular phenomena modes of motion. The main goal of many studies devoted to asymptotic analysis of various physical systems is to derive a simplified set of equations - simpler for mathematical or numerical analysis. Such systems may be derived in a very formal way, however we will concentrate on rigorous mathematical analysis. I will concentrate on low Mach number limits with so called ill-prepared data and I will present some results wich concerns passage from compressible to incompressible models of fluid flow emphasising difficulties characteristic for particular problems. In particular we will discuss Navier–Stokes–Fourier system on varying domains, a multi-scale problem for viscous heat-conducting fluids in fast rotation and the incompressible limit of compressible finitely extensible nonlinear bead-spring chain models for dilute polymeric fluids.

[1] D. Del Santo, F. Fanelli, G. Sbaiz, A. Wróblewska-Kamińska. On the influence of gravity in the dynamics of geophysical flows. *Mathematics in Engineering.* 2023, Volume 5, Issue 1: 1–33.

[2] D. Del Santo, F. Fanelli, G. Sbaiz, A. Wróblewska-Kamińska. A multi-scale problem for viscous heat-conducting fluids in fast rotation. *Journal of Nonlinear Science*, 2021.

[3] O. Kreml, V. Mácha, Š. Nečasová, A. Wróblewska-Kamińska. Low stratification of a heat-conducting fluid in time-dependent domain. *Journal of Evolution Equations.* 2021.

[4] E. Süli, A. Wróblewska-Kamińska, The incompressible limit of compressible finitely extensible nonlinear bead-spring chain models for dilute polymeric fluids. *Journal of Differential Equations*, 2020.

[5] A. Wróblewska-Kamińska, Asymptotic analysis of complete fluid system on varying domain: form compressible to incompressible flow. *SIAM Journal on Mathematical Analysis.* 49(5), 3299–3334, 2017.

Consensus of a fractional model in attitude dynamics

Małgorzata Wyrwas
Faculty of Computer Science, Bialystok University of Technology, Białystok, Poland

This is a joint work with Professors: **Jerzy Baranowski** (Department of Automatic Control and Robotics, AGH University of Science & Technology, Kraków, Poland), **Waldemar Bauer** (Department of Automatic Control and Robotics, AGH University of Science & Technology, Kraków, Poland), **Karolina Dukała** (SWPS University of Social Sciences and Humanities, Katowice, Poland) and **Dorota Mozyrska** (Faculty of Computer Science, Bialystok University of Technology, Białystok, Poland). The work of M. Wyrwas and D. Mozyrska was supported by Bialystok University of Technology grant No. WZ/WI-IIT/1/2020 and funded by the resources for research by Ministry of Science and Higher Education.

The shaping of attitudes and opinions in societies and groups is a topic of constant interest, practically from the dawn of time. How people vote, what they support, who do they like, who do they hate - these questions are investigated both by researchers and by users of such knowledge. Possible modelling tools joining psychology, consensus modelling and fractional calculus are discussed. We base on the methods of consensus modelling which can be traced to works of Krause, see for instance [1]. This classical model introduced by Krause is sometimes referred to as the Hegselmann–Krause model, see for instance [2]. Use of classical models is however not adequate, as it does not capture the memory based influences on human behavior. That is why we have used fractional calculus, which is an emerging topic in multi-agent modelling. Fractional models have infinite memory, and can be understood as relatively simple extension of traditional calculus. We use the methods of system analysis, in particular stability analysis in order to formulate some results on possibilities of consensus arising in the modeled group of agents. To do so we use fractional difference equations, which illustrate our considerations in different levels of agent complication. The most important goal in opinion dynamics is to find a model with which recognize and potentially predict the tendency of a group of individuals into the direction of common opinion. When a group of a team, a committee or consumers, called generally agents, takes with time the same or very close similar opinion or way of behaviors we can say that consensus is reached by the group.

We propose a new interdisciplinary model for attitude dynamics and study of its certain properties in continuous and discrete time domains. We use the knowledge from psychology to determine the parametric structure of the model. Such model is formulated as a system of nonlinear fractional differential equations which can be used to study the behaviour of individual agents. We also formulate the model of opinion differences, which along with the use of stability theory allows determination of whether consensus among the agents in the system is possible.

[1] U. Krause, A discrete nonlinear and non-autonomous model of consensus formation, Proc. Commun. Difference Equations (2000) 227–236.

[2] V. D. Blondel, J. M. Hendrickx, F. N. Tsitsiklis, On Krause's multi-agent consensus model with state-dependent connectivity, IEEE Transactions on Automatic Control 5 (11) (2009) 2586–2597.

Infinitely many solutions to the isentropic system of gas dynamics

Cheng Yu
University of Florida, USA

In this talk, I will discuss the non-uniqueness of global weak solutions to the isentropic system of gas dynamics. In particular, I will show that for any initial data belonging to a dense subset of the energy space, there exists infinitely many global weak solutions to the isentropic Euler equations for any $1 < \gamma \leq 1 + 2/n$. The proof is based on a generalization of convex integration techniques and weak vanishing viscosity limit of the Navier-Stokes equations. This talk is based on the joint work with M. Chen and A. Vasseur.

Nonisothermal Richards flow in porous media with cross diffusion

Nicola Zamponi
TU Vienna, Vienna, Austria

This is a joint work with Dr. Esther Daus (formerly TU Vienna, Vienna, Austria) and Prof. Pina Milišić (University of Zagreb, Zagreb, Croatia).

The existence of large-data weak solutions to a nonisothermal immiscible compressible two-phase unsaturated flow model in porous media is proved. The model is thermodynamically consistent and includes temperature gradients and cross-diffusion effects. Due to the fact that some terms in the energy flux are not integrable, so-called "variational entropy solutions" are considered, which satisfy the partial mass balance equations, the *integrated* total energy equation, and the entropy balance *inequality*. A priori estimates are derived from the entropy balance and the total energy balance. A sequence of approximated solutions is constructed by introducing a time semi-discretization as well as higher order and lower order regularising terms. Finally, the compactness of the sequence of approximated solutions is showed by employing the Div-Curl lemma.

Positive solutions for second order damped boundary value problems

Mirosława Zima
University of Rzeszów, Rzeszów, Poland

We study the existence of positive solutions to the damped regular differential equation

$$x'' + cx' = g(t)x^{\lambda} - h(t)x^{\mu} \tag{1}$$

under periodic or Neumann boundary conditions, where $c \in \mathbb{R}$, $g, h \in C([0,T])$ and λ, $\mu > 0$. In particular, we apply an averaging type method developed in [1] for the equation

$$x'' + cx' = \nu g(t)x^{\lambda} - h(t)x^{\mu},$$

where $\nu > 0$ is a small parameter. Following the approach initiated in [2], we also use a lower and upper solutions technique for (1). The applications for a periodic problem related to the Liebau phenomenon and for a Neumann boundary value problem arising in fluid dynamics will be discussed. The talk is based on the joint paper [3].

[1] J. A. Cid, J. Mawhin, M. Zima, *An abstract averaging method with applications to differential equations* J. Differ. Equ. **274** (2021), 231–250.

[2] R. Hakl, P. J. Torres, *Existence and stability of periodic solutions of a Duffing equation by using a new maximum principle* J. Differ. Equ. **248** (2010), 111–126.

[3] F. Wang, J. A. Cid, S. Li, M. Zima, *Existence results for damped regular equations under periodic or Neumann boundary conditions* J. Math. Anal. Appl. **509** (2022), 125978.

Abstracts – contributed talks

Finite-approximate controllability of fractional functional evolution equation via variational approach

Sumit Arora

Indian Institute of Technology Roorkee, Roorkee, India

This is a joint work with Prof. Manil T. Mohan (Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, 247667, India) and Prof. Jaydev Dabas (Department of Applied Mathematics and Scientific Computing (Former name-Applied Science and Engineering), Indian Institute of Technology Roorkee, Roorkee, 247667, India).

In this talk, we discuss the finite approximate controllability of fractional functional evolution equations in Hilbert spaces. Here, we establish sufficient conditions for the finite-approximate controllability of the semilinear fractional evolution equations with delay via variational approach. We also find an appropriate control to get the approximate controllability of the linear system, in addition, it also ensures that the finite-approximate controllability of that problem. Moreover, we provide an example of fractional wave equation to illustrate the efficiency of the developed results.

Existence and regularity results for a class of parabolic problems with double phase flux of variable growth

Rakesh Arora
Masaryk University, Brno, Czech Republic

This is a joint work with Prof. Sergey Shmarev (University of Oviedo, Oviedo, Spain).

This talk presents existence and regularity results of homogeneous Dirichlet problem for the equation

$$u_t - \operatorname{div}\left((a(z)|\nabla u|^{p(z)-2} + b(z)|\nabla u|^{q(z)-2})\nabla u\right) = f \quad \text{in } Q_T = \Omega \times (0,T),$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with $\partial\Omega \in C^2$. The variable exponents p, q and the nonnegative modulating coefficients a, b are given Lipschitz-continuous functions of the argument $z = (x,t) \in Q_T$. It is assumed that $\frac{2N}{N+2} < p(z),\ q(z)$ and that the modulating coefficients and growth exponents satisfy the balance conditions

$$a(z) + b(z) \geq \alpha > 0 \text{ in } \overline{Q}_T,\ \alpha = const; \qquad |p(z) - q(z)| < \frac{2}{N+2} \text{ in } \overline{Q}_T.$$

We find conditions on the source f and the initial data $u(\cdot, 0)$ that guarantee the existence of a unique strong solution u with $u_t \in L^2(Q_T)$ and $a|\nabla u|^p + b|\nabla u|^q \in L^\infty(0,T; L^1(\Omega))$. The solution possesses the property of global higher integrability of the gradient,

$$|\nabla u|^{\min\{p(z),q(z)\}+r} \in L^1(Q_T) \quad \text{with any } r \in \left(0, \frac{4}{N+2}\right),$$

which is derived with the help of new interpolation inequalities in the variable Sobolev spaces. The second-order differentiability of the strong solution is proven:

$$D_{x_i}\left(\left(a|\nabla u|^{p-2} + b|\nabla u|^{q-2}\right)^{\frac{1}{2}} D_{x_j} u\right) \in L^2(Q_T), \quad i,j = 1,2,\ldots,N.$$

Stokes problem with dynamic boundary conditions

Tomáš Bárta
Charles University, Prague, Czech Republic

This is a joint work with Page Davies and Petr Kaplický.

We study the Stokes problem with dynamic boundary conditions

$$\partial_t u - \Delta u + \nabla p = f \qquad \text{in } I \times \Omega, \tag{P1.1}$$

$$\operatorname{div} u = 0 \qquad \text{in } I \times \Omega, \tag{P1.2}$$

$$\partial_t u + (2Du \cdot \nu)_\tau + \alpha u_\tau = f_\tau \qquad \text{in } I \times \partial\Omega, \tag{P1.3}$$

$$u_\nu = 0 \qquad \text{in } I \times \partial\Omega, \tag{P1.4}$$

$$u = u_0 \qquad \text{in } \{0\} \times \Omega \tag{P1.5}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. As usually, $Du = \frac{1}{2}(\nabla u + (\nabla u)^T)$ denotes the symmetric gradient of u. Further, ν is the unit outer normal vector, the subscripts τ, resp. ν denote tangent, resp. normal part of a vector. The unknowns are u and p representing velocity and presure of a fluid.

Since the problem combines evolutionary equations in the interior of Ω and on its boundary, the abstract reformulation works on a space which is a product of spaces in the interior and on the boundary of Ω. We show that the corresponding operator is the generator of an analytic semigroup and since we work in Hilbert spaces we obtain maximal L^p regularity.

Existence and uniqueness of a generalized solution to the one-dimensional flow and thermal explosion micropolar reactive real gas model

Angela Bašić-Šiško
University of Rijeka, Croatia

In this work, we consider the model of one-dimensional flow and thermal explosion of reactive micropolar gas, taking into account the equation of state of real gas given by

$$P = R\rho^p\theta, \tag{1}$$

where P is the pressure, ρ is the mass density, θ is the absolute temperature, and $R > 0$ is a constant. Exponent $p \geq 1$ is called pressure exponent.

First, the corresponding boundary-initial value problem with homogeneous boundary conditions in mass Lagrangian coordinates is derived, then the notion of a generalized solution is defined and its existence and uniqueness are considered. Namely, it was first proved that the described problem has a solution locally in time [4]. To prove this result, a constructive technique involving Faedo-Galerkin approximations was used. It was also shown that the problem admits at most one generalized solution. Finally, using the theorems on local existence and uniqueness, it is shown that the solution can be extended to an interval of arbitrary finite length. The problem is also solved numerically, using the same construction as in the proof of the theorem on local existence.

[1] G. Lukaszewicz, Micropolar Fluids Theory and Applications. Modeling and Simulation in Science, Engineering and Technology. Birkhäuser, Boston, 1999.

[2] S. N. Antontsev, A. V. Kazhikhov, V. N. Monakhov, Boundary value problems in mechanics of nonhomogeneous fluids, Studies in Mathematics and its Applications (vol. 22). North-Holland Publishing Co., Amsterdam, 1990.

[3] M. Lewicka, P. Mucha, *On temporal asymptotics for the pth power viscous reactive gas*, Nonlinear Anal. **57** (2004), 951–969.

[4] A. Bašić-Šiško, I. Dražić, *Local existence for viscous reactive micropolar real gas flow and thermal explosion with homogeneous boundary conditions*, J. Math. Anal. Appl. **509**(2) (2022), 125988.

[5] A. Bašić-Šiško, I. Dražić, L. Simčić, *One-dimensional model and numerical solution to the viscous and heat-conducting micropolar real gas flow with homogeneous boundary conditions*, Math. Comput. Simulation **195**(2) (2022), 71–87.

[6] A. Bašić-Šiško, I. Dražić, *Uniqueness of generalized solution to micropolar viscous real gas flow with homogeneous boundary conditions*, Math. Methods Appl. Sci. **44**(6) (2021), 4330–4341.

[7] A. Bašić-Šiško, I. Dražić, *Global solution to a one-dimensional model of viscous and heat-conducting micropolar real gas flow*, J. Math. Anal. Appl. **495**(1) (2021), 124690.

Global stability for price models with delay

István Balázs
Bolyai Institute, University of Szeged, Hungary

This is a joint work with prof. Tibor Krisztin (Bolyai Institute, University of Szeged, Hungary).

After a short introduction to the field of delay and neutral differential equations, we consider the delay equation

$$\dot{x}(t) = a \int_0^r x(t-s)\, d\eta(s) - g(x(t)) \tag{1}$$

and the neutral type equation

$$\dot{y}(t) = a \int_0^r \dot{y}(t-s)\, d\mu(s) - g(y(t)) \tag{2}$$

where $a > 0$, $g : \mathbb{R} \to \mathbb{R}$ is smooth, $ug(u) > 0$ for $u \neq 0$, $\int_0^s g(u)\, du \to \infty$ as $|s| \to \infty$, $r > 0$, η and μ are nonnegative functions of bounded variation on $[0, r]$, $\eta(0) = \eta(r) = 0$, $\int_0^r \eta(s)\, ds = 1$, μ is nondecreasing, μ does not have a singular part, $\int_0^r d\mu = 1$. Both equations can be interpreted as price models. Global asymptotic stability of $y = 0$ is obtained, in case $a \in (0, 1)$, for the neutral equation by using a Lyapunov functional. Then this result is applied to get global asymptotic stability of $x = 0$ for the (non-neutral) delay differential equation provided $a \in (0, 1)$. As particular cases, two related global stability conjectures are solved, with an affirmative answer.

[1] I. Balázs, T. Krisztin, *Global stability for price models with delay,* J. Dynam. Differential Equations. **31(3)** (2019), 1327–1339.

Normal and binormal motion of interacting curves in space

Michal Beneš
Faculty of Nuclear Sciences and Physical Engineering Czech Technical University in Prague, Prague, Czech Republic

We investigate a system of geometric evolution equations describing the curvature driven motion of a family of space curves in the normal and binormal directions. Evolving curves can interact with each other locally or nonlocally. The evolution of curves $\Gamma^i, i = 1, \ldots, n$ is treated parametrically by the geometric motion law

$$\partial_t \mathbf{X}^i = v_N^i \mathbf{N^i} + v_B^i \mathbf{B^i} + v_T^i \mathbf{T}^i, \quad i = 1, \ldots, n, \tag{1}$$

where the unit tangent $\mathbf{T}^i$, normal $\mathbf{N^i}$ and binormal $\mathbf{B^i}$ vectors form the Frenet frame, and $\mathbf{X}^i : S^1 \times [0, \infty) \to \mathbb{R}^3$ are the parametrizations of Γ_t^i. The velocity expressions v_N^i, v_B^i, v_T^i may be functions of the position vector $\mathbf{X}^i \in \mathbb{R}^3$, the curvature κ^i, the torsion τ^i, and of all parametrized curves $\Gamma^i, i = 1, \ldots, n$.

As indicated in [1, 2], motion (1) of one-dimensional structures forming space curves can be identified in variety of problems arising in science and engineering. Among them, one of the oldest is the dynamics of one-dimensional vortex structures forming a vortex ring. One-dimensional structures can also be formed within the crystalline lattice of solid materials. Certain class of nano-materials is produced by electrospinning - jetting polymer solutions in high electric fields into ultrafine nanofibers behaving according to (1). Some linear molecular structures with specific properties exist inside cells and exhibit a dynamics in terms of (1).

Equations (1) form a system of nonlinear parabolic partial differential equations for $v_N > 0$. Using the theory of nonlinear analytic semi-flows, we prove local existence, uniqueness and continuation of its classical Hölder smooth solutions. Using the finite volume method, we construct an efficient scheme for numerical solution of (1). Moreover, a nontrivial tangential velocity is considered allowing for the redistribution of discretization nodes. We demonstrate behavior of the solution on several computational studies of the flow combining the normal and binormal velocity and considering nonlocal interactions.

[1] M. Beneš, M. Kolář and D. Ševčovič, *Qualitative and Numerical Aspects of a Motion of a Family of Interacting Curves in Space*, SIAM J. Appl. Math. **82**(2) (2022), 10.1137/21M1417181.

[2] M. Beneš and M. Kolář and D. Ševčovič, *Curvature driven flow of a family of interacting curves with applications*, Math. Meth. Appl. Sci. **43** (2020), 4177–4190.

Maximum principles for parabolic p-Laplacian problems

Jiří Benedikt
University of West Bohemia, Plzeň, Czechia

This is a joint work with Profs. Peter Takáč (University of Rostock, Germany), Petr Girg and Lukáš Kotrla (University of West Bohemia, Plzeň, Czechia).

We are concerned with maximum and comparison principles for both the elliptic and the parabolic p-Laplacian in a domain in $\mathbb{R}^N$, with zero Dirichlet boundary conditions. We discuss several aspects that affect the validity of the principle:

- Whether we speak about the maximum or the comparison principle since in the nonlinear case they are not equivalent.
- Whether we speak about the weak or the strong version of a principle.
- Whether we consider the elliptic (time-independent) or the parabolic case. In particular, in the parabolic case there may be a difference between the time-local and the time-global principle.
- Whether we have the singular ($1 < p < 2$, fast diffusion), the linear ($p = 2$, standard diffusion), or the degenerate ($p > 2$, slow diffusion) case.

We also discuss the influence of a non-lipschitz reaction term.

New asymptotic stability conditions for linear time-varying fractional systems

Swaroop Nandan Bora

Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati, India

This is a joint work with Dr. Bichitra Kumar Lenka, Post Doctoral Fellow, Department of Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati-781039, India.

This article gives a representation of linear time-varying incommensurate fractional-order systems defined in the light of Caputo fractional derivative that involves real orders lying in the interval $(0, 1)$. By making some elementary assumptions on the coefficient matrix of such a system and following a new fractional comparison method, new asymptotic stability results are put forward for such systems. Two new distinct theorems and several new analytic criteria are proposed to verify the asymptotic behaviours of the zero solutions of such systems. Some examples are provided for such systems.

Mathematically, the initial value problem of linear time-varying fractional system can be represented as [2]:

$$ {}^{C}D^{\widehat{\alpha}}_{0,t}x(t) = A(t)x(t) \tag{1} $$

subject to the initial condition $x(0) = x_0$, where state variable $x(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{R}^n$, vector Caputo derivative operator ${}^{C}D^{\widehat{\alpha}}_{0,t}x(t) = \left({}^{C}D^{\alpha_1}_{0,t}x_1(t), \ldots, {}^{C}D^{\alpha_n}_{0,t}x_n(t)\right)^T \in \mathbb{R}^n$ [3], fractional orders $\alpha_1, \ldots, \alpha_n \in (0, 1]$ and the matrix $A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}$ is continuous on $[0, \infty)$. When $A(t) = A$ is a constant matrix, the system (1) reduces to an autonomous system [1].

We set some basic elementary assumptions on $A(t)$, i.e., $a_{ii}(t) \le \delta_{ii}$, $\delta_{ii} > 0$ and $|a_{ij}(t)| \le \delta_{ij}$, $\delta_{ij} \ge 0$, $\forall t \ge 0$. We let the constants $c_{ii} = 2\delta_{ii} - \sum_{j=1, j\neq i}^{n} \delta_{ij}$ for $i = 1, 2, \ldots, n$ and matrix $\Delta = \begin{pmatrix} -c_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{21} & -c_{22} & \cdots & \delta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n1} & \delta_{n2} & \cdots & c_{nn} \end{pmatrix}$.

One of our main results is now stated below.

Theorem 1. *If the criteria*

(i) $c_{11}, c_{22}, \ldots, c_{nn}$ *are positive for* $i = 1, 2, \ldots, n$.

(ii) every root of $\det\left[diag\left(s^{\alpha_1}, s^{\alpha_2}, \ldots, s^{\alpha_n}\right) - \Delta\right] = 0$ *lies in the sector* $|\arg(s)| > \frac{\pi}{2}$,

are satisfied, then the zero solution to system (1) *is globally asymptotic stable.*

[1] W. Deng, C. Li, J. Lü, *Stability analysis of linear fractional differential system with multiple time delays.* Nonlinear Dyn, **48** (2007), 409–416.

[2] B. K. Lenka, S. N. Bora, *New asymptotic stability conditions for linear time-varying fractional systems.* Submitted to a Journal for the publication, (2022).

[3] I. Podlubny, Fractional differential equations. Academic Press, San Diego, 1999.

Bouncing solutions of generalized Lazer–Solimini equation

Jana Burkotová
Faculty of Science, Palacký University Olomouc, Czech Republic

This is a joint work with Jan Tomeček and Irena Rachůnková (Faculty of Science, Palacký University Olomouc, Czech Republic) and Jakub Stryja (VŠB - Technical University Ostrava, Czech Republic). We investigate the singular nonlinear differential equation of the second order

$$x'' + g(x) = p(t),$$

where g has a weak singularity at $x = 0$ and p is a continuous 2π-periodic function. We consider the case of the attractive singularity of g, that is $\lim_{x\to 0+} g(x) = \infty$. We derive sufficient conditions for the existence of 2π-periodic bouncing solutions which reach the singularity at isolated points. The results are based on the Generalized Poincaré–Birkhoff Twist Map Theorem for a successor mapping with the area–preserving property.

[1] J. Tomeček, I. Rachůnková, J. Burkotová, J. Stryja, *Coexistence of bouncing and classical periodic solutions of generalized Lazer–Solimini equation*, Nonlinear Analysis **196** (2020), 111783.

Problem-dependent formulas for solving stiff problems

Shuenn-Yih Chang

Department of Civil Engineering, National Taipei University of Technology, Taipei, Taiwan, ROC

This is a joint work with Chiu-Li Huang (Fu-Jen Catholic University, New Taipei City, Taiwan, ROC).

Unlike classical formulas, a series of linear multi-step formulas in a problem-dependent form are proposed for solving initial value problems. They are derived from an eigenmode concept and can be characterized by problem dependency. This is because that their coefficients can be functions of the product of initial Jacobian matrix for defining the problem under analysis and step size. Thus, coefficients can be in a matrix form. The development of this series of problem-dependent formulas is presented and their numerical properties are intensively studied. The first member of this series is a one-step formula and it can integrate A-stability, second order accuracy and explicitness of each step. Consequently, it is best suited to solve stiff problems. It is also found that the two-step method can have a third accuracy and three-step method can have a fourth order accuracy. It is also affirmed that thee problem-dependent formulas can have either a non-iterative or iterative implementation in the solution of initial value problems. The combination of A-stability, second order accuracy and explicitness of each step for the first member of this series of formulas is computationally efficient for solving stiff initial value problems. All the numerical properties and computational efficiency of the proposed formulas are substantiated by numerical tests. It should be mentioned that no classical formulas that are A-stable, second order accurate and explicit.

Behavior of solutions to a quasilinear attraction-repulsion chemotaxis system

Yutaro Chiyo
Tokyo University of Science, Japan

This talk deals with global existence, boundedness and finite-time blow-up in a quasilinear parabolic–elliptic–elliptic attraction-repulsion chemotaxis system with/without logistic source. This system consists of the following three equations: a parabolic partial differential equation, for the density of organisms, involving the nonlinear diffusion, attraction and repulsion terms; elliptic partial differential equations for the chemoattraction and the chemorepulsion. Here, it is expected that the attraction term promotes blow-up, and the repulsion term leads to boundedness, where blow-up represents the concentration of organisms, boundedness idealizes absence of chemotactic collapse. In support of this, as to the *linear* version of the system, boundedness has been proved in the case that the repulsion is stronger than the attraction, while blow-up has been shown in the opposite case. Therefore it is expected that the balance between the attraction and repulsion terms classifies boundedness and blow-up in the *quasilinear* system. More precisely, let p and q be the powers appearing in the attraction and repulsion terms involved by the *quasilinear* system. Then the following question arises.

Are boundedness and blow-up in the quasilinear system classified by the sizes of p and q?

The purpose is to give an answer to the question. The results presented in this talk are based on [1]. This is a joint work with Professor Tomomi Yokota (Tokyo University of Science).

[1] Y. Chiyo, T. Yokota, *Boundedness and finite-time blow-up in a quasilinear parabolic–elliptic–elliptic attraction-repulsion chemotaxis system.* Z. Angew. Math. Phys. **73** (2022), Paper No. 61, 27 pages.

Iterative discontinuous Galerkin finite element method for strongly monotone quasi-linear PDEs

Scott Congreve
Charles University, Czech Republic

In this talk we extend our previous work on the numerical approximation of monotone, quasi-linear elliptic boundary value problems as obtained from combining Banach fixed-point iterations and *hp*-type finite element methods [1] to *hp*-version discontinuous Galerkin finite element methods (DGFEMs).

As opposed to Newton's method, which requires information from the previous iteration in order to linearise the iteration matrix (and thereby to recompute it) in each step, the alternative fixed-point method used exploits the monotonicity properties of the problem, and only needs the iteration matrix calculated once for all iterations of the fixed-point method. We outline the *a priori* and *a posteriori* error estimates for iteratively obtained solutions, and show both theoretically as well as numerically how the number of iterations of the fixed-point method can be restricted in dependence of the mesh size, or of the polynomial degree, to obtain optimal convergence.

Specifically, we focus on finding an approximate solution of the problem

$$-\nabla\cdot(\mu(\boldsymbol{x},|\nabla u|)\nabla u)+f(\boldsymbol{x},u)=0 \qquad \text{in } \Omega, \tag{1}$$

$$u=0 \qquad \text{on } \Gamma, \tag{2}$$

where Ω is a bounded, open, polygonal domain in $\mathbb{R}^d$, $d=2,3$, with boundary Γ. We assume certain conditions on the non-linearities such that the resulting formulation is *strongly monotone* and *Lipschitz continuous*.

[1] S. Congreve and T. P. Wihler. Iterative Galerkin discretizations for strongly monotone problems. *J. Comput. Appl. Math.* 311 (2017), 457–472.

D-stability of the initial value problem for symmetric nonlinear functional differential equations

Natalia Dilna

Institute of Mathematics, Slovak Academy of Sciences, Slovakia

This is a joint work with Prof. M. Fečkan (Department of Math. Analysis and Numerical Mathematics, Comenius University, Bratislava, Slovakia) and Dr. Mykola Solovyov (Superconductors Department, Institute of Electrical Engineering, Slovak Academy of Sciences, Bratislava, Slovakia).

We offer a method of establishing the D-stability terms of the symmetric solution of scalar symmetric linear and nonlinear functional differential equations (NFDEs):

$$x'(t)+\sum_{i=1}^{m} g_i(t)x(\nu_i(t))-\sum_{i=1}^{m} p_i(t)x(\mu_i(t))-f(x(\tau_1(t)),x(\tau_2(t)),\dots,x(\tau_m(t)),x(t),t)=q(t), \quad (1)$$

with initial value condition

$$x(t_0)=\alpha, \quad (2)$$

where $t\in\mathbb{R}$, function $f:\mathbb{R}^{m+2}\to\mathbb{R}$ is continuous, $m\geq 1$, $\mu_i,\nu_i,\tau_i:\mathbb{R}\to\mathbb{R}$ are measurable functions, and $p_i,g_i,q \in L(\mathbb{R},\mathbb{R})$, $i=1,2,\dots,m$ and with the symmetric property $x(t)=x(\psi(t))$, $t\in\mathbb{R}$, where ψ is a monotonously increasing C^1-function.

Definition, [1]. The NFDE (1) has a D-stable property in the vicinity of the trivial solution $x\equiv 0$ if there exist such $\delta_0>0$ that for every pair $\{q,\alpha\}\in L\times\mathbb{R}$, satisfying conditions $\|q\|_L<\delta_0$, $|\alpha|<\delta_0$, the problem (1), (2) has a unique solution $x\in D$ and for arbitrary $\varepsilon>0$ there exists such $\delta=\delta(x,\varepsilon)>0$ that $\|x_1-x\|_D<\varepsilon$ if $\|q_1-q\|_L<\delta$, $|\alpha_1-\alpha|<\delta$, where x_1 is the solution of the problem (1), (2) with $q=q_1$, $\alpha=\alpha_1$ and $\|q_1\|_L<\delta_0$, $|\alpha_1|<\delta_0$.

Theorem, [2]. Assume there exist $K>0$, $M>0$ and $\delta>0$ such that, for all $\|y_i\|_D\leq\delta$, $i=1,2$, the inequalities

$$\Big\|\sum_{i=1}^{m}(p_i(t)(\xi_i y_1)(t)-g_i(t)(\kappa_i y_1)(t))-\sum_{i=1}^{m}(p_i(t)(\xi_i y_2)(t)-g_i(t)(\kappa_i y_2)(t))\Big\|_L\leq K\|y_1-y_2\|_D,$$

$$\|f\Big((\sigma_1 y_1),(\sigma_2 y_1),\dots,(\sigma_m y_1),y_1,t\Big)-f\Big((\sigma_1 y_2),(\sigma_2 y_2),\dots,(\sigma_m y_2),y_2,t\Big)\|_L\leq M\|y_1-y_2\|_D$$

are true and $(\psi(t_0)-t_0)(K+M)<1$. If $f(0,0,\dots,0,t)=0$, then the problem (1), (2) is D-stable in the vicinity of the trivial solution, where for the number $l(t)$ of the intervals that contain a point $t\in\mathbb{R}$ operators $\{\xi_i,\kappa_i,\sigma_i\}:C([t_0,\psi(t_0)],\mathbb{R})\to L([t_0,\psi(t_0)],\mathbb{R})$, $i=1,2,\dots,m$, are defined:

$$(\xi_i x)(t):=\begin{cases} x(\mu_i(t)),\mu_i(t)\in[t_0,\psi(t_0)],\\ x\left(\psi^{-l(\mu_i(t))}(\mu_i(t))\right),\mu_i(t)\notin[t_0,\psi(t_0)],\end{cases} \qquad (\kappa_i x)(t):=\begin{cases} x(\nu_i(t)),\nu_i(t)\in[t_0,\psi(t_0)],\\ x\left(\psi^{-l(\nu_i(t))}(\nu_i(t))\right),\nu_i(t)\notin[t_0,\psi(t_0)],\end{cases}$$

$$(\sigma_i x)(t):=\begin{cases} x(\tau_i(t)),\tau_i(t)\in[t_0,\psi(t_0)],\\ x\left(\psi^{-l(\tau_i(t))}(\tau_i(t))\right),\tau_i(t)\notin[t_0,\psi(t_0)].\end{cases}$$

[1] N.V. Azbelev, M.B. Ermolaev, P.M. Simonov, *On the stability of functional-differential equations with respect to the first approximation.* Izv. Vyss. Uchebnykh Zaved. Mat. **39** (1995), 3–9

[2] N. Dilna, M. Fečkan, M. Solovyov, *D-stability of the initial value problem for symmetric nonlinear functional differential equations.* Symmetry **12** (2020), 1761; https://doi.org/10.3390/sym12111761

Discontinuous Galerkin method for nonlinear partial differential equations: goal-oriented error estimates

Vít Dolejší
Charles University, Prague, Czech Republic

We deal with the numerical solution of nonlinear partial differential equations (PDEs) by the discontinuous Galerkin (DG) method. The DG discretization leads to a system of nonlinear algebraic equations which have to be solved iteratively by a suitable solver. Although the Newton method is frequently used, it is not always advanatgeous, namely in situations where the problem lacks the regularity.

In practical applications, we are frequently not interested in the approximate solution of PDEs itself but in the value of a certain solution-dependent functional. The error of the corresponding quantity of interest can be approximated by a posteriori goal-oriented error estimates which exhibit an efficient tool for the numerical solution of PDEs. The framework of the goal-oriented error estimates is based on the formulation and solution of adjoint problem [1]. For nonlinear PDEs, the adjoint problem is usually based on the Jacobian of the primal problem.

We develop another approach where the adjoint problem employs the linearization used in the iterative solver of the DG discretization [2]. However, the linearization have to guarantee the adjoint consistency of the numerical scheme. Extending the results of [3], we propose stopping criteria for iterative solvers which balances the discretization and algebraic errors. Furthermore, the derived estimates are combined with anisotropic *hp*-mesh adaptation method. Several numerical examples demonstrate the efficiency of this algorithm.

[1] Wolfgang Bangerth and Rolf Rannacher. Adaptive finite element methods for differential equations. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2003.

[2] V. Dolejší, O. Bartoš, and F. Roskovec. *Goal-oriented mesh adaptation method for nonlinear problems including algebraic errors.* Comput. Math. Appl. **93** (2021), 178–198

[3] R. Rannacher and J. Vihharev. *Adaptive finite element analysis of nonlinear problems: Balancing of discretization and iteration errors.* J. Numer. Math. **2** (2013), 23–61.

On Szegö–Weinberger inequalities for special type of domains

Pavel Drábek

University of West Bohemia in Pilsen, Czech Republic

This is a joint work with T. V. Anoop (Indian University of Technology, Madras, India) and Vladimir Bobkov (Ufa Federal Research Centre, RAS, Ufa, Russia).

We consider the Neumann eigenvalue problem in a bounded Lipschitz domain $\Omega \subset \mathbf{R}^N$:

$$-\Delta u = \mu u \quad \text{in} \quad \Omega, \qquad \partial \mathrm{u}/\partial \mathrm{n} = 0 \quad \text{on} \quad \partial\Omega$$

with a discrete sequence of eigenvalues $0 = \mu_1(\Omega) < \mu_2(\Omega) \leq \mu_3(\Omega) \leq \ldots$. Then the Szegö–Weinberger inequality states that $\mu_2(\Omega) \leq \mu_2(B)$, where B is an open ball of the same Lebesgue measure as Ω, $|\Omega| = |B|$ and if equality holds then $\Omega = B$ up to a set of zero Lebesgue measure. This inequality has been refined and improved in different ways, in particular, in the dimension $N = 2$ and for domains with symmetries, see the references in [1].

The purpose of our work is to generalize the Szegö–Weinberger inequality for domains with symmetries in higher dimension $N \geq 2$ which have "holes". Namely, we are interested in domains $\Omega = \Omega_{out} - \overline{\Omega}_{in}$, where Ω_{in} is (the hole) compactly contained in the domain Ω_{out} and $0 \in \Omega_{in}$. We say that the domain $\Omega \subset \mathbf{R}^N$ is symmetric of order q if for any couple of indices i, j such that $1 \leq i < j \leq N$ this domain is invariant under the rotation by angle $\frac{2\pi}{q}$ in every coordinate plane (x_i, x_j). Let $x \in \Omega$ if and only if $-x \in \Omega$. Then Ω is centrally symmetric. In what follows B_γ stands for the open ball of radius $\gamma > 0$ centered at the origin.

The main results of our work can be stresed as follows:

Let Ω be a domain with hole, $\Omega = \Omega_{out} - \overline{\Omega}_{in}$, $0 \leq \alpha < \beta$ be such that $B_\alpha \subset \Omega_{in}$ and $|\Omega| = |B_\beta - \overline{B}_\alpha|$. Then the following assertions hold:

(i) If Ω is either symmetric of order 2 or centrally symmetric, then

$$\mu_2(\Omega) \leq \mu_2(B_\beta - \overline{B}_\alpha).$$

(ii) If Ω is symmetric of order 4, then

$$\mu_i(\Omega) \leq \mu_i(B_\beta - \overline{B}_\alpha) = \mu_2(B_\beta - \overline{B}_\alpha)$$

for $i = 3, \ldots, N + 1$, and

$$\mu_{N+2}(\Omega) \leq \mu_{N+2}(B_\beta - \overline{B}_\alpha).$$

(iii) If $N = 2$ and Ω is symmetric of order 8, then

$$\mu_5(\Omega) \leq \mu_4(B_\beta - \overline{B}_\alpha) = \mu_5(B_\beta - \overline{B}_\alpha).$$

If equality holds in above inequalities, then Ω coincides a.e. with $B_\beta - \overline{B}_\alpha$.

Detailed proofs as well as references, illustrations and examples can be found in [1].

[1] T. V. Anoop, Vladimir Bobkov and Pavel Drábek, *Szegö–Weinberger type inequalities for symmetric domains with holes*, SIAM J. Math. Anal. **54** (2022), 389–422.

Critical points for reaction diffusion system with unilateral conditions

Jan Eisner

Faculty of Science, University of South Bohemia, České Budějovice, Czech Republic

This is a joint work with Martin Väth (Free University, Berlin, Germany).

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary. Consider the reaction-diffusion system

$$u_t = d_1 \Delta u + g_1(u,v)\,, \quad v_t = d_2 \Delta v + g_2(u,v) \qquad \text{on } \Omega \tag{1}$$

with boundary conditions

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \setminus \Gamma_1\,, \quad \frac{\partial u}{\partial n} \in m_1(u) \text{ on } \Gamma_1\,, \quad \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \setminus \Gamma_2\,, \quad \frac{\partial v}{\partial n} \in m_2(v) \text{ on } \Gamma_2 \tag{2}$$

with measurable given sets $\Gamma_1, \Gamma_2 \subseteq \partial\Omega$.

We are interested in bifurcation of stationary solutions of (1), (2) near a constant equilibrium $(\overline{u}, \overline{v})$ which is linearly stable without diffusion terms (i.e. if $d_1 = d_2 = 0$).

In case $\Gamma_1 = \Gamma_2 = \emptyset$, Turing's famous effect of "diffusion-driven instability" [1] implies that for the above problem the equilibrium $(\overline{u}, \overline{v})$ becomes instable for certain values of diffusion speeds $d_1, d_2 > 0$ and spatially nonconstant stationary solutions bifurcate from the equilibrium when $d = (d_1, d_2)$ crosses certain hyperbolas C_n.

We are interested in the question of what changes in these observations when we impose on Γ_i obstacles described by the (in general multivalued) functions m_i. More precisely, we will assume $(\overline{u}, \overline{v}) = (0,0)$ and $m_i = m$ or $m_i(u) = -m(-u)$ where m is (or behaves in a certain weak sense near 0 qualitatively similar to) the function

$$m_0(u) := \begin{cases} [0,\infty) & \text{if } u = 0, \\ \{0\} & \text{if } u > 0, \\ \emptyset & \text{if } u < 0 \end{cases} \tag{3}$$

or that $m(v) \neq \emptyset$,

$$\lim_{v \to 0^-} (\inf m(v))/v = -\infty \qquad \text{or} \qquad \limsup_{v \to 0^-} (\inf m(v))/v < 0, \tag{4}$$

$$\inf m(0) \geq 0 \quad \text{and} \quad \lim_{v \to 0^+} \sup \pm m(v) = 0. \tag{5}$$

We show the nonexistence of bifurcation of stationary solutions near certain critical parameter values. The result implies assertions about a related mapping degree which in turn implies for "small" obstacles the existence of a new branch of bifurcation points (spatial patterns) induced by the obstacle.

[1] A. M. Turing, The chemical basis of morphogenesis, Phil. Trans. R. Soc. London Ser. B 237 (641) (1952) 37–72.

[2] J. Eisner, M.Väth, *Degree, instability and bifurcation of reaction–diffusion systems with obstacles near certain hyperbolas.* Nonlinear Anal. **135** (2016), 158–193.

Error estimates of a semi-discrete numerical scheme for nonlocally regularized KdV-type equations

Husnu Ata Erbay
Ozyegin University, Istanbul, Turkey

This is a joint work with Prof Saadet Erbay (Ozyegin University, Istanbul, Turkey) and Prof Albert Erkip (Sabanci University, Istanbul, Turkey).

Numerical approximation of a general class of nonlinear unidirectional wave equations with a convolution-type nonlocality in space is considered:

$$u_t + \alpha * \left((f(u))_x + \kappa u_{xxx}\right) = 0.$$

Here the symbol $*$ denotes the convolution operation in space, $(\alpha * v)(x) = \int_{\mathbb{R}} \alpha(x-y)v(y)\mathrm{d}y$, α is a sufficiently smooth kernel function and κ is a positive constant. Our calculations closely follow the approaches in [1, 2] where error analysis of a similar semi-discrete method was conducted for the nonlocal bidirectional and unidirectional wave equations, respectively. To solve the Cauchy problem, a semi-discrete numerical method based on both a uniform space discretization and the discrete convolution operator is introduced. The method is proved to be uniformly convergent as the mesh size goes to zero. The order of convergence for the discretization error depends on the smoothness of the convolution kernel. The discrete problem defined on the whole spatial domain is then truncated to a finite domain. Restricting the problem to a finite domain introduces a localization error and it is proved that this localization error stays below a given threshold if the finite domain is large enough. To illustrate the theoretical results, some numerical experiments concerning the propagation of a single solitary wave are carried out for special forms of the kernel function. The experiments conducted confirm the error estimates provided. (See [3] for more detailed results.)

[1] H.A. Erbay, S. Erbay and A. Erkip, *Convergence of a semi-discrete numerical method for a class of nonlocal nonlinear wave equations*, ESAIM Math. Model. Numer. Anal. **52** (2018), 803–826.

[2] H.A. Erbay, S. Erbay and A. Erkip, *A semi-discrete numerical method for convolution-type unidirectional wave equations*, J. Comput. Appl. Math. **387** (2021), Article number: 112496.

[3] H.A. Erbay, S. Erbay and A. Erkip, *A semi-discrete numerical scheme for nonlocally regularized KdV-type equations*, Appl. Numer. Math. **175** (2022), 29–39.

Convergence of the nonlocal nonlinear equation to the nonlinear elasticity equation

Saadet Erbay
Ozyegin University, Istanbul, Turkey

This is a joint work with Prof Husnu Ata Erbay (Ozyegin University, Istanbul, Turkey) and Prof Albert Erkip (Sabanci University, Istanbul, Turkey).

We consider a general class of convolution-type nonlocal wave equations modeling bidirectional propagation of nonlinear waves in a continuous medium:

$$u_{tt} = \beta * (u + g(u))_{xx},$$

where the symbol $*$ denotes the convolution operation in space, $(\beta * v)(x) = \int_{\mathbb{R}} \beta(x-y)v(y)\mathrm{d}y$, In the limit of vanishing nonlocality we study the behavior of solutions to the Cauchy problem. We prove that, as the kernel functions of the convolution integral approach the Dirac delta function, the solutions converge strongly to the corresponding solutions of the nonlinear elasticity equation:

$$u_{tt} = u_{xx} + g(u)_{xx}.$$

An energy estimate with no loss of derivative plays a critical role in proving the convergence result. As a typical example, we consider the continuous limit of the discrete lattice dynamic model (the Fermi-Pasta-Ulam-Tsingou model) and show that, as the lattice spacing approaches zero, solutions to the discrete lattice equation converge to the corresponding solutions of the nonlinear elasticity equation. (More details can be found in [1].)

[1] H.A. Erbay, S. Erbay and A. Erkip, *On the convergence of the nonlocal nonlinear model to the classical elasticity equation*, Phys. D **427** (2021), Article number: 133010.

Theory and applications of the DGM in time dependent domains

Miloslav Feistauer[1], **Monika Balázsová**[2]
[1]*Charles University, Prague,* [2]*Czech Technical University, Prague*

Most of the results on the solvability and numerical analysis of nonstationary partial differential equations (PDEs) are obtained under the assumption that a space domain is independent of time. However, in practice it is necessary to work out an accurate, efficient and robust, theoretically based numerical method for the solution of nonlinear initial boundary value problems in time dependent domains. A typical example is the solution of the interaction between compressible flow described by the Navier-Stokes equations and an elastic body.

The subject of this paper is the description of the analysis of a simplified model of compressible flow represented by a nonlinear parabolic problem in a time dependent domain solved by the space-time discontinuous Galerkin method (STDGM) and then applied to the interaction of compressible flow and dynamic elasticity.

Stationary solutions of semilinear Schrödinger equations with trapping potentials in supercritical dimensions

Filip Ficek
Jagiellonian University, Kraków, Poland

The Schrödinger-Newton-Hooke equation, having the form

$$i\,\partial_t\psi = -\Delta\psi + |x|^2\psi - \left(\int_{\mathbb{R}^d} \frac{|\psi(t,y)|^2}{|x-y|^{d-2}}dy\right)\psi, \tag{1}$$

is usually encountered as a description of various quantum-mechanical systems but it can also be obtained as a nonrelativistic limit of small scalar perturbations of the anti-de Sitter spacetime. This observation gives us a motivation to investigate this equation in higher dimensions. However, in energy supercritical dimensions ($d > 6$) not very much is known about it, since the usually employed approach based on the variational methods ceases to work.

Alternatively, one can focus on spherically symmetric solutions and use the classical methods coming from the field of ordinary differential equations [1]. In this short talk I want to present results regarding existence and uniqueness of the stationary solutions to (1) that can be obtained with this approach [2]. In the end I will also mention some dynamical properties of this system. Most of the presented methods and results hold also for other similar systems with trapping potentials [3].

[1] F. H. Selem, H. Kikuchi, J. Wei, *Existence and uniqueness of singular solution to stationary Schrödinger equation with supercritical nonlinearity*, Discrete Contin. Dyn. Syst. **33** (2013), 4613–4626.

[2] F. Ficek, *Schrödinger–Newton–Hooke system in higher dimensions: Stationary states*, Phys. Rev. D **103** (2021), 104062.

[3] P. Bizoń, F. Ficek, D. E. Pelinovsky, S. Sobieszek, *Ground state in the energy super-critical Gross–Pitaevskii equation with a harmonic potential*, Nonlinear Anal. **210** (2021), 112358.

A topological degree theory for rotating solutions of planar systems

Paolo Gidoni

Institute of Information Theory and Automation of the Czech Academy of Sciences, Prague, Czech Republic

We present a generalized notion of degree for rotating solutions of planar systems. We prove a formula for the relation of such degree with the classical use of Brouwer's degree and obtain a twist fixed-point theorem providing information of the rotation number of the associated periodic solutions [1]. We then apply the result in a short proof of the sharp lower bound on the number of periodic solutions of planar Hamiltonian systems asymptotically linear at zero and infinity [2], illustrating the complementarity of our theorem with the Poincaré–Birkhoff Theorem.

[1] P. Gidoni, *A topological degree theory for rotating solutions of planar systems*, preprint arXiv:2108.13722.

[2] P. Gidoni and A. Margheri, *Lower bound on the number of periodic solutions for asymptotically linear planar Hamiltonian systems*, Discrete Contin. Dyn. Syst. **39** (2019), 585–605.

Towards global existence in the 3D chemorepulsion system

Karol Hajduk
Institute of Mathematics Polish Academy of Sciences, Poland

This is joint work with Tomasz Cieślak (Institute of Mathematics Polish Academy of Sciences, Warsaw, Poland), Mario Fuest (Institute of Applied Mathematics, Leibniz University Hannover, Hannover, Germany) & Mikołaj Sierżega (Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Warsaw, Poland).

The chemo-repulsion system

$$\begin{cases} \partial_t u = \nabla \cdot (\nabla u + u \nabla v) \\ \partial_t v = \Delta v - v + u \end{cases} \tag{1}$$

is similar in its form to the more known Keller–Segel model. The difference is the '+' sign on the RHS of the first equation in (1), which corresponds to the repulsion mechanism (the opposite phenomenon to the chemo-attraction). Due to that alteration, the properties of the solutions are expected to be different from those of the Keller–Segel system, and it is a common belief that there is no finite-time blow-up of solutions. Indeed, this is true in a two-dimensional case, as was shown in 2008 by Cieślak et al. [2]. In particular, 2D solutions of (1) are regular, global-in-time and bounded. However, the question of whether the same holds in three-dimensional space remains unanswered. In this talk, we will present a step towards answering that question, providing a condition which guarantees the global-in-time existence of classical solutions. Our method is based on the energy-like identity introduced in 2019 by Cieślak & Fujie [1] and uses the functional inequality of Bernis-type given by Winkler [3].

[1] T. Cieślak, and K. Fujie, *Some remarks on well-posedness of the higher-dimensional chemorepulsion system.* Bull. Pol. Acad. Sci. Math. **67**, 2 (2019), 165–178.

[2] T. Cieślak, P. Laurençot, and C. Morales-Rodrigo, *Global existence and convergence to steady states in a chemorepulsion system.* In *Parabolic and Navier-Stokes equations. Part 1*, Banach Center Publ. **81**, Warsaw, 2008, 105–117.

[3] M. Winkler, *Global large-data solutions in a chemotaxis-(Navier–)Stokes system modeling cellular swimming in fluid drops.* Comm. Partial Differential Equations **37**, 2 (2012), 319–351.

Riccati technique and conditional oscillation of second order equations

Petr Hasil
Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Czech Republic

The talk is based on the joint work with Michal Veselý, see [1, 2, 3, 4]. The main subject is to present the results concerning the conditional oscillation of second order linear and half-linear differential equations obtained by variants of Riccati technique. The obtained criteria identify certain equations as conditionally oscillatory, i.e., that there exists a certain threshold value given by their coefficients which separates the oscillatory equations from the non-oscillatory ones. This borderline is explicitly identified for the considered equations. We discuss the difference equations as well and we point out the similarities and differences in comparison with the continuous case.

[1] P. Hasil, M. Veselý, *New conditionally oscillatory class of equations with coefficients containing slowly varying and periodic functions.* J. Math. Anal. Appl. **494** (2021), 1–22.

[2] P. Hasil, M. Veselý, *Positivity of solutions of adapted generalized Riccati equation with consequences in oscillation theory.* Appl. Math. Lett. **117** (2021), 1–7.

[3] P. Hasil, M. Veselý, *Oscillation of linear and half-linear differential equations via generalized Riccati technique.* Rev. Mat. Complut. (accepted 2021).

[4] P. Hasil, M. Veselý, *Riccati technique and oscillation of linear second-order difference equations.* Arch. Math. **117** (2021), 657–669.

Simplest bifurcation diagram of vector fields on a torus

Marc Homs-Dones
University of Warwick, United Kingdom

This is joint work with Prof R. S. MacKay and Dr C. Baesens (University of Warwick, UK).

Despite the large body of literature on bifurcation theory and on qualitative theory of flows on surfaces, there are surprisingly few concrete results on global bifurcations on surfaces. To this end, we aim to understand full bifurcation diagrams on a torus, where the most natural type of families are given by,

$$\mathbf{x}' = \boldsymbol{\Omega} + f(\mathbf{x}), \tag{1}$$

where $\boldsymbol{\Omega} \in \mathbb{R}^2$ are the parameters. These are a subclass of monotone families of vector fields on a torus, and they can produce very complex bifurcation diagrams. As a stepping stone to understanding these diagrams, we will present the simplest possible one.

A two-step proof was required to show the simplicity. First, in [1] it was shown that any bifurcation diagram of a monotone family would have at least a particular number of bifurcation points and curves of different types. Then in our current work, we showed that a specific family of the form (1) has exactly the minimal number of these bifurcations.

[1] C Baesens and R S MacKay, *Simplest bifurcation diagrams for monotone families of vector fields on a torus.* Nonlinearity **31** (2018), 2928–2981.

Semilinear nonlocal elliptic equations with source term and measure data

Phuoc-Truong Huynh
Alpen-Adria-Universität Klagenfurt, Klagenfurt, Austria

In this paper, we study the Dirichlet problem for superlinear equation

$$\mathbb{L}u = u^p + \lambda\mu \text{ in } \Omega \tag{1}$$

with homogeneous boundary or exterior Dirichlet condition, where Ω is a bounded domain, $p > 1$ and $\lambda > 0$. The operator $\mathbb{L}$ belongs to a class of nonlocal operators including typical types of fractional Laplacians and the datum μ is taken in the optimal weighted measure space. Our technique is based on a fine analysis on the Green kernel, which enables us to construct a theory for semilinear equation (1) in measure frameworks. In particular, we show that there exist a critical exponent p^* and a threshold value λ^* such that the multiplicity holds for $1 < p < p^*$ and $0 < \lambda < \lambda^*$, the uniqueness holds for $1 < p < p^*$ and $\lambda = \lambda^*$, and the nonexistence holds in other cases. Various types of nonlocal operators are discussed to exemplify the wide applicability of our theory.

This is a joint work with Phuoc-Tai Nguyen.

Predator-prey slow-fast cycles and Hilbert's 16th problem

Renato Huzak
Hasselt University, Campus Diepenbeek, Agoralaan Gebouw D, 3590 Diepenbeek, Belgium

The goal of this talk is to study the cyclicity of planar canard cycles containing hyperbolic saddles away from critical curves, under very general conditions. We use geometric singular perturbation theory. We apply our theory to polynomial quadratic perturbations of the following degenerate graphic of predator-prey type

$$x' = -ax^2 - xy, \; y' = x + xy$$

where a is a positive constant. This problem is related to the well-known Hilbert's 16th problem and Dumortier–Roussarie–Rousseau program.

[1] R. Huzak, *Cyclicity of canard cycles with hyperbolic saddles located away from the critical curve*, J. Differential Equations **320** (2022), 479–509. https://doi.org/10.1016/j.jde.2022.02.050

Kneser-type oscillation criterion for second-order half-linear delay differential equations

Irena Jadlovská
Mathematical Institute, Slovak Academy of Sciences, Slovakia

Consider the second-order half-linear delay differential equation

$$\left(r\Phi(y')\right)'(t) + q(t)\Phi(y(\tau(t))) = 0, \quad \Phi(y) := |y|^{\alpha-1}y, \quad \alpha > 0, \quad t \geq t_0 > 0, \tag{1}$$

where r, q, τ are continuous functions defined on $[t_0, \infty)$ such that $r(t) > 0$, $q(t) \geq 0$ for large t, $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$, and

$$R(t) := \int_{t_0}^{t} r^{-1/\alpha}(s)\mathrm{d}s \to \infty \quad \text{as } t \to \infty.$$

By using a method of iteratively improved monotonicity properties of nonoscillatory solutions (see [3] for a detailed description), we propose a sharp extension of the classical (Hille-)Kneser oscillation criterion. In particular, it is shown that all solutions of (1) are oscillatory if

$$\liminf_{t\to\infty} r^{1/\alpha}(t)R(t)R^{\alpha}(\tau(t))q(t) > \begin{cases} 0 & \text{for } \lambda_* = \infty, \\ c(\alpha, \lambda_*) & \text{for } \lambda_* < \infty, \end{cases} \tag{2}$$

where

$$\lambda_* := \liminf_{t\to\infty} \frac{R(t)}{R(\tau(t))} \quad \text{and} \quad c(\alpha, \lambda_*) = \max\{\alpha m(1-m)^{\alpha}\lambda_*^{-\alpha m} : 0 < m < 1\},$$

see [1, Theorem 2, Theorem 4] for $\alpha \geq 1$ and [2, Theorem 3] or [3, Corollary 2] for general α. The oscillation constant $c(\alpha, \lambda_*)$ is best possible in the sense that the strict inequality in (2) cannot be replaced by a nonstrict one without affecting validity of the criterion. Numerous possible extensions of the method are discussed as well.

[1] I. Jadlovská, J. Džurina, *Kneser-type oscillation criteria for second-order half-linear delay differential equations*, Appl. Math. Comput. **380** (2020), 1–15.

[2] G. E. Chatzarakis, S. R. Grace, I. Jadlovská, *On the sharp oscillation criteria for half-linear second-order differential equations with several delay arguments*, Appl. Math. Comput. **397** (2021), 1–9.

[3] I. Jadlovská, *New Criteria for Sharp Oscillation of Second-Order Neutral Delay Differential Equations*, Mathematics **9**(17) (2021), –23.

Recent results in the theory of p-critical linear even-order difference equations

Jan Jekl
Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Brno,Czech Republic

The theory of p-critical equations has been developed relatively recently in [1]. Article [1] deals with linear even-order Sturm-Liouville difference equations of the form

$$\sum_{i=0}^{k} (-\triangle)^i \left(r_n^{[i]} \triangle^i y_{n-i} \right) = 0, \quad n \in \mathbb{Z} \tag{1}$$

and states that Eq. (1) is p-critical if spaces of recessive solution at $\pm\infty$ intersect to create a p dimensional space. Since the article [1] there has appeared some progress in the field of p-critical equations. For example, article [2] formulates the equivalent conditions under which the equation

$$\triangle^k \left(r_n \triangle^k y_n \right) = 0, \quad r_n > 0, k \in \mathbb{N}, \tag{2}$$

is at least m-critical for $m \in \{1, \dots, k\}$.

We present recent results contained in articles [3] and [4]. Among others we show that the equation

$$\triangle^4 y_n - \triangle \left(r_{n+1} \triangle y_{n+1} \right) = 0, \quad r_n > 0,$$

is always 1-critical. Then we present the result that if for any $\varepsilon > 0$ and $H \in \mathbb{Z}$ exists a nontrivial $u_n \in l_0^2(\mathbb{Z})$ such that

$$\sum_{n=-\infty}^{\infty} r_n \left(\triangle^k u_{n-k} \right)^2 < \varepsilon \left(\triangle^{m-1} u_{H-(m-1)} \right),$$

then Eq. (2) is at least m-critical. Furthermore, we also present a condition on the sequences $r_n^{[i]}$ so that Eq. (1) is at most m-critical.

[1] O. Došlý, P. Hasil, *Critical higher order Sturm-Liouville difference operators*, J. Difference Equ. Appl., **17**(9), 1351–1363.

[2] P. Hasil, *Conjugacy of self-adjoint difference equations of even order*, Abstr. Appl. Anal., **2011**, 1–16.

[3] J. Jekl, *Properties of critical and subcritical second order self-adjoint linear equations*, Math. Slovaca, **71**(5), 1149-1166.

[4] J. Jekl, *Special cases of critical linear difference equations*, Electron. J. Qual. Theory Differ. Equ.,**2021**, 1–17.

Computation of flow past cactus-shaped cylinders: A hybrid immersed interface approach

Jiten C Kalita
Department of Mathematics, Indian Institute of Technology Guwahati INDIA

Raghav Singhal
Department of Mathematics, Indian Institute of Technology Guwahati INDIA

In the present work, we simulate the flow past cactus-shaped stationary cylinders immersed in fluids in uniform flow. A recently developed hybrid explicit jump immersed interface approach [1] in conjunction with a higher order compact (HOC) scheme developed by the authors has been employed for simulating the transient complex flows governed by the streamfunction-vorticity (ψ-ω) formulation of the Navier-Stokes (N-S) equations for incompressible viscous flows. Opposed to most of the earlier computations which were performed by either finite volume or finite element approach in extremely finer grids, our approach accomplishes the same in relatively coarse grids in Finite Difference set-up. Moreover, to the best of our knowledge, such simulations have been performed for the first time by such an approach employing the ψ-ω formulation of the N-S equations against the primitive variable formulation in earlier simulations. The drag reduction induced by the edges of the cactus in comparison with smooth stationary cylinders has also been studied in our studies. In particular, the simulation of flow past a twenty four edge cactus exemplifies the efficiency and robustness of the recently developed explicit jump IIM approach for transient flows past bluff bodies of complex shapes.

[1] Raghav Singhal and Jiten C Kalita, *An efficient explicit jump HOC immersed interface approach for transient incompressible viscous flows* DOI: 10.48550/arXiv.2205.10080.

Variational methods for non-convex problems involving inertia

Malte Kampschulte
Faculty of Mathematics and Physics, Charles University, Czech Republic

When considering problems in continuum mechanics, in particular those involving large deformations of a solid, one is generally faced with a highly non-linear, non-convex state-space of admissible solutions. As a result, many of the standard PDE-methods can only be applied after performing some problem-specific adaptations, if at all. In contrast to this, methods from the calculus of variations have no difficulties with those non-convexities, however they in turn cannot be easily applied to problems involving inertia, where the solutions do not have the structure of local minima.

The aim of this contribution is to present a general approach to such kind of problems that relies on concepts from both sides and can be used to show existence of weak solutions for varied classes of solutions. We illustrate this at some problems, in particular those arising in fluid-structure interaction.

This is based on joint works with B. Benešová (Charles University), S. Schwarzacher (Uppsala) as well as in parts on other works also involving D. Breit (Heriot-Watt), A. Češík (Charles University) and G. Gravina (Temple University).

Local pressure-corrections for incompressible flows

Utku Kaya
Otto-von-Guericke-Universität Magdeburg, Germany

This is a joint work with Thomas Richter (Otto-von-Guericke-Universität Magdeburg) and Malte Braack (Christian-Albrechts-Universität zu Kiel).

Pressure-correction methods facilitate approximations of solutions to time-dependent incompressible fluid flows by decoupling the momentum equation from the continuity equation [1]. A common strategy used by several pressure-correction methods is:

- compute a (not necessarily divergence-free) predictor velocity field,
- solve a Poisson problem for the pressure,
- project the predictor velocity field onto a divergence-free one.

In cases, where an explicit time-stepping scheme is employed for the momentum equation, the Poisson problem for the pressure remains to be the most expensive step. We here present a domain decomposition method that replaces the pressure Poisson problem from step (ii) with local pressure Poisson problems on non-overlapping subregions [2, 3]. No communication between the subregions is needed, thus the method is favorable for parallel computing. We illustrate the effectivity of the method via numerical results.

[1] J.L. Guermond, P. Minev, J. Shen, *An overview of projection methods for incompressible flows.* Comput. Methods Appl. Mech. Engrg. **195**:44-47 (2006), 6011-6045.

[2] M. Braack, U. Kaya, *Local pressure-correction for the Stokes system.* J. Comput. Math. **38**:1 (2020), 125-141.

[3] U. Kaya, R. Becker, M. Braack, *Local pressure-correction for the Navier-Stokes equations.* Internat. J. Numer. Methods Fluids. **93**:4 (2021), 1199-1212.

Operator estimates for homogenization in perforated domains

Andrii Khrabustovskyi
University of Hradec Králové, Czech Republic

Let $\varepsilon > 0$ be a small parameter. We consider the perforated domain $\Omega_\varepsilon = \Omega \setminus D_\varepsilon$, where Ω is an open domain in $\mathbb{R}^n$, and D_ε is a family of small identical balls of the radius $d_\varepsilon = o(\varepsilon)$ distributed periodically with period ε. Let Δ_ε be the Laplace operator in Ω_ε subject to the Robin condition $\frac{\partial u}{\partial n} + \gamma_\varepsilon u = 0$ with $\gamma_\varepsilon \geq 0$ on the boundary of the holes and the Dirichlet condition on the exterior boundary. Kaizu [1] and Brillard [2] have shown that, under appropriate assumptions on d_ε and γ_ε, the operator Δ_ε converges in strong resolvent sense to the sum of the Dirichlet Laplacian in Ω and a constant potential.

In the talk we discuss some improvements of this result. We derive the estimates on the rate of convergence in terms of $L^2 \to L^2$ and $L^2 \to H^1$ operator norms. As a byproduct we establish the estimate on the distance between the spectra of the associated operators. Our proofs rely on the abstract scheme developed by Post in [4] concerning the resolvent and spectral convergence in varying Hilbert spaces.

This is a joint work with Michael Plum (Karlsruhe Institute of Technology, Germany) [3].

[1] A. Brillard, *Asymptotic analysis of two elliptic equations with oscillating terms.* RAIRO, Modélisation Math. Anal. Numér. **22** (1988), 187–216.

[2] S. Kaizu, *The Poisson equation with semilinear boundary conditions in domains with many tiny holes.* J. Fac. Sci., Univ. Tokyo, Sect. I A **36** (1989), 43–86.

[3] A. Khrabustovskyi, M. Plum, *Operator estimates for homogenization of the Robin Laplacian in a perforated domain.* arXiv:2106.10216 [math.AP].

[4] O. Post, *Spectral convergence of quasi-one-dimensional spaces.* Ann. Henri Poincaré **7** (2006), 933–973.

Oscillatory properties of fractional delay differential equations

Tomas Kisela
Brno University of Technology, Czech Republic

This is a joint work with Prof. Jan Čermák (Brno University of Technology, Czech Republic).

The talk discusses oscillatory properties of a test delay differential system

$$\mathrm{D}^{\alpha} y(t) = Ay(t-\tau)\,, \quad t > 0 \tag{1}$$

where $\alpha > 0$ is a real scalar and the symbol D^{α} is the Caputo derivative of order α. We formulate corresponding criteria as explicit necessary and sufficient conditions that enable direct comparisons with the results known for classical integer-order delay differential equations. The talk is concluded by a few illustratory figures including phase portraits that show behaviour that is not present in the classical integer-order case.

Periodic solutions of nonlinear damped wave equations on $\mathbb{R}^N$

Władysław Klinikowski
Nicolaus Copernicus University in Toruń, Poland

This is a joint work with A. Ćwiszewski (Nicolaus Copernicus University in Toruń, Poland).

We shall present results of the existence of periodic solutions for nonlinear damped wave equations:

$$\frac{\partial^2}{\partial t^2}u(t,x) + \alpha \cdot \frac{\partial}{\partial t}u(t,x) = \Delta u(t,x) - V(x)u(t,x) + f(t,x,u(t,x)), \tag{1}$$

that are forced by periodic nonlinear Lipschitz function f. Here $t \geq 0$, $x \in \mathbb{R}^N$, $\alpha > 0$ is a damping coefficient and V is a Kato–Rellich type functional.

Two distinct cases are studied: resonant and non-resonant ones. In the resonant case sufficient criteria for the existence of periodic solutions in terms of both Landesman–Lazer type and sign conditions are provided. In the non-resonant situation the interplay of the spectra of the wave operator for the linearized equations either at infinity or zero are exploited.

The approach is based on the translation along trajectories operator and fixed point index for k-set-contractions, therefore due compactness properties and estimates are provided.

Presented results are inspired by following works concerning parabolic equations, both [1] in resonant case and [2] in non-resonant situation.

[1] A. Ćwiszewski, R. Łukasiak, A Landesman–Lazer type result for periodic parabolic problems on $\mathbb{R}^N$ at resonance, Nonlinear Anal. 125 (2015), 608–625.

[2] A. Ćwiszewski, R. Łukasiak, Periodic solutions for nonresonant parabolic equations on $\mathbb{R}^N$ with Kato-Rellich type potentials, J. Fixed Point Theory Appl. 23 (2021).

Area preserving geodesic curvature driven flow of closed curves on a surface

Miroslav Kolář
Czech Technical University in Prague, Czech Republic

This is a joint work with Prof. Michal Beneš (Czech Technical University in Prague) and Prof. Daniel Ševčovič (Comenius University, Bratislava, Slovakia).

We investigate a non-local geometric flow preserving surface area enclosed by a curve on a given surface evolved in the normal direction by the geodesic curvature and the external force. Our contribution is based on [1] and we show how such a geodesic curvature driven flow of surface curves can be projected into a curvature driven flow of planar curves with the non-local normal velocity. We also discuss the importance of the appropriate choice of the tangential velocity functional, which doesn't affect the shape of the evolved curve. However, if chosen properly, it plays a crucial role in the design of the numerical approximation scheme and its stability. Then we review some theoretical properties of such a flow. We show that the surface area preserving flow decreases the length of the evolved surface curves. Local existence and continuation of classical smooth solutions to the governing system of partial differential equations is reviewed as well. Furthermore, we propose a numerical method based on flowing finite volume for spatial discretization in combination with the Runge–Kutta method for solving the resulting system. Several computational examples demonstrate variety of evolution of surface curves and the order of convergence.

[1] M. Kolář, M. Beneš and D. Ševčovič, *Area preserving geodesic curvature driven flow of closed curves on a surface*, Discrete and Continuous Dynamical Systems - B. **22** (2017), 3671–3689.

Nonlocal heat equations with generalized fractional Laplacian

Igor Kossowski
Institute of Mathematics, Lodz University of Technology, Poland

This is a joint work with Bogdan Przeradzki (Institute of Mathematics, Lodz University of Technology, Lodz, Poland).

Let us consider initial-boundary value

$$u_t + g(-\Delta)u = f(t, x, u), \qquad u(t, \cdot)|_{\partial\Omega} = 0, \qquad u(0, \cdot) = u_0, \tag{1}$$

where $g(-\Delta)$ is the generalized fractional Laplacian, which is defined by spectral theorem (compare in [1]).

We aim to show that the problem (1) has a solution. To obtain the existence of solution we use two methods:

- direct method, which relies on using Fourier series;
- semigroup method.

Moreover, we present some numerical simulations of solutions to our problem. Above results was obtained by using Python.

[1] I. Kossowski, B. Przeradzki, *Nonlinear equations with a generalized fractional Laplacian*, Rev. R. Acad. Cienc. Exactas Fis. Nat. 115 (2), 115, art. 58 (2021).

Comparison principles in problems with p-Laplacian

Lukáš Kotrla
University of West Bohemia, Plzeň, Czech Republic

This is a joint work with Peter Takáč (University of Rostock, Germany), Petr Girg, and Jiří Benedikt (both University of West Bohemia, Plzeň, Czech Republic).

The contributed talk is devoted to comparison principles for weak solutions to parabolic and elliptic problems involving p-Laplacian in one dimension. We consider a parabolic problem with both convection and reaction terms in full generality. More precisely, let $p > 1$, $T > 0$, and for $i = 1, 2$ consider

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial u_i}{\partial x}\right|^{p-2}\dfrac{\partial u_i}{\partial x}\right) - b(x,t)\dfrac{\partial u_i}{\partial x} + \varphi(x,t,u_i) = f_i(x,t), & x \in (-1,1)\times(0,T)\,, \\ u_i(\pm 1, t) = 0\,, & t \in (0,T)\,, \\ u_i(x,0) = g(x)\,, & x \in (-1,1)\,. \end{cases}$$

Here b, g, φ, and f_i are given sufficient smooth functions such that $f_1 \le f_2$. Under the assumption

$$\frac{1}{2}\frac{\partial b}{\partial x}(x,t) + \frac{\partial \varphi}{\partial u}(x,t,s) \ge 0 \quad \text{for all } (x,t,s) \in (-1,1)\times(0,T)\times\mathbb{R}\,,$$

the proof of the weak comparison principle is matter of application of variational calculus with the tests function $(u_1 - u_2)^+$. Further we focus on the strong comparison principles for two partial problems.

At first we are interested in a stationary problem with $p > 2$ which is in fact ODE. We provide a reasonable assumption on the difference $f_1 - f_2$ near the end points ± 1 to ensure the validity of the strong comparison principle and show some counterexamples to support the need for the assumption.

Then we assume the problem without convection ($b = 0$) and reaction ($\varphi = 0$) terms and we compare a stationary and a time dependent solution. Once again, we show a counterexample to validity of the strong comparison principle and provide a sufficient conditions to ensure its validity. We take an advantage concavity of the stationary solution u_1 and formulate a condition on difference of f_1 and f_2 on the interval where u_1 possesses its maximum.

Multiplicity results for bouncing solutions of generalized Lazer–Solimini equation

Věra Krajščáková
Faculty of science, Palacký University Olomouc, Czech Republic

This is a joint work with Jan Tomeček (Faculty of science, Palacký University Olomouc).

We investigate the singular differential equation

$$x'' + g(x) = p(t),$$

with g having a weak, attractive singularity at $x = 0$ and 2π–periodic function p. We prove the coexistence of a classical positive periodic solution and $2m\pi$–periodic bouncing solutions having n impacts with the singularity for any positive integers m and n.

Observer-based data assimilation for isothermal gas transport using distributed measurements

Teresa Kunkel
Technical University of Darmstadt, Germany

This is a joint work with Prof Jan Giesselmann (Technical University of Darmstadt, Germany) and Prof Martin Gugat (Friedrich-Alexander-University Erlangen-Nürnberg, Germany).

For an efficient operation of gas pipe networks it is essential to know the current state of the gas in the pipes. In order to reconstruct the system state from incomplete measurements we set up an observer system that is based on the isothermal Euler equations. Our main goal is to show exponential synchronization of the observer towards the original system state.

We model the gas transport by the isothermal Euler equations

$$\begin{aligned} \partial_t \rho + \partial_x(\rho v) &= 0, \\ \partial_t v + \partial_x\left(\tfrac{1}{2}v^2 + P'(\rho)\right) &= -\gamma |v| v \end{aligned} \tag{1}$$

for $0 < x < \ell,\ t > 0$ with smooth and strictly convex pressure potential $P = P(\rho)$ and associated energy functional $\mathcal{H}(\rho, v) := \int_0^\ell \left(\frac{1}{2}\rho v^2 + P(\rho)\right) dx$ complemented with suitable boundary conditions. The stability of the equations (1) with respect to parameters and initial data in the practically relevant case of small (subsonic) velocities is investigated in [1] by exploiting the Hamiltonian structure of the equations and estimating the relative energy

$$\mathcal{H}(\hat{u}|u) := \mathcal{H}(\hat{u}) - \mathcal{H}(u) - \mathcal{H}'(u)(\hat{u} - u)$$

between two solutions $u := (\rho, v)$ and $\hat{u} := (\hat{\rho}, \hat{v})$.

Based on a modification of the relative energy framework that is inspired by an extension of the energy that was used in [2] we investigate convergence for long times of the observer system

$$\begin{aligned} \partial_t \hat{\rho} + \partial_x(\hat{\rho}\hat{v}) &= R_\rho, \\ \partial_t \hat{v} + \partial_x\left(\tfrac{1}{2}\hat{v}^2 + P'(\hat{\rho})\right) &= -\gamma |\hat{v}| \hat{v} + R_v \end{aligned} \tag{2}$$

for $0 < x < \ell,\ t > 0$, where distributed measurements of one of the fields mass density, velocity or mass flow are inserted into the observer through source terms R_ρ, R_v of Luenberger type, e.g., $R_\rho = 0,\ R_v = \mu(v - \hat{v}),\ \mu > 0$ for measurements of the velocity v. We can show exponential convergence of the observer towards the original solution under technical assumptions that are reasonable in practice and can partially be verified theoretically, see [3]. We focus on results on single pipes. They can be extended to star-shaped networks provided energy consistent coupling conditions are used.

[1] H. Egger and J. Giesselmann, *Stability and asymptotic analysis for instationary gas transport via relative energy estimates.* arXiv:2012.14135 (2020).

[2] H. Egger and T. Kugler, *Damped wave systems on networks: Exponential stability and uniform approximations.* Numer. Math. **138** (2018), 839–867.

[3] M. Gugat and S. Ulbrich, *Lipschitz solutions of initial boundary value problems for balance laws.* Math. Models Methods Appl. Sci. **28** (2018), 921–951.

Travelling wave solutions of the suspension bridge type equation

Hana Levá
Faculty of Applied Sciences, University of West Bohemia, Pilsen, Czech Republic

This is a joint work with Gabriela Holubová (University of West Bohemia, Pilsen, Czech Republic). The subject of our study is the existence of travelling wave solutions of the fourth–order partial differential equation

$$u_{tt} + u_{xxxx} + \alpha u^+ - \beta u^- + g(u) = 1, \quad x \in \mathrm{R},\, t > 0, \tag{1}$$

where α, $\beta > 0$, $u^{\pm} = \max\{\pm u, 0\}$ and $g\left(\frac{1}{\alpha}\right) = 0$. This type of equation can be used as a model of an asymmetrically supported beam or as a generalized model of a suspension bridge.

Using variational methods we show that the equation (1) posseses infinitely many homoclinic travelling wave solutions with arbitrary wave speed from the interval $\left(\sqrt[4]{12\beta}, \sqrt[4]{4\alpha}\right)$. Namely, we use the Mountain Pass Theorem and the method of nonzero weak convergence after some suitable translation. The existence result holds under considerably weakened assumptions on nonlinearity g than those formerly used in the literature. However, allowing the presence of sign preserving nonlinearities seems to limit the possible values of the wave speed. Lastly, we present some numerical experiments that are necessary to find particular forms of the classical solutions.

Lower and upper functions method for even order boundary value problems

Lucía López-Somoza
Universidade de Santiago de Compostela, Spain

This is a joint work with Prof Alberto Cabada (Universidade de Santiago de Compostela).

This talk will be devoted to prove the existence of solution of arbitrary even order boundary value problems via the lower and upper functions method. In particular, we will use some relations between Green's functions of the same problem coupled to various boundary conditions to show how the existence of a pair of lower and upper functions of a problem under certain boundary conditions will imply the existence of a solution of the same problem coupled with differente boundary conditions.

The obtained results can be found in [1].

[1] A. Cabada, L. López-Somoza, *Lower and upper solutions for even order boundary value problems.* Mathematics **2019**, 7, 878 (2019).

On implicit constitutive relations to parabolic problems

Erika Maringová
Institute of Science and Technology, Austria

This is a joint work with Josef Málek and Miroslav Bulíček (Charles University, Prague, Czech Republic).

We study systems of nonlinear partial differential equations of parabolic type, in which the elliptic operator is replaced by the first order divergence operator acting on a flux function, which is related to the spatial gradient of the unknown through an additional implicit equation. Formulating four conditions concerning the form of the implicit equation, we first show that these conditions describe a maximal monotone p-coercive graph. We then establish the global-in-time and large-data existence of a (weak) solution and its uniqueness. The theory is tractable from the point of view of numerical approximations. For details, we refer to [1].

[1] M. Bulíček, E. Maringová, J. Málek, *On nonlinear problems of parabolic type with implicit constitutive equations involving flux,* Math. Models Methods Appl. Sci. **31** (2021), no.10, 2039–2090.

Second order Stieltjes derivatives and differential equations

Ignacio Márquez Albés
Universidade de Santiago de Compostela, Spain

This is a joint work with Prof. F.J. Fernández (Universidade de Santiago de Compostela, Santiago de Compostela, Spain) and Prof. F.A.F. Tojo (Universidade de Santiago de Compostela, Santiago de Compostela, Spain), see [1].

The Stieltjes derivative is generalization of the usual derivative defined in terms of nondecreasing and left-continuous function, $g : \mathbb{R} \to \mathbb{R}$. Roughly speaking, the Stieltjes derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$f'_g(t) = \lim_{s \to t} \frac{f(s) - f(t)}{g(s) - g(t)},$$

whenever that limit exists, see [2, 4] for more details. This definition, of course, is not valid for all points and, in particular, cannot be used at any point of the set

$$C_g = \{t \in \mathbb{R} : g \text{ is constant on } (t - \varepsilon, t + \varepsilon) \text{ for some } \varepsilon > 0\},$$

as, for such points, the quotient function in the limit above is not defined on a neighbourhood of the points.

In this talk, we provide a definition of Stieltjes derivatives that avoids this limitation and discuss the importance of doing so for considering second (and higher) order derivatives. This definition will be consistent with the Lebesgue-Stieltjes integral and a special type of continuity introduced in [3]. This allows us to extend the concept of differentiability class to the context of Stieltjes derivatives and, thus, consider second order differential equations with Stieltjes derivatives.

[1] Fernández, F.J., Márquez Albés, I., Tojo, F.A.F.: *On first and second order linear Stieltjes differential equations*. J. Math. Anal. Appl. **511**(1), 126,010 (2022).

[2] López Pouso, R., Rodríguez, A.: *A new unification of continuous, discrete, and impulsive calculus through Stieltjes derivatives*. Real Anal. Exchange **40**(2), 319-353 (2014/15).

[3] M. Frigon, R. López Pouso, *Theory and applications of first-order systems of Stieltjes differential equations*, Adv. Nonlinear Anal. **6**(1) (2017) 13-36.

[4] Márquez Albés, I.: *Notes on the linear equation with Stieltjes derivatives*. Electron. J. Qual. Theory Differ. Equ. **42**, 1-18 (2021).

Parabolic boundary-value problems in generalized Sobolev spaces

Volodymyr Mikhailets
Institute of Mathematics of the Czech Academy of Sciences, Czech Republic

This is a joint work with Prof Aleksandr Murach (Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine) and Prof Valerii Los (National Technical University of Ukraine, Kyiv, Ukraine).

We consider a general inhomogeneous parabolic initial-boundary value problem for a $2b$-parabolic differential equation given in a finite multidimensional cylinder. We investigate the solvability of this problem in some generalized anisotropic Sobolev spaces. They are parametrized with a pair of positive numbers s and $s/(2b)$ and with a function $\varphi : [1, \infty) \to (0, \infty)$ that varies slowly at infinity. The function parameter φ characterizes subordinate regularity of distributions with respect to the power regularity given by the number parameters. We prove that the operator corresponding to this problem is an isomorphism on appropriate pairs of these spaces. As an application, we give a theorem on the local regularity of the generalized solution to the problem. We also obtain sharp sufficient conditions under which chosen generalized derivatives of this solution are continuous on a given set.

[1] V. Los, V. Mikhailets, A. Murach, *Parabolic problems in generalized Sobolev spaces*, Comm. Pure Appl. Anal. **20** (2021), 3605–3636.

[2] V. Los, V. Mikhailets, A. Murach, *An isomorphism theorem for parabolic problems in Hörmander spaces and its applications*, Comm. Pure Appl. Anal. **16** (2017), 69–97.

[3] V. Los, V. Mikhailets, A. Murach, Parabolic Problems and Generalized Sobolev Spaces. Naukova Dumka, Kyiv, 2021. (arXiv:2109.03566)

Blow-up in a two-species chemotaxis-competition system

Masaaki Mizukami
Kyoto University of Education, Japan

This talk deals with existence of blow-up solutions for a two-speceies chemotaxis-competition system, and gives an important progress in the study of chemotaxis systems. This system describes a situation in which multi species move towards higher concentration of a chemoattractant, and moreover compete each other. In mathematical view, this system is an interesting problem on account of the influence of the diffusion, the Lotka–Volterra type competitive kinetics and the chemotaxis. Here, the diffusion term and the competition terms represent stabilization of species; on the other hand, the chemotaxis term shows concentration of species. Thus "*whether the solution is bounded or not*" is one of the fundamental themes in the study of the two-species chemotaxis-competition system.

In the previous works about the two-species chemotaxis system it is shown that some smallness of the chemotactic effect leads to boundedness and stabilization (see, e.g., [1, 2]). Bai–Winkler [1] established global existence and boundedness under the smallness conditions for the chemotactic effect. They also obtained additional smallness conditions for the chemotactic effect which derives asymptotic stability. Recently, these conditions for asymptotic stability were improved in [2].

However, the case that the chemotactic effect is large seems to be not studied yet; therefore, it still remains to analyze on the following question:

Is the solution bounded also in the case that the chemotactic effect is large?

The purpose of this talk is to give a *negative* answer to this question. This is a joint work with Mr. Yuya Tanaka and Professor Tomomi Yokota.

[1] X. Bai, M. Winkler, *Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics*, Indiana Univ. Math. J. **65** (2016), 553–583.

[2] M. Mizukami, *Improvement of conditions for asymptotic stability in a two-species chemotaxis-competition model with signal-dependent sensitivity*, Discrete Contin. Dyn. Syst. Ser. S **13** (2020), 269–278.

On discontinuous sweeping processes and vanishing viscosity approximations

Giselle A. Monteiro
Institute of Mathematics, Czech Academy of Sciences, Prague, Czechia

This is a joint work with P. Krejčí (Czech Technical University, Prague, Czechia) and V. Recupero (Politecnico di Torino, Torino, Italy).

A sweeping process, as introduced by J.-J Moreau in 1971, is a first order differential inclusion of the form

$$-\dot{\xi}(t) \in N_{C(t)}(\xi(t)), \tag{1}$$

which describes the problem of finding a function ξ such that $\xi(t) \in C(t)$ for all $t \in [0, T]$ and whose derivative at time t points in the inward normal direction to $C(t)$ at the point $\xi(t)$. Under weaker regularity assumptions though, the inclusion has to be properly interpreted and the "time derivative" $\dot{\xi}(t)$ has to be given an appropriate meaning. The functional framework of regulated functions is rather convenient to incorporate discontinuities and allows for a compatible reformulation of (1) in terms of the Kurzweil integral.

In this talk, we discuss some issues related to viscous approximations of discontinuous sweeping processes. Our investigation relies on the analysis of the limiting behavior of a regularized problem obtained by introducing a viscous dissipation mechanism which stabilizes the process.

Acknowledgement: Supported by the GAČR project 20-14736S.

Almost oscillatory fractional differential equations

M. Guadalupe Morales
Department of Mathematics and Statistics, Masaryk University, Brno, Czech Republic

This is a joint work with Prof. Zuzana Došlá (Masaryk University, Brno, Czech Republic).

Very interesting and novel applications of fractional differential equations appear in physics, chemistry, engineering, and other sciences. Some early examples are given in [1], [2], [3], [7], [9] dealing with the modeling of the mechanical properties of materials. Thus, the fractional differential equations turn out an innovative way to describe macroscopically complex behavior.

This work shows asymptotic properties of the fractional differential equation

$$D_0^{\alpha} x(t) + q(t)x(t) = 0, \quad t > 0, \tag{1}$$

where $2 < \alpha \leq 3$ is fixed, q is a real-valued locally bounded measurable function defined on $(0, \infty)$ such that

$$q(t) \neq 0 \ \text{ for large } \ t$$

and D_0^{α} denotes the Riemann-Liouville fractional differential operator, see [8].

Equation (1) is linked to the modeling of the relaxation processes, see [4]. Thus, from the computational and theoretical approach, it is natural to understand the possible types of solutions and the effect of varying the derivative order of the differential equations and compare it with very well-known classical results, [5], [6]. Our results explain the discrepancy of qualitative properties among differential equations of the second, third and fractional orders.

[1] D. Băleanu, K. Diethelm, E. Scalas, J.J. Trujillo, Fractional calculus models and numerical methods, Series on Complexity, Nonlinearity and Chaos: Volume 5, Second edition. World Scientific, Boston, 2016.

[2] R. Bagley, *On the equivalence of the Riemann-Liouville and the Caputo fractional order derivatives in modeling of linear viscoelastic materials.* Fract. Calc. Appl. Anal. **10** (2007), 123–126.

[3] M. Caputo, F. Mainardi, *A new dissipation model based on the memory mechanism.* Pure Appl. Geophys. **91** (1971), 134–147, reprinted in Fract. Calc. Anal. **10** (2007), 310-323.

[4] R. Hilfer, Applications of Fractional Calculus in Physics. World Scientific, River Edge, 2000.

[5] I. T. Kiguradze, T. A. Chanturia, Asymptotic properties of solutions of non-autonomous ordinary differential equations. Kluwer, Dordrecht, 1993.

[6] T. Kusano, M. Naito, K. Tanaka, *Oscillatory and asymptotic behaviour of solutions of a class of linear ordinary differential equations.* Proc. Royal Soc. Edinburgh, **90** (1981), 25-40.

[7] F. Mainardi, Fractional viscoelasticity in: Handbook of fractional calculus with applications, Volume 4 Applications in Physics, Part A, Edited by: Vasily E. Tarasov. De Gruyter, Berlin, 2019.

[8] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional integrals and derivatives. Theory and applications. Gordon and Breach Science Publishers, Yverdon, 1993.

[9] Y. Yu, P. Perdikaris, G. E. Karniadakis, *Fractional modeling of viscoelasticity in 3D cerebral arteries and aneurysms.* J. Comput. Phys., **323** (2016), 219-242.

Linearized stability for fractional differential and difference equations

Luděk Nechvátal
Brno University of Technology, Czech Republic

This is a joint work with Prof. Jan Čermák (Brno University of Technology, Brno, Czech Republic).

The classical principle of linearized stability states that a hyperbolic equilibrium point x_0 of the autonomous system $x'(t) = f(x(t))$, f being a sufficiently smooth vector field, is either asymptotically stable or unstable and its stability is governed by stability of the zero solution of the linear system $x'(t) = J_f(x_0)x(t)$ where J_f is the Jacobian matrix of f. The natural question is whether this principle can be applied also in the case of fractional differential systems (involving the Caputo differential operator). The answer is positive due to the papers [1] and [2]. Moreover, the proof technique suggested in [1] can be successfully extended for the demands of fractional difference equations. It was recently shown in the paper [3] where the Caputo backward fractional difference is considered in a system of autonomous difference equations.

[1] N. D. Cong, T. S. Doan, S. Siegmund, and H. T. Tuan, *Linearized asymptotic stability for fractional differential equations*, Electron. J. Qual. Theory Differ. Equ. **2016** (2016), Paper No. 39, 13 pp.

[2] N. D. Cong, T. S. Doan, S. Siegmund, and H. T. Tuan, *An instability theorem for nonlinear fractional differential systems*, Discrete Contin. Dyn. Syst. Ser. B **22** (2017), 3079–3090.

[3] J. Čermák and L. Nechvátal, *On a problem of linearized stability for fractional difference equations*, Nonlinear Dyn. **104** (2021), 1253–1267.

Neutral half-linear differential equations and the modified Riccati technique

Zuzana Pátíková
Tomas Bata University in Zlín, Czech Republic

This is a joint work with Dr. Simona Fišnarová (Mendel University in Brno, Czech Republic).

In the contribution, we present the results of the paper [1]. We study the second-order neutral half-linear differential equation and formulate new oscillation criteria for this equation, which are obtained through the use of the modified Riccati technique. In the first statement, the oscillation of the equation is ensured by the divergence of a certain integral. The second one, which can be seen as a Hille–Nehari-type criterion, provides the condition of the oscillation in the case where the relevant integral converges. The use of the results is to be shown in several examples, in which the Euler-type equation and its perturbations are considered.

[1] Z. Pátíková, S. Fišnarová, *Use of the Modified Riccati Technique for Neutral Half-Linear Differential Equations.* Mathematics **9, 235** (2021).

Analytical and numerical solution of curve dynamics modeling dislocation dynamics and topological changes

Petr Pauš

Czech Technical University in Prague, Czech Republic

This is a joint work with Prof Michal Beneš and Miroslav Kolář (both Czech Technical University in Prague), and Prof Shigetoshi Yazaki (Meiji University, Tokyo, Japan).

In material science, dislocations can be regarded as open plane curves evolving according to the curvature with an external force. In the present contribution, evolving curves connecting two circular obstacles are treated from both analytical and numerical viewpoints: An exact solution curve is constructed with sliding endpoints along obstacles, and all important and typical phenomena including touching-splitting, non-touching and Orowan island can be treated numerically.

The mathematical model of the dislocation dynamics is based on the curvature flow equation with forcing term schematically written as

$$velocity = curvature + forcing\ term$$

for open or closed curves [1].

From the numerical point of view, the curve is described parametrically and treated by a method of finite volumes [2, 3]. Numerical simulations are presented to illustrate the analytical solutions.

The results of analytical and numerical studies of curve bowing and topological changes of parametric curves (see [4]) will be presented in the contribution.

[1] P. Pauš, M. Beneš, and J. Kratochvíl. Simulation of dislocation annihilation by cross-slip. *Acta Phys. Pol. A*, 122 (2012), 509–511.

[2] P. Pauš and M. Beneš. Direct approach to mean-curvature flow with topological changes. *Kybernetika* 45(2009), 591–604.

[3] M. Kolář, P. Pauš, J. Kratochvíl, M. Beneš. Improving method for deterministic treatment of double cross-slip in FCC metals under low homologous temperatures. *Comput. Mater. Sci.* 189, 110251.

[4] P. Pauš, S. Yazaki. Exact solution for dislocation bowing and a posteriori numerical technique for dislocation touching-splitting. *JSIAM Letters*, 7 (2015), 57–60.

Semilinear second-order differential inclusions in abstract spaces

Martina Pavlačková
Dept. of Appl. Mathematics, Moravian Business College Olomouc, Olomouc, Czech Republic

This is a joint work with Valentina Taddei (University of Modena and Reggio Emilia, Italy).

We study the existence of a mild solution to the Cauchy problem for semilinear second-order differential inclusions in abstract spaces in the case when the right-hand side depends also on the first derivative. The results are obtained by combining the Kakutani fixed point theorem with the approximation solvability method and the weak topology. This combination enables getting the results without any requirements for compactness of the right-hand side.

Bifurcations of neural fields on the sphere

Mónika Polner
Bolyai Institute, University of Szeged, Szeged, Hungary

This is a joint work with Len Spek[1], Yuri A. Kuznetsov[2,1], Stephan A. van Gils[1] ([1]Department of Applied Mathematics, University of Twente, Enschede, The Netherlands and [2]Department of Mathematics, Utrecht University, Utrecht, The Netherlands).

A natural model to study pattern formation in large groups of neurons is the neural field. We investigate a neural field model on a sphere, with excitatory and inhibitory neurons, with space-dependent delays and gap junctions. This work is an extension of [1] in the following directions: we add a diffusion term to the model to simulate gap junctions. Moreover, we consider two distinct populations of excitatory and inhibitory neurons in a Wilson-Cowan type model, instead of an Amari type model. The main focus is on the investigation of pattern formation in these systems on the sphere. Specifically, we look in detail at the periodic and quasi periodic orbits which are generated by Hopf bifurcation in the presence of spherical symmetry. We derive formulas to compute the normal form coefficients of these bifurcations and predict the stability of the resulting branches. All these results are used to study the effect of the gap junctions on the resulting patterns of the neural field.

We will demonstrate that predictions of the emerging spatio-temporal patterns are found to be in excellent agreement with the results of our direct numerical simulations.

[1] S. Visser, R. Nicks, O. Faugeras and S. Coombes, *Standing and travelling waves in a spherical brain model: The Nunez model revisited,* Physica D. **349** (2017), 27–45.

On the 2nd order linear ODEs with singular terms

Dalibor Pražák
Charles University, Prague, Czech Republic

Joint work with Vít Průša, Karel Tůma (Charles University, Prague).

The classical problem of stability of pendulum with a periodically oscillating pivot, where the motion of the pivot is not smooth, leads to the ODE of the form

$$x'' + (\alpha + \beta\Delta(t))x = 0 \tag{1}$$

where $\Delta(t)$ is a singular term, as for example the sum of Dirac measures

$$\Delta(t) = \delta_0(t) - \delta_\pi(t) + \delta_{2\pi}(t) + \dots \tag{2}$$

The characteristic equation (and hence the boundary of the stability region) can be computed explicitly in terms of α, β, in contrast of the classical situation (Mathieu's equation).

This motivates further interest in more general problems of similar form, as for example the Sturm–Liouville type equation

$$\big(p(t)x'\big)' + \big(\lambda r(t) + q(t)\big)x = 0 \tag{3}$$

with possibly singular terms $r(t)$ and $q(t)$. Also here, results analogous to the classical situation can be obtained (Sturm comparison theorems, Lyapunov theorem, . . .)

[1] D. Pražák, V. Průša, K. Tůma, *A note on parametric resonance induced by a singular parameter modulation.* Int. J. Non-Linear Mech., 139 (2022), 103893.

[2] D. Žárský, *Lineární diferenciální rovnice se singulárními členy.* Diploma thesis MFF UK, in preparation.

Identification of material parameters for Gao beam

Jana Radová
Faculty of Science, Palacký University in Olomouc, Czech Republic

Data identification belongs to an important class of problems with many practical applications. From a set of measured data one tries to identify some characteristic quantities which are not known a-priori in corresponding mathematical models.

This contribution is focused on the identification of material coefficients of a contact problem for the nonlinear Gao beam which is unilaterally supported by a foundation. The beam model is governed by a nonlinear fourth-order differential equation. We consider two types of foundations, perfectly rigid and elastic deformable foundation. The identification problem is formulated as optimal control problem with an appropriate least squares cost functional. The unknown material coefficients of the Gao beam and modulus of the foundation play the role of control variables if the elastic deformable foundation is considered. In the case of the identification problem with perfectly rigid foundation only material parameters of Gao beam represent control variables. The existence of a solution to the identification problem is analyzed. The presented theory is illustrated by several numerical examples.

[1] Gao, D. Y.: *Nonlinear elastic beam theory with application in contact problems and variational approaches.* Mechanics Research Communications, Vol. 23, No. 1, pp. 11–17, 1996.

[2] Machalová, J., Netuka, H.: *Comments on the large deformation elastic beam model developed by D.Y. Gao.* Mechanics Research Communications, Vol. 110, 103607, 2020.

[3] Machalová, J., Netuka, H.: *Solution of contact problems for Gao beam and elastic foundation.* Mathematics and Mechanics of Solids, Special Issue on Inequality Problems In Contact Mechanics, Vol. 23, No. 3, pp. 473–488, 2018.

[4] Radová, J. Machalová, J. Burkotová J.: *Identification Problem for Nonlinear Gao Beam.* Mathematics, Vol. 8, Issue 11, 1916, 2020.

The effects of viscous dissipation on the Darcy–Brinkman flow

Marko Radulović
University of Zagreb, Faculty of Science, Department of Mathematics, Croatia

This is a joint work with Professor Igor Pažanin (University of Zagreb, Croatia).

In this talk, we consider the non–isothermal flow through a thin domain occupied by a saturated porous medium. The flow is governed by the Darcy–Brinkman system and the heat equation including the viscous dissipation term proposed by Al–Hadhrami et al. in [1]. Using the methods of asymptotic analysis with respect to the domain's thickness, a second–order asymptotic approximation of the problem is built. The formally derived model is rigorously justified via error estimates.

The presented results were published in [2].

[1] A.K. Al–Hadhrami, L. Elliott, D.B. Ingham, *A new model for viscous dissipation in porous media across a range of permeability values.* Transp. Porous Media **53** (2003), 117–122.

[2] I. Pažanin, M. Radulović, *Effects of the viscous dissipation on the Darcy–Brinkman flow: Rigorous derivation of the higher–order asymptotic model.* Appl. Math. Comput. **386** (2020), 125479.

A numerical method to solve Maxwell's equations in 3D singular geometry

Irina Raichik
Bar Ilan University, Ramat Gan, Israel

This is a joint work with Prof Franck Assous (Ariel University, Ariel, Israel).

We propose a new variational method to compute the three-dimensional Maxwell equations in an axisymmetric singular domain, generated by the rotation of a singular polygon around one of its sides, namely containing reentrant corner or edges. Due to the axisymmetric assumption, the singular computational domain boils down to a subset of R^2. However, the electromagnetic field and other vector quantities still belong to R^3.

Taking advantage that the domain becomes a two-dimensional one, by doing Fourier analysis in the third dimension, one arrives to a sequence of singular problems set in a 2D domain, depending on the Fourier variable k. In these conditions, one will solve the 3D solution by solving several 2D problems, each one depending on k.

Furthermore, for each mode k, we can show that the solution can be decomposed into a regular and a singular part. Therefore, the regular part can be computed with a classical finite element method. The singular part belongs to a finite-dimensional subspace, its dimension being equal to the number of reentrant corners and edges of the 2D polygon that generates the 3D domain.

We first compute this singular part by an *ad hoc* numerical method only for $k = 0, \pm 1, 2$, the mode $k = 2$ appearing as a "stabilization" mode for all other k. Then, the total solution will be computed, based on a non stationary variational formulation. Numerical examples will be shown, that illustrate that the proposed method can capture the singular part of the solution. This approach can also be seen as the generalization of the Singular Complement Method to three-dimensional axisymmetric problem [1], [2].

[1] F. Assous, P. Ciarlet, Jr., S. Labrunie and J. Segré, *Numerical solution to the time-dependent Maxwell equations in axisymmetric singular domains: The Singular Complement Method.* J. Comput. Phys. **191** (2003) 147–176.

[2] F. Assous, P. Ciarlet, Jr., and S. Labrunie, Mathematical Foundations of Computational Electromagnetism. Appl. Math. Sc., AMS 198, Springer, 2018.

About sign-constancy of Green's functions of two-point impulsive boundary value problems

Vladimir Raichik
Ariel University, Ariel, Israel

This is a joint work with Prof A. Domoshnitsky and Dr S. Malev (Ariel University, Ariel, Israel).

We consider the following second order impulsive differential equation

$$\begin{cases} x''(t) + \sum_{j=1}^{m} p_j(t)x_j(h_j(t)) = f(t), \quad t \in [0,\omega], \\ x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1,2,\ldots,n, \end{cases}$$

where $\delta_k > 1$ for $k = 1,2,\ldots,n$.

We obtain sufficient conditions of semi-nonoscillation of the solution of the homogeneous equation on the interval $[0,\omega]$ (i.e. the fact that nontrivial solution having zero of its derivative on this interval does not have zero itself there).

Using these results we formulate theorems on differential inequalities and sign-constancy of Green's functions for two-point impulsive boundary value problems.

This work is based, in particular, on the results obtained in [1], [2], [3].

[1] A. Domoshnitsky, Iu. Mizgireva, V. Raichik, *Seminonoscillation intervals and sign-constancy of Green's functions of two-point impulsive boundary value problems*, Ukrainian Mathematical Journal **73** (2021), # 7, DOI 10.1007/s11253-021-01975-2.

[2] A. Domoshnitsky, G. Landsman, *Semi-nonoscillation Intervals in the Analysis of Sign Constancy of Green's Functions of Dirichlet, Neumann and Focal Impulsive Problems*, Advances in Difference Equations **81** (2017), # 1, DOI:10.1186/s13662-017-1134-1.

[3] A. Domoshnitsky, V. Raichik, *The Sturm Separation Theorem for Impulsive Delay Differential Equations*, Tatra Mt. Math. Publ. **71** (2018), 65–70.

Normalised solutions to semilinear equations with potential

Matteo Rizzi
Justus Liebig University, Germany

This is a joint work with Thomas Bartsch (Justus Liebig University, Giessen, Germany), Riccardo Molle (University of Roma 2, Rome, Italy) and Gianmaria Verzini (Politecnico di Milano, Milano, Italy)

In [1] we consider a Schrödinger type equation of the form

$$-\Delta u + (\lambda + V(x))u = |u|^{p-2}u \tag{1}$$

in $\mathbb{R}^N$, with a non radial potential V under the mass constraint

$$\int_{\mathbb{R}^N} v^2 = \rho^2. \tag{2}$$

We provide some sufficient conditions about V for existence of solutions $(u, \lambda) \in H^1(\mathbb{R}^N) \times (0, \infty)$ for powers $2 + \frac{4}{N} < p < \frac{2N}{N-2}$. The potential is allowed to have singularities. λ appears as a Lagrange multiplier, due to the mass constraint (2). The proof is variational, based on a min-max argument.

[1] T. Bartsch, R. Molle, M. Rizzi, G. Verzini, *Normalized solutions of mass supercritical Schrödinger equations with potential.* Comm. Partial Differential Equations **46** (2021), no. 9, 1729–1756.

Transversality conditions for discontinuous differential equations

Jorge Rodríguez-López
Universidade de Santiago de Compostela, Spain

This is a joint work with Prof Rodrigo López Pouso (Universidade de Santiago de Compostela, Spain).

We study the existence of absolutely continuous solutions for systems of discontinuous ordinary differential equations. More precisely, we are concerned with the existence of Carathéodory solutions to the initial value problem

$$x' = f(t,x) \quad \text{for a.a. } t \in [0,L], \quad x(0) = x_0 \in \mathbb{R}^n, \tag{1}$$

where $L > 0$ is fixed and the function $f : [0,L] \times \mathbb{R}^n \to \mathbb{R}^n$ need not be continuous. Obviously, the lack of continuity makes necessary to impose another condition on f in order to guarantee the existence of solutions for (1). Inspired by an earlier theorem due to Bressan and Shen [1], we assume that f satisfies a certain *transversality condition* which, unlike [1], is only imposed at the discontinuity points of f. Our approach involves differential inclusions and, in particular, the notion of Krasovskij solution of (1).

The results are included in the paper [2].

[1] A. Bressan and W. Shen, On discontinuous differential equations, *Differential Inclusions and Optimal Control*, J. Andres, L. Gorniewicz and P. Nistri Eds., Julius Schauder Center, Lect. Notes Nonlinear Anal. **2** (1998), 73–87.

[2] R. López Pouso and J. Rodríguez-López, Existence and uniqueness of solutions for systems of discontinuous differential equations under localized Bressan-Shen transversality conditions, *J. Math. Anal. Appl.* **492** (2020), 124425.

On the error analysis of the time-continuous Discontinuous Galerkin scheme for degenerate parabolic equations

Sunčica Sakić
Charles University, Czech Republic

This is a joint work with Vít Dolejší (Charles University, Prague, Czech Republic) and Scott Congreve (Charles University, Prague, Czech Republic).

We deal with an error analysis of a semidiscrete scheme for a doubly nonlinear parabolic partial differential equation (PDE), which admits the fast-diffusion type of degeneracy. The corresponding solutions of this equation are usually not smooth. These equations have been studied in papers on the existence, uniqueness, and regularity of solutions to elliptic-parabolic differential equations (see, e.g., [1] and [2]). A well-known representative of such a PDE type is the Richards' equation, which is widely used to model porous media flow.

A vast number of numerical methods have been proposed for solving such problems. We mention [3], which demonstrated the higher-order space-time discontinuous Galerkin finite element method as a promising tool for solving of the Richards' equation in an efficient, robust and accurate way. However, the rigorous mathematical theory for this method is still missing.

Therefore, in this talk, we focus on the error analysis for the time-continuous scheme. Due to its favorable features in this class of problems, we choose the incomplete interior penalty Galerkin (IIPG) method for spatial discretisation; cf. [3]. We pay special attention to the estimation of the accumulation term, which possibly can vanish. Furthermore, we use continuous mathematical induction to derive a priori error estimates in the L^2-norm and the so-called DG-norm with respect to the spatial discretisation parameter and the Hölder coefficient of the accumulation term derivative.

[1] H.W. Alt, S. Luckhaus, *Quasilinear elliptic-parabolic differential equations.* Math. Z. **183** (1983), 311–342.

[2] F. Otto , *L^1-contraction and uniqueness for quasilinear elliptic-parabolic equations.* J. Differ. Equ. **131**(1) (1996), 20–38.

[3] V. Dolejší, M. Kuraz, P. Solin, *Adaptive higher-order space-time discontinuous Galerkin method for the computer simulation of variably-saturated porous media flows.* Appl. Math. Model. (2019), 276–305

The existence and dimension of the attractor for the 3D flow of a non-Newtonian fluid subject to dynamic boundary conditions

Sembukutti Liyanage Buddhika Priyasad
Mathematical Institute of Charles University, Czech Republic

We investigate a generalized form of Navier-Stokes equations suggested by Ladyzhenskaya with dynamic boundary conditions. These equations govern an incompressible, homogenous fluid occupied in a bounded domain in 3 dimension. The stress tensor of such model, has several polynomial dependence on the symmetric velocity gradient. The goal is to estimate the dimension of the global attractor in terms of relevant physical constants

Non-oscillation of linear and half-linear Euler type differential equations

Jiřina Šišoláková
Faculty of Science, Masaryk University, Brno, Czech Republic

This talk is based on the joint work [1] with P. Hasil and M. Veselý and on the paper [2]. We deal with Euler type half-linear second order differential equations. In [2], there are derived conditions in order their non-trivial solutions are non-oscillatory. These conditions form the core of the talk. The corresponding oscillatory counterpart is studied and an oscillation criterion is established in [1]. The used effective technique for this investigation is the combination of the generalized adapted Prüfer angle and the modified Riccati transformation. The presented results are new even in the linear case.

[1] P. Hasil, J. Šišoláková, M. Veselý, *Averaging technique and oscillation criterion for linear and half-linear equations*, Appl. Math. Lett. **92** (2019), 62–69.

[2] J. Šišoláková, *Non-oscillation of linear and half-linear differential equations with unbounded coefficients*, Math. Methods Appl. Sci. **44** (2021), no. 2, 1285–1297.

Vallée-Poussin theorem for fractional functional differential equations

Satyam Narayan Srivastava
Department of Mathematics, Ariel University, Ariel, Israel

This is a joint work with Prof Alexander Domoshnitsky (Department of Mathematics, Ariel University, Ariel-40700, Israel) and Prof Seshadev Padhi (Department of Mathematics, Birla Institute of Technolology Mesra, Ranchi-835215, India)

An analog of the classical Vallée-Poussin theorem about differential inequality in the theory of ordinary differential equations is developed for fractional functional differential equations. The main results are obtained in a form of a theorem about several equivalent assertions. Among them solvability of two-point boundary value problems with fractional functional differential equation, negativity of Green's function, and its derivatives and existence of a function $v(t)$ satisfying a corresponding differential inequality. Thus the Vallée-Poussin theorem presents one of the possible "entrances" to assertions on nonoscillating properties and assertions about the negativity of Green's functions and their derivatives for various two-point problems. Choosing the function $v(t)$ in the condition, we obtain explicit tests of sign-constancy of Green's functions and their derivatives. It can be stressed that a choice of a corresponding function in the Vallée-Poussin theorem leads to explicit criteria in the form of algebraic inequalities, which, as we demonstrate with examples, cannot be improved. Replacing strict inequalities with non-strict ones will already lead to incorrect statements. In some cases, the well-known results obtained in the form of the Lyapunov inequalities for fractional differential equations can be improved based on ours. Another development, we propose, is a concept to consider fractional functional differential equations. The basis of this concept is to reduce boundary value problems for fractional functional differential equations to operator equations in the space of essentially bounded functions. Fractional functional differential equations can appear in various applications and in the process of construction of equations for a corresponding component of a solution-vector of a system of fractional equations.

Periodic solutions to symmetric Newtonian systems in neighborhoods of orbits of equilibria

Piotr Stefaniak
Faculty of Mathematics and Computer Science, Nicolaus Copernicus University in Toruń, Toruń, Poland

This is a joint work with Anna Gołębiewska, Marta Kowalczyk, Sławomir Rybicki (Nicolaus Copernicus University in Toruń, Poland).

The aim of this talk is to discuss the existence of periodic solutions to Newtonian systems of the form

$$\ddot{u}(t) = -\nabla U(u(t)). \tag{1}$$

in neighborhoods of equilibria. Allowing the potential U to be symmetric, we consider equilibria which are not necessarily isolated. More precisely, if the potential U is Γ-invariant for a compact Lie group Γ, the equilibria form orbits of the action of this group. Consequently, if $\dim \Gamma \geq 1$, then it can happen that $\dim \Gamma(u_0) \geq 1$, i.e., the critical point u_0 is not isolated in $(\nabla U)^{-1}(0)$.

Assuming these orbits of equilibria to be isolated, we apply equivariant bifurcation techniques to obtain a generalization of the classical Lyapunov center theorem. Our tool is an equivariant version of the Conley index given in [1]. To compare the indices we compute cohomological dimensions of some orbit spaces. To this end we use the results given in [2].

The talk is based on the paper [3].

[1] M. Izydorek, *Equivariant Conley index in Hilbert spaces and applications to strongly indefinite problems*, *Nonlinear Anal.*, **51** (2002), 33–66.

[2] T. Kawasaki, *Cohomology of twisted projective spaces and lens complexes*, *Math. Ann.*, **206** (1973), 243–248.

[3] A. Gołębiewska, M. Kowalczyk, S. Rybicki, P. Stefaniak, *Periodic solutions to symmetric Newtonian systems in neighborhoods of orbits of equilibria*, Electron. Res. Arch. **30(5)** (2022), 1691–1707.

Discrete Riccati matrix equation and the order preserving property

Viera Štoudková Ružičková
Brno University of Technology, Czech Republic

If a symmetric matrix differential equation

$$Q'(t) = G(t, Q),$$

where $Q(t)$ and $G(t, Q)$ are real symmetric $n \times n$ matrix functions, has the order preserving property and the matrix dimension is at least 2, then this equation is the Riccati matrix differential equation

$$Q'(t) + A^T(t)Q(t) + Q(t)A(t) + Q(t)B(t)Q(t) - C(t) = 0,$$

see [1]. A similar statement holds for discrete matrix equations as well, with the discrete Riccati matrix equation,

$$R[Q]_k := Q_{k+1}(\mathcal{A}_k + \mathcal{B}_k Q_k) - (\mathcal{C}_k + \mathcal{D}_k Q_k) = 0,$$

see [2]. In the proof we extend a discrete function to a continuous one by using the iteration theory and then apply the known result for the continuous case.

[1] A. N. Stokes, *A special property of the matrix Riccati equation.* Bull. Austral. Math. Soc. **10** (1974), 245–253.

[2] V. Štoudková Ružičková, *Discrete Riccati matrix equation and the order preserving property.* Linear Algebra Appl. **618** (2021), 58–75.

Equivalence of ill-posed dynamical systems

Tomoharu Suda
Faculty of Science and Technology, Keio University, Japan

The problem of topological classification is fundamental in the study of dynamical systems. However, when we consider systems without well-posedness, it is unclear how to generalize the notion of equivalence. For example, when a system has trajectories distinguished only by parametrization, we cannot apply the usual definition of equivalence based on the phase space, which presupposes the uniqueness of trajectories.

In this study, we formulate a notion of “topological equivalence" using the axiomatic theory of topological dynamics proposed by Yorke [1], where dynamical systems are considered to be shift-invariant subsets of a space of partial maps. In particular, we study how the type of problems can be regarded as invariants under the morphisms between systems and how the usual definition of topological equivalence can be generalized.

The materials in this talk are mainly based on [2].

[1] J.A. Yorke, *Spaces of solutions.* Mathematical Systems Theory and Economics I/II (1969), 383–403.

[2] T. Suda, *Equivalence of topological dynamics without well-posedness.* Topology Appl. **312** (2022), 108045.

Blow-up phenomena in a quasilinear chemotaxis system with logistic source

Yuya Tanaka
Tokyo University of Science, Japan

This talk reports a blow-up result in a parabolic–elliptic system involving a nonlinear diffusion term and a nonlinear chemotactic term as well as logistic source. This system describes a part of the life cycle of the cellular slime molds with chemotaxis. In particular, the diffusion and chemotactic terms show the random motion and concentration of cells, respectively, and the logistic source represents the proliferation and death of the cells. From the mathematical point of view, the chemotactic term leads to blow-up, whereas the diffusion term and the logistic source suppress it.

In the case that the diffusion and chemotactic terms are *linear*, Winkler [3] and Fuest [1] showed that if the logistic damping effect is small, then finite-time blow-up is possible. On the other hand, in the case that the diffusion and chemotactic terms are *nonlinear*, it is unknown how the effects of diffusion and chemotaxis affect blow-up.

The purpose of this talk is to give conditions for blow-up to a quasilinear chemotaxis system with logistic source. The talk is based on a result in [2].

[1] M. Fuest, *Approaching optimality in blow-up results for Keller–Segel systems with logistic-type dampening,* NoDEA Nonlinear Differential Equations Appl. **28** (2021), 17 pages.

[2] Y. Tanaka, *Boundedness and finite-time blow-up in a quasilinear parabolic–elliptic chemotaxis system with logistic source and nonlinear production*, J. Math. Anal. Appl. **506** (2022), 29 pages.

[3] M. Winkler, *Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction*, J. Math. Anal. Appl. **384** (2011), 261–272.

Numerical solution of Caputo fractional differential equations initial value problems

Petr Tomášek
Brno University of Technology, Brno, Czech Republic

The talk discusses numerical solution of the Caputo type fractional differential equation initial value problem

$$\begin{cases} D^{\alpha} y(t) = f(t, y(t)), & t \in [0, t_f]. \\ y(0) = y_0, \end{cases} \tag{IVP}$$

where $\alpha \in (0, 1)$. Basic approaches to numerical solution of (IVP) will be introduced. There arise specific problems in numerical solution of fractional order differential equations in comparison with ordinary differential equations. Fundamental properties of several numerical methods will be presented. A comparison of numerical solution and analytical solution will be demonstrated on several test equations too. The presented algorithms are developed in Python environment.

[1] K. Diethelm, V. Kiryakova, Y. Luchko, J.A.T. Machado, V.E. Tarasov, *Trends, directions for further research, and some open problems of fractional calculus*, Nonlinear Dyn. **107** (2022), 3245–3270.

[2] R. Garrappa, *Numerical solution of fractional differential equations: A survey and software tutorial*, Mathematics **2018** 6,16 (2018), 1–23.

[3] R. Garrappa, *Neglecting nonlocality leads to unreliable numerical methods for fractional differential equations*, Commun. Nonlinear Sci. Numer. Simulat. **70** (2019), 302–306.

[4] C.P. Li, F.H.Zeng, Numerical methods for fractional calculus. Chapman & Hall/CRC, Boca Raton, 2015.

Numerical approaches to the modelling of quasi-britte crack propagation

Jiří Vala

Brno University of Technology, Faculty of Civil Engineering, Brno, Czech Republic

Numerous materials of civil engineering structures, as cementitious composites with various reinforcements, suffer from initiation and propagations of micro- and macro-fractured zones, resulting in decrease of their load bearing ability and durability. Therefore deeper physical and mathematical modelling of related processes is needed, covering both formal verification and practical validation of such models, including reliable identification of material characteristics. The relevance of such modelling increases in the design of advanced materials, structures and technologies where no long-time practical experience is available. Using the nomenclature of [1], unlike pure ductile or brittle crack propagation, the computational analysis of the so-called quasi-brittle fracture in reinforced cement-based and similar composites is more difficult because of the initiation of zones of micro-defects, followed by the formation and propagation of a large number of macroscopic cracks, whose complete deterministic description is rarely available. Thus some non-local semi-heuristic model like [2] is needed, although the solvability of related problems, as studied by [3], may be not transparent.

As a model problem we can introduce the principle of conservation of energy in the form

$$\rho\, \partial^2 u(x,t)/\partial t^2 + \nabla\sigma(x,t) = f(x,t) \;\text{ on } \Omega\,, \qquad \sigma(x,t)\cdot n = g(x,t) \;\text{ on } \partial\Omega$$

on certain domain Ω in the 3-dimensional Euclidean space (with unit normal vectors n on its boundary $\partial\Omega$), supplied by the Cartesian coordinated system x, in any time $t \geq 0$; here ρ is the material density, f and g are the vectors of volume and surface loads and u is the unknown vector of displacements (related to the initial configuration). The symmetric stress matrix σ must be evaluated from some constitutive equation, namely as $\sigma(\nabla u, C, \mathcal{D}(u))$, C being the $(3\times 3)\times(3\times 3)$-stiffness tensor (by the Hooke law: 21 independent factors in general, only 2 for isotropic continua); certain smeared cracking is then incorporated by [4] using a non-local multiplicative factor $\mathcal{D}(u)$. Related mathematical and computational problems have been discussed by [5] and by [6]. More general formulations are needed; some possibilities will be demonstrated in this contribution.

[1] Y. Sumi, Mathematical and Computational Analyses of Cracking Formation. Springer, Tokyo, 2014.

[2] C. Giry, F. Dufour and J. Mazars, *Stress-based nonlocal damage model.* Int. J. Solids and Struct. **48** (2011), 3431–3443.

[3] A. Evgrafov and J. C. Belido, *From nonlocal Eringen's model to fractional elasticity.* Mathematics and Mechanics of Solids **24** (2019), 1935–1953.

[4] P. Havlásek, P. Grassl and M. Jirásek, *Analysis of size effect on strength of quasi-brittle materials using integral-type nonlocal models.* Eng. Fract. Mech. **157** (2016), pp. 72–85.

[5] J. Vala and V. Kozák, *Computational analysis of quasi-brittle fracture in fibre reinforced cementitious composites.* Theor. Appl. Fract. Mech. **107** (2020), 102486/1–8.

[6] J. Vala and V. Kozák, *Nonlocal damage modelling of quasi-brittle composites.* Appl. Math. **66** (2021), 701–721.

Asymptotic behavior of solutions to an advance-delay differential equation

Gabriela Vážanová
Faculty of Electrical Engineering and Communication, Brno University of Technology, Brno, Czech Republic

This is a joint contribution with Professor Josef Diblík (Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Brno University of Technology, Veveří 331/95, 602 00 Brno, Czech Republic).

In the contribution, a scalar differential equation of an advance-delay type

$$\dot{y}(t) = -\left(a_0 + \frac{a_1}{t}\right) y(t-\tau) + \left(b_0 + \frac{b_1}{t}\right) y(t+\sigma), \tag{1}$$

where $a_0, b_0, \tau,\ \sigma \in (0,\infty)$ and a_1, b_1 are constants, is considered. The behavior of solutions for $t \to \infty$ is examined provided that the transcendental equation

$$\lambda = -a_0 \mathrm{e}^{-\lambda\tau} + b_0 \mathrm{e}^{\lambda\sigma}$$

has a real root. The form of a solution is searched as an exponential-type function and its existence is proved by a monotone iterative sequences method. Moreover, an estimation is given of a semi-global solution to equation (1).

[1] J. Diblík, M. Kúdelčíková, *Inequalities for positive solutions of the equation $\dot{y}(t) = -(a_0 + a_1/t)x(t - \tau_1) - (b_0 + b_1/t)x(t - \tau_2)$*. Stud. Univ. Zilina Math. Ser. **17** (2003), no. 1, 27–46.

[2] J. Diblík, M. Kúdelčíková, *Inequalities for the positive solutions of the equation $\dot{y}(t) = -\sum_{i=1}^{n}(a_i + b_i/t)y(t - \tau_i)$*. Differential and Difference Equations and Applications, Hindawi Publ. Corp., New York, (2006), 341–350.

[3] J. Diblík, M. Kúdelčíková, *Nonoscillating solutions of the equation $\dot{x}(t) = -(a + b/t)x(t - \tau)$*. Stud. Univ. Zilina Math. Ser. **15** (2002), no. 1, 11–24.

[4] J. Diblík, G. Vážanová, *Lower and upper estimates of semi-global and global solutions to mixed-type functional differential equations.* Adv. Nonlinear Anal. **11** (2022), no. 1, 757–784.

[5] R. D. Driver, Ordinary and Delay Differential Equations. Springer-Verlag, 1977.

[6] I. Györi, G. Ladas, Oscillation Theory of Delay Differential Equations. Clarendon Press, Oxford, 1991.

[7] J. K. Hale, S. M. V. Lunel, Introduction to Functional Differential Equations. Springer-Verlag, 1993.

[8] E. Zeidler, Nonlinear Functional Analysis and its Application, part I, Fixed-Point Theorems. Springer-Verlag, 1985.

Perturbations of homogeneous linear difference systems with coefficient matrices from commutative groups

Michal Veselý

Department of Mathematics and Statistics, Faculty of Science, Masaryk University

This is a joint work with Petr Hasil (see [1, 2, 3, 4]). We consider limit periodic perturbations of homogeneous linear difference systems over infinite fields with absolute values. Using iterative methods, we study the solution spaces of limit periodic and almost periodic systems, where the coefficient matrices of the considered systems are taken from a given commutative (or bounded) group. In particular, we analyse such systems whose fundamental matrices are not almost periodic. We identify conditions on the matrix group which guarantee that these systems form a dense subset in the space of all considered systems.

[1] P. Hasil, M. Veselý, *Limit periodic homogeneous linear difference systems.* Appl. Math. Comput. **265** (2015), 958–972.

[2] P. Hasil, M. Veselý, *Solution spaces of homogeneous linear difference systems with coefficient matrices from commutative groups.* J. Difference Equ. Appl. **23** (2017), no. 8, 1324–1353.

[3] P. Hasil, M. Veselý, *Asymptotically almost periodic solutions of limit periodic difference systems with coefficients from commutative groups.* Topol. Meth. Nonlinear Anal. **54** (2019), no. 2A, 515–535.

[4] P. Hasil, M. Veselý, *Limit periodic perturbations of difference systems with coefficients from commutative groups.* Submitted (2022).

Stabilization in degenerate parabolic equations in divergence form and application to chemotaxis systems

Tomomi Yokota
Tokyo University of Science, Tokyo, Japan

This is a joint work with Professor Sachiko Ishida (Chiba University, Chiba, Japan).

This talk deals with the quasilinear degenerate parabolic equation

$$u_t = \nabla \cdot (f(u)\nabla u + g(u,x,t)), \quad x \in \Omega,\ t > 0,$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ under the no-flux boundary condition with non-negative initial data $u_0 \in L^\infty(\Omega)$. Here f is a non-negative function belonging to $C([0,\infty)) \cap C^2((0,\infty))$, and g is a vector-valued function on $[0,\infty) \times \Omega \times (0,\infty)$. It is known that this equation has a global-in-time weak solution under some conditions for f and g by the well-known parabolic theory.

The purpse of this talk is to present a stabilization result in [1]; in detail, the above equation admits a global weak solution which fulfills

$$u(\cdot,t) \to \overline{u_0} \quad \text{weakly}^* \text{ in } L^\infty(\Omega) \text{ as } t \to \infty,$$

where $\overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0$. The result can be applied to degenerate Keller–Segel systems.

[1] S. Ishida, T. Yokota, *Weak stabilization in degenerate parabolic equations in divergence form: application to degenerate Keller–Segel systems.* Calc. Var. Partial Differential Equations **61** (2022), Paper No. 105.

Fisher–Kolmogorov equation with discontinuous density dependent diffusion

Michaela Zahradníková
Department of Mathematics and NTIS, Faculty of Applied Sciences
University of West Bohemia, Pilsen, Czech Republic

This is a joint work with Pavel Drábek (University of West Bohemia, Pilsen, Czech Republic). We are concerned with the quasilinear reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(d(u)\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + g(u), \quad (x,t) \in \mathbb{R} \times [0,+\infty), \quad p > 1, \tag{1}$$

and its travelling wave solutions $u(x,t) = U(x - ct)$, where c denotes the (unknown) speed of propagation. We consider a Fisher-KPP type reaction term $g \in C[0,1]$ (also called monostable), i.e., $g(0) = g(1) = 0$, $g > 0$ in $(0,1)$. The diffusion coefficient $d = d(s)$ is a rather general function in the sense that it need not be continuous in $[0,1]$ or even in $(0,1)$. In particular, d may vanish or be singular at one or both endpoints and it may also have discontinuities of the first kind at a finite number of points in $(0,1)$.

We obtain existence and nonexistence results for (1) by investigating the equivalent first order problem

$$\begin{cases} y'(t) = p'\left[c\left(y^{+}(t)\right)^{\frac{1}{p}} - f(t)\right], \quad t \in (0,1), \\ y(0) = y(1) = 0 \end{cases} \tag{2}$$

where $f(t) = (d(t))^{\frac{1}{p-1}}\, g(t)$, $f \in L^1(0,1)$. Our method is based on comparison results for ODEs in the sense of Carathéodory. We present sufficient condition which guarantees the existence of $c^* > 0$ such that (2) possesses a unique solution $y_c = y_c(t)$, $y_c > 0$ in $(0,1)$, for each $c \geq c^*$. We discuss the joint influence of d and g on the existence of travelling waves and how it affects the minimal wave speed c^*. In the case of power-type behaviour of the reaction and diffusion terms near 0 and 1, we also study asymptotic properties of travelling wave profiles. Detailed proofs and discussion can be found in [1].

Our approach provides a broad theoretical background for mathematical treatment of various phenomena arising in population dynamics, chemistry and physics. In [2] we used similar methods to derive existence and uniqueness results for (1) with a bistable reaction term g.

[1] P. Drábek, M. Zahradníková, *Travelling waves for generalized Fisher–Kolmogorov equation with discontinuous density dependent diffusion* (under review). Preprint available on Authorea: `https://doi.org/10.22541/au.165208146.61972405/v1`

[2] P. Drábek, M. Zahradníková, *Traveling waves for unbalanced bistable equations with density dependent diffusion.* Electron. J. Differential Equations **76** (2021), 1–21.

On the uniqueness of the solution and finite-dimensional attractors for the 3D flow with dynamic boundary condition

Michael Zelina
Charles University, Faculty of Mathematics and Physics, Prague, Czech Republic

This is a joint work with Dalibor Pražák (Charles University, Prague, Czech Republic).

We consider a general class of incompressible, non-Newtonian fluids of power-law model occupying a bounded domain in $\mathbb{R}^3$. We work with a rather general class of the so-called dynamic boundary condition

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{n} &= 0, \\ \beta \partial_t \boldsymbol{u} + s(\boldsymbol{u}) &= -\big[\boldsymbol{S}\boldsymbol{n}\big]_\tau, \end{aligned}$$

where $\boldsymbol{n}$ is the outer normal, τ the tangential projection, $\boldsymbol{S}$ the Cauchy stress and β some positive constant. The function $s(\cdot)$ is non-linear, or even possibly implicit in the sense of corresponding to a maximal monotone graph. A weak solution to such a problem exists according to [1].

Our aim here is to extend some results which are well-known for the same model subject to a homogenenous Dirichlet boundary condition. Firstly, we use the approach of [2] to obtain additional time regularity of arbitrary weak solution, provided that the power-law exponent r is at least equal to the critical value $\frac{11}{5}$.

As a second and natural corollary, we establish the existence of global attractor for all $r \geq \frac{11}{5}$. Assuming further that the boundary nonlinearity is Lipschitz continuous, we also show that the attractor is finite-dimensional. Moreover, the exponential attractor can also be constructed. It extends the result of [3].

[1] Abbatiello, A and Bulíček, M. and Maringová, E., *On the dynamic slip boundary condition for Navier–Stokes-like problems*, Math. Models Methods Appl. Sci. **31** (2021), 2165–2212.

[2] Bulíček, M. and Kaplický, P. and Pražák, D., *Uniqueness and regularity of flows of non-Newtonian fluids with critical power-law growth*, Math. Models Methods Appl. Sci. **29** (2019), 1207–1225.

[3] Bulíček, M. and Ettwein, F. and Kaplický, P. and Pražák, D., *On uniqueness and time regularity of flows of power-law like non-Newtonian fluids*, Math. Methods Appl. Sci. **33** (2010), 1995–2010.

Discrete symplectic systems and eigenfuctions expansion

Petr Zemánek

Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Czech Republic

Eigenfunctions expansion is a very classical topic in the spectral theory of linear operators on a Hilbert space and its origin can be traced back at least to works of Euler, d'Alembert, D. Bernoulli and, especially, of Fourier. Subsequently, in the works of Sturm and Liouville, the theory of the eigenfunctions expansion was built in a more general fashion of regular boundary value problems associated with the second order linear differential equations. Its generalization to the case of a system of the first order differential equations was initiated by Hurwitz and all of these results can be unified by using linear Hamiltonian differential systems.

Although difference equations play also an important role in mathematical physics or continuum mechanics, the literature covering discrete counterparts of expansion theorems seems to be, surprisingly, quite humbler. In this talk we focus on an eigenfunctions expansion for a class of boundary value problems determined by the regular discrete symplectic system depending linearly on the spectral parameter $\lambda \in \mathbb{C}$, i.e.,

$$z_k(\lambda) = (\mathcal{S}_k + \lambda\, \mathcal{V}_k)\, z_{k+1}(\lambda), \quad k \in \mathcal{I}_{\mathbb{Z}}, \tag{S_λ}$$

where $\mathcal{I}_{\mathbb{Z}} = [0, N]_{\mathbb{Z}} := [0, N] \cap \mathbb{Z}$ is a finite discrete interval and, for every $k \in \mathcal{I}_{\mathbb{Z}}$, the coefficients $\mathcal{S}_k, \mathcal{V}_k$ are $2n \times 2n$ matrices such that

$$\mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{V}_k^* \mathcal{J} \mathcal{S}_k \ \text{ is Hermitian,} \quad \text{and} \quad \mathcal{V}_k^* \mathcal{J}\, \mathcal{V}_k = 0 \tag{1}$$

for $\mathcal{J}$ being the $2n \times 2n$ orthogonal and skew-symmetric matrix $\mathcal{J} := \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$ with $n \times n$ blocks of zero and identity matrices. The first condition in (1) means that $\mathcal{S}_k$ are symplectic matrices and all conditions in (1) can be simultaneously written by using the matrix $\mathbb{S}_k(\lambda) := \mathcal{S}_k + \lambda\, \mathcal{V}_k$ as the symplectic-type identity

$$\mathbb{S}_k^*(\bar{\lambda})\, \mathcal{J}\, \mathbb{S}_k(\lambda) = \mathcal{J},$$

which is valid for all $k \in \mathcal{I}_{\mathbb{Z}}$ and $\lambda \in \mathbb{C}$. The main result represents a significant generalization of the Expansion theorem given by Bohner, Došlý and Kratz in [1] for the case of system (S_λ) with a very special linear dependence on λ. The talk is based on [2].

[1] M. Bohner, O. Došlý, and W. Kratz, *Sturmian and spectral theory for discrete symplectic systems*, Trans. Amer. Math. Soc. **361** (2009), no. 6, 3109–3123.

[2] P. Zemánek, *Eigenfunctions expansion for discrete symplectic systems with general linear dependence on spectral parameter*, J. Math. Anal. Appl. **499** (2021), no. 2, Art. No. 125054, 37 pp. (electronic).

Abstracts – posters

On a shape derivative formula for the Robin p-Laplace eigenvalue problem

Ardra A
Department of Mathematics, Indian Institute of Technology Palakkad, Palakkad, India

This is a joint work with Dr. Sarath Sasi (IIT Palakkad, Palakkad, India) and Dr. Mohan Mallick (SRM University AP, Amaravati, India).

We consider the following Robin eigenvalue problem

$$\begin{aligned} \Delta_p u + \lambda |u|^{p-2} u = 0 \quad & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \eta} + \beta |u|^{p-2} u = 0 \quad & \text{on } \partial\Omega, \end{aligned}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplace operator for $1 < p < \infty$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, η denotes the outward unit normal and β is a positive real constant. We find a shape derivative formula for the first eigenvalue of this problem. This formula is used to show that the domain derivative of the first eigenvalue of a ball is negative for certain volume increasing smooth perturbations. Analogous results are proved for certain volume decreasing smooth perturbations. Domain monotonicity results in the existing literature assumes convexity of at least one of the domains even in the case of the Laplace operator. Our results indicate that, under certain conditions on β, domain monotonicity might be valid for more general domains.

Rigid body in compressible flow with general inflow-outflow boundary data

Šimon Axmann
University of Chemistry and Technology, Prague, Czechia

This is a joint work with Šárka Nečasová (Czech Academy of Sciences, Prague, Czechia) and Ana Radošević (University of Zagreb, Croatia).

We consider the initial-boundary value problem for a system of partial differential equations describing the motion of a rigid body placed in a compressible newtonian fluid. We do not include the thermal effects, assuming the pressure given by the isentropic equation of state.

The main novelty with respect to the known result of Feireisl [1] lies in the presence of general nonhomogenous Dirichlet boundary data for the velocity field, in the spirit of [2].

The main result states the existence of a weak solution to the problem on a time interval as long as the rigid body does not collide with the boundary of the domain.

[1] E. Feireisl, *On the motion of rigid bodies in a viscous compressible fluid.* Arch. Ration. Mech. Anal. **167** (2003), 281–308.

[2] T. Chang, B. J. Jin, and A. Novotný, *Compressible Navier–Stokes system with general inflow-outflow boundary data.* SIAM J. Math. Anal. **51** (2019), 1238–1278.

Oscillation of third-order neutral differential equation with oscillatory operator

Miroslav Bartušek
Masaryk University, Czech Republic

A third-order nonlinear neutral differential equation

$$z''' + q(t)z' + r(t)|x(\sigma(t))|^{\lambda} \operatorname{sgn} x(\sigma(t)) = 0\,, \quad t \geq 0$$

with $z(t) = x(t) + a(t)x(\tau(t))$ is studied where $\lambda \in (0,1]$, $q \geq 0$, $r > 0$, $a \geq 0$, $\sigma(t) \leq \tau(t) \leq t$ and the associated equation $h'' + q(t)h = 0$ is oscillatory. Sufficient conditions are given under which every solution x is either oscillatory or z' oscillates. Hence, the very known Property A (for $a \equiv 0$) is generalized (in some sence) to the above given equation.

Periodic and connecting orbits for delay differential equations

Gábor Benedek
University of Szeged, Bolyai Institute, Szeged, Hungary

This is a joint work with Tibor Krisztin (University of Szeged, Bolyai Institute, Szeged, Hungary).

We consider a nonlinear differential equation with delayed feedback. Numerical results suggest that the equation can generate complex dynamics. An example is the famous *Mackey-Glass equation* modeling physiological processes in which time lag plays a significant role. The nonlinearity in the equation allows to model the so-called Allee effect in population dynamics.

First, a limiting version of the equation is introduced with discontinuous nonlinearity. A combination of analytical and verified numerical tools gives the existence of an orbitally asymptotically stable periodic orbit. Then it is shown that near this periodic orbit the original equation has a periodic orbit, as well.

For nonlinearities, allowing to model the Allee effect, an additional unstable equilibrium occurs. From this equilibrium point, and from periodic orbits close to this equilibrium there exists a connecting orbit to the stable periodic orbit obtained in the first step.

Approximation of systems with delay their stability

Igor Cherevko, Iryna Tuzyk
Yuriy Fedkovych Chernivtsi National University, Ukraine

In this paper, the schemes of approximation of linear systems with delays by special systems of ordinary differential equations are considered and the connections between their solutions are investigated. This allowed us to build algorithms for studying the stability of linear systems with delay.

Consider the initial problem for a linear system of differential-difference equations

$$\frac{dx}{dt} = Ax(t) + \sum_{i=1}^{k} B_i x(t - \tau_i), \tag{1}$$

$$x(t) = \varphi(t), t \in [-\tau, 0], \tag{2}$$

where $A, B_i, i = \overline{1,k}$ fixed $n \times n$ matrix $x \in R^n$, $0 < \tau_1 < \tau_2 < ... < \tau_k = \tau, \varphi(t) \in [-\tau, 0]$.

Let us correspond to the initial problem (1) – (2) the system of ordinary differential equations [1-3]

$$\frac{dz_0(t)}{dt} = A(t)z_0(t) + \sum_{i=1}^{k} B_i z_{l_i}(t), l_i = [\frac{\tau_i m}{\tau}],$$
$$\frac{dz_j(t)}{dt} = \mu(z_{j-1}(t) - z_j(t)), j = \overline{1,m}, \mu = \frac{\mu}{\tau}, \mu \in N,$$

with initial conditions

$$z_j(0) = \varphi(-\frac{\tau j}{m}), j = \overline{0,m}.$$

Theorem [1]. *If the zero solution of the system with delay (1) is exponentially stable (not stable), then there is $m_0 > 0$ such that for all $m > m_0$, the zero solution of the approximating system (3) is also exponentially stable (not stable).*

If for all $m > m_0$ zero solution of approximation system (3) is exponentially stable (not stable) then the zero solution of the system with a delay (1) is exponentially stable (not stable).

It follows from Theorem that for a sufficiently large m the asymptotic stability (instability) of the zero solution of a linear system with a delay is equivalent to the asymptotic stability (instability) of the zero solution of the system of approximate ordinary differential equations.

[1] Matviy O.V., Cherevko I.M., *About approximation of system with delay and them stability*, Nonlinear oscilations **2** (2004), 208–216.

[2] Ilika S.A., Piddubna L.A., Tuzyk I. I., Cherevko I.M., *Approximation of linear differential-difference equations and their application*, Bukovinian Mathematical Journal **6** (2018), 80–83 .

[3] Cherevko I., Tuzyk I., Ilika S., Pertsov A., *Approximation of Systems with Delay and Algorithms for Modeling Their Stability,* 2021 11th International Conference on Advanced Computer Information Technologies ACIT'2021 (2021), 49–52.

The application of Koopman operator-based algorithms to dynamical systems

Nelida Črnjarić-Žic
University of Rijeka, Faculty of Engineering, Croatia

The goal of this paper is to analyze various dynamical systems using algorithms based on the Koopman operator. The Koopman operator provides an alternative approach to describing the evolution of a dynamical system. It is a linear infinite-dimensional operator, so the analysis is based on the study of the spectral properties of this operator. The Koopman operator-based data-driven algorithms make it possible to identify the spectral elements of the Koopman operator from data. They can then be used to analyze the behavior of the system under consideration, to reconstruct or predict the system dynamics, and as a tool to reduce the dimensionality of the data provided. In this work, the algorithms are applied to data obtained from various dynamical systems, e.g., data from HVAC systems, fluid flow systems, epidemiological data, etc. The corresponding discussions and analyzes were performed on the observed examples.

Numerical solution of the micropolar reactive real gas model for one-dimensional flow and thermal explosion

Ivan Dražić

University of Rijeka, Faculty of Engineering, Rijeka, Croatia

This is a joint work with Angela Bašić-Šiško (University of Rijeka, Faculty of Engineering, Rijeka, Croatia).

In this work, we consider the unsteady one-dimensional flow of a viscous micropolar and heat-conducting real gas, i.e., the fluid with the pressure given by $\rho^p\theta$, where ρ is the mass density, θ is the temperature, and $p \geq 1$ is the pressure exponent. In addition to the usual variables of the micropolar fluid model, a new variable describing the amount of unburned fuel and an equation describing the chemical reaction were added to this model. The mathematical model is derived in the Lagrangian description and presented in [1].

For the corresponding initial boundary problem with homogeneous boundary conditions as well as with sufficiently smooth initial conditions, we introduce a system of approximate equations using the Faedo-Galerkin method and construct its solutions. We describe the results of numerical tests on a number of examples, presenting the computed solutions, and test convergence and stabilization.

[1] A. Bašić-Šiško, I. Dražić, I., *Local existence for viscous reactive micropolar real gas flow and thermal explosion with homogeneous boundary conditions*, J. Math. Anal. Appl., **509** (2) (2022), 125988

Multipoint problem for higher-order Euler partial differential equations

Volodymyr Ilkiv[1], **Zinovii Nytrebych**[1], **Yaroslav Slonovskyi**[1], **Mykhailo Symotiuk**[1,2]
[1]*Lviv Polytechnic National University, Lviv, Ukraine,*
[2]*Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, NASU, Lviv, Ukraine*

In recent decades, multipoint problems for differential equations have been actively studied. For certain classes of lower-order equations, the conditions for the existence and uniqueness of the solution of such problems are established, the Green's function is constructed and its properties are studied, methods for the approximate solution are described in [1, 2]. For the higher-order partial differential equations, problems with multipoint conditions on the selected variable and periodicity conditions on the rest of the coordinates are, in general, ill-posed according to Hadamard, and their solvability in many cases is related to the problem of the small denominators [1, 2, 3]. Using the metric approach, B.Y. Ptashnyk and his Ph.D. students proved that the conditions for the unique solvability of multipoint problems are satisfied for almost all (with respect to the Lebesgue measure) vectors constructed by the coefficients of an equation and the values of interpolation nodes.

Multipoint problems for partial differential equations with variable coefficients with power degenerations remain weakly studied. We consider the next multipoint problem for higher-order partial differential equations of Euler-type

$$\Big[t^n\partial_t^n+\sum_{j=1}^{n}t^{n-j}a_j(\partial_x)\partial_t^{n-j}\Big]u(t,x)=0,\qquad (t,x)\in(t_1^*,t_2^*)\times\Omega_{2\pi}^p, \tag{1}$$

$$u(t_j,x)=\varphi_j(x),\qquad j=0,1,\ldots,n-1,\quad x\in\Omega_{2\pi}^p, \tag{2}$$

where $n,p\in\mathbb{N},n>2$, $\Omega_{2\pi}^p$ is a p-dimensional torus $(\mathbb{R}/2\pi\mathbb{Z})^p$, $x=(x_1,\ldots,x_p)\in\Omega_{2\pi}^p$,

$$\partial_t=\partial/\partial t,\qquad \partial_x^s=\partial_{x_1}^{s_1}\ldots\partial_{x_p}^{s_p},\quad a_j(\partial_x)=\sum\nolimits_{|s|\le j}a_{js}\partial_x^s,\quad a_{js}\in\mathbb{C},\quad s=(s_1,\ldots,s_p)\in\mathbb{Z}_+^p,$$

$$0<t_1^*\le t_0<t_1<\ldots<t_{n-1}\le t_2^*<\infty.$$

The conditions for the correctness of the problem (1), (2) in the spaces of smooth functions, which Fourier coefficients on variables $x_1,\ldots,x_p$ have power or exponential decay, are established. The properties of the solution of the problem (1), (2) at $t\to 0$ are investigated. The partial cases of the problem are considered when 1) nodes $t_0,\ldots,t_{n-1}$ are logarithmically equidistant or 2) roots of the characteristic equations are equidistant. The metric theorems about lower bounds estimates of the small denominators of the problem (1), (2) are proved.

This work was supported by the budget program of Ukraine "Support for the development of priority research areas" (CPCEC 6451230).

[1] O. Vejvoda, L. Herrmann, V. Lovicar, M. Sova, I. Straskraba, M. Stedry, Partial differential equations: time periodic solutions. Sijthoff and Noordhoof International Publishers B.V., Alphen aan den Rijn, 1981.

[2] B. Y. Ptashnyk, Ill-posed boundary value problems for partial differential equations, Nauk. Dumka, Kyiv, 1984.

[3] M. Berti, P. Bolle, *Cantor families of periodic solutions for completely resonant nonlinear wave equations.* Duke Math. J. **134** (2006), 359-419.

Singular semilinear elliptic problems on unbounded domains in $\mathbb{R}^n$

Anumol Joseph
Indian Institute of Technology Palakkad, India

This is a joint work with Lakshmi Sankar (Indian Institute of Technology Palakkad, Palakkad, India).

We prove the compactness of the solution operator for a class of singular semilinear elliptic problems on the exterior of a ball in $\mathbb{R}^n$, $n \geq 3$. Compactness of solution operators for similar problems in $\mathbb{R}^n$, $n \geq 2$ are also established. Further, using these compactness results and employing Schauder fixed point theorem, we prove the existence of a positive solution to classes of semipositone problems with asymptotically linear reaction terms.

Uniqueness and regularity of flows of non-Newtonian fluids around the critical power-law growth

Petr Kaplický
Charles University, Prague, Czech Republic

This is a joint work with Miroslav Bulíček, Frank Ettwein and Dalibor Pražák.

We deal with the flows of non-Newtonian fluids in three dimensional setting subjected to the homogeneous Dirichlet boundary condition. Under the natural monotonicity, coercivity and growth condition on the Cauchy stress tensor expressed by a critical power index $p = 11/5$ we show that a Gehring type argument is applicable which allows to improve regularity of any weak solution. Improving further the regularity of weak solutions along a regularity ladder allows to show that actually solution belongs to a uniqueness class provided data of the problem are sufficiently smooth.

We also discuss if the similar technique can work below the critical value of the growth exponent.

The results are based on [1] and [2].

[1] M. Bulíček, F. Ettwein, P. Kaplický, D. Pražák: On uniqueness and time regularity of flows of power-law like non-Newtonian fluids. Math. Methods Appl. Sci. 33 (2010).

[2] M. Bulíček, P. Kaplický, D. Pražák: Uniqueness and regularity of flows of non-Newtonian fluids with critical power-law growth. Math. Models Methods Appl. Sci. 29 (2019), no. 6, 1207–1225.

Uniform attractors in sup-norm for semilinear parabolic problem and application to the robast stability theory

Olena Kapustian[1], Oleksiy Kapustyan[1], Oleksandr Stanzytskiy[1], Ihor Korol[2]

[1] *Taras Shevchenko National University of Kyiv, Ukraine*

[2] *Uzhhorod National University, Uzhhorod, Ukraine*

Stability property of stationary points plays an important role in robust control theory. The notion of input-to-state stability, firstly appeared in [1] now is widely used to nonlinear systems of different nature. In recent years there have appeared many papers devoted to adaptation of input-to-state stability theory to infinite dimensional case [2]. In particular, in [3] there were obtained results about input to state stability and asymptotic gain properties with respect to global attractors of semi linear heat and wave equations in L^2 space. This results requires that the corresponding non autonomous problem generated semi process family with uniform attractor which tends to the global attractor of undisturbed system.

In the present paper we apply this scheme to the case of the phase space $\mathbb{C}_a$ of continuous functions supplied with sup-norm. Also we prove that under suitable assumptions, mild solutions generate a semi process family on $\mathbb{C}_0$ with uniform attractors. Finally we establish local input-to-state stability and asymptotic gain properties w.r.t. global attractors of the unperturbed system.

Most precisely we consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} = Au + f(u) + h(t,x), \quad (t,x) \in (0,\infty) \times \Omega \\ u(0,x) = u_0(x), \\ u|_{\delta\Omega} = 0, \end{cases} , \tag{1}$$

where A is a strongly elliptic self-adjoint operator, f is locally Lipschitz, h is bounded disturbance. Considering mild solutions in the phase space C_0, we prove that

$$\overline{\lim_{t\to\infty}} \inf_{\xi\in\Theta} \|\xi - u(t)\|_\infty \le \gamma(\|h\|_\infty),$$

where Θ is a global attractor of undisturbed system ($h \equiv 0$), γ is a strictly increasing function, vanishing at the origin.

[1] E.D. Sontag, *Smooth stabilization implies coprime factorization*, IEEE Trans. Automat. Control, **34** N**4** (1989), 435–443.

[2] S. Dashkovskiy, A. Mironchenko *Input-to-state stability of infinite-dimensional control systems.* Math. Control Signal Systems **25** (2013), 1–35.

[3] S. Dashkovskiy, O.Kapustyan, J. Shmid, *A local input-to-state stability result w.r.t. attractors of nonlinear reaction-diffusion equations,* Math. Control Signal Systems **32 N 3** (2020), 309–326.

Neumann Laplacian in a perturbed domain

Hong Hai Ly
Department of Mathematics, University of Ostrava, Ostrava, Czech Republic

This is a joint work with Prof. Diana Barsegyan, Prof. Baruch Schneider (University of Ostrava, Czech Republic).

We consider a domain with a small compact set of zero Lebesgue measure of removed. Our main result concerns the spectrum of the Neumann Laplacian defined on such domain. We prove that the spectrum of the Laplacian converges in the Hausdorff distance sense to the spectrum of the Laplacian defined on the unperturbed domain.

[1] D. Barseghyan, B Schneider, L. H. Hai, Neumann Laplacian in a perturbed domain, Mediterranean Journal of Mathematics, 19 (2022), 1–17.

[2] O. Post, *Spectral convergence of quasi-one-dimensional spaces.* Annales. Henri Poincare 7 (2006), 933—973.

[3] J. Rauch, M. Taylor, *Potential and scattering theory on wildly perturbed domains*, Journal of Functional Analysis 18(1975), 27—59.

[4] S. Ozawa, *Spectra of domains with small spherical Neumann boundary*, Proceedings of the Japan Academy, Series A, Mathematical Sciences 58 (1982),

Theoretical guarantees for the statistical finite element method

Yanni Papandreou
Imperial College London, UK

This is joint work with Dr. Jon Cockayne (University of Southampton, UK), Prof. Mark Girolami (University of Cambridge, UK), and Dr. Andrew Duncan (Imperial College London, UK).

The statistical finite element method (StatFEM) is an emerging probabilistic method that allows observations of a physical system to be synthesised with the numerical solution of a PDE intended to describe it in a coherent statistical framework, to compensate for model error.

This work presents a new theoretical analysis of the statistical finite element method demonstrating that it has similar convergence properties to the finite element method on which it is based.

Our results constitute a bound on the Wasserstein-2 distance between the ideal prior and posterior and the StatFEM approximation thereof, and show that this distance converges at the same mesh-dependent rate as finite element solutions converge to the true solution. Several numerical examples are presented to demonstrate our theory, including an example which test the robustness of StatFEM when extended to nonlinear quantities of interest.

Nonconstant periodic solutions of a nonlinear delay equation

Pham Le Bach Ngoc
University of Szeged, Szeged, Hungary

This is a joint work with Prof Tibor Krisztin (University of Szeged, Szeged, Hungary).

The delay differential equation

$$x'(t) = -ax(t) + bf\left(x(t-1)\right) \tag{E}$$

is considered where $a > 0,\ b > 0$ and a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 0,\ f'(\xi) > 0$ for $\xi \in \mathbb{R}$. It is well-know that if 0 is hyperbolic then it has a neighborhood in which there exists no nontrivial periodic orbit. By using the exponential dichotomy constants, we focus on the estimation of the optimal size of this neighborhood. The aim is to construct the neighborhood as large as possible in order to be able to carry out a verified numerical step for equation (E).

[1] O. Diekmann, et al., Delay Equations: Functional-, Complex-, and Nonlinear Analysis. Springer-Verlag, New York, 1995.

[2] A. B. Ferenc, T. Krisztin, A. Vigh, *Stable periodic orbits for the Mackey–Glass equation,* J. Differential Equations **296** (2021), 15–49.

[3] J. Hale, S. M. V. Lunel, Introduction to Functional Differential Equations. Springer-Verlag, New York, 1993.

[4] T. Krisztin, H. O. Walther, J. Wu, Shape, Smoothness and Invariant Stratification of an Attracting Set for Delayed Monotone Positive Feedback. American Mathematical Society, Rhode Island, 1999.

[5] T. Krisztin, *Periodic solutions with long period for the Mackey–Glass equation,* Electron. J. Qual. Theory Differ. Equ. **83** (2020), 1–12.

Mathematical modeling of COVID-19 transmission and intervention strategies

Eunha Shim
Soongsil University, South Korea

This is a joint work with Wongyeong Choi (Soongsil University, Seoul, South Korea).

The approved coronavirus disease (COVID-19) vaccines reduce the risk of disease by $70 - 95\%$; however, their efficacy in preventing COVID-19 is unclear. Moreover, the limited vaccine supply raises questions on how they can be used effectively. To examine the optimal allocation of COVID-19 vaccines in South Korea, we constructed an age-structured mathematical model, calibrated using country-specific demographic and epidemiological data. The optimal control problem was formulated with the aim of finding time-dependent age-specific optimal vaccination strategies to minimize costs related to COVID-19 infections and vaccination, considering a limited vaccine supply and various vaccine effects on susceptibility and symptomatology. Our results suggest that "susceptibility-reducing" vaccines should be relatively evenly distributed among all age groups, resulting in more than 40% of eligible age groups being vaccinated. In contrast, "symptom-reducing" vaccines should be administered mainly to individuals aged 20–29 and ≥ 60 years. Thus, our study suggests that the vaccine profile should determine the optimal vaccination strategy. Our findings highlight the importance of understanding vaccine's effects on susceptibility and symptomatology for effective public health interventions.

[1] W. Choi, E. Shim. *Vaccine effects on susceptibility and symptomatology can change the optimal allocation of COVID-19 vaccines: South Korea as an example.* Journal of Clinical Medicine **10** (2021): 2813.

A posteriori error estimates of the Richards equation by the spatial and temporal flux reconstructions

Hyun-Geun Shin
Charles University, Prague, Czech Republic

This is a joint work with Prof Vít Dolejší (Charles University Prague).

We deal with the numerical solution of the Richards equation [1, 2] which describes the flow through variably saturated porous media. The Richards equation is a nonlinear parabolic equation which can degenerate. We discretized the Richards equation using the space-time discontinuous Galerkin method [4]. In this contribution, we present a posteriori error estimates where the error is measured in a dual norm as in [3]. We employ the technique of the spatial and temporal flux reconstructions following the approach from [5]. The spatial reconstruction is based on the solution of local space-time Neumann problems defined on patches that consist of elements sharing a point. The spatial reconstructed fluxes belong to the Raviart-Thomas-Nedelec finite element spaces. The right Radau polynomials are used for temporal flux reconstruction. The resulting error estimates provide an upper bound of the error without any unknown constants. We present a numerical experiment showing the performance of the developed a posteriori error estimates with *hp* adaptations.

[1] L.A. Richards, Capillary conduction of liquids through porous mediums, J. Appl. Phys. 1 (5), (1931), 318–333.

[2] P.S. Huyakorn, S.D. Thomas, B.M. Thompson, Techniques for making finite elements competitive in modeling flow in variably saturated porous media, Water Resour. Res. 20 (8), (1984), 1099–1115.

[3] A. Ern, I. Smears, and M. Voralik, Guaranteed, locally space-time efficient, and polynomial-degree robust a posteriori error estimates for high-order discretizations of parabolic problems., SIAM J. Numer. Anal., 55, (2017), pp. 2811-2834.

[4] V. Dolejší , M. Kuráž, P. Solin, Adaptive higher-order space-time discontinuous Galerkin method for the computer simulation of variably-saturated porous media flows, Applied Mathematical Modelling, (2019), 72:276-305.

[5] V. Dolejší, F. Roskovec, M. Vlasák, A posteriori error estimates for higher order space-time Galerkin discretizations of nonlinear parabolic problems, SIAM J. Numer. Anal., (2021), 59: 1486-1509.

Homogenization of the evolutionary compressible Navier–Stokes–Fourier system in domains with tiny holes

Emil Skříšovský

Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

This is a joint work with doc. Mgr. Milan Pokorný, Ph.D., DSc. (Mathematical Institute of Charles University).

We study the homogenization of the evolutionary compressible Navier–Stokes–Fourier system in a bounded three-dimensional domain, which is perforated with a large number of very tiny holes. We can show that under suitable assumptions on the smallness and the distribution of the holes, the limit system in the unperforated domain remains the same.

One of the main novelties here is in the treatment of the entropy inequality, which also improves the related result for the steady case treated in [1].

[1] Y. Lu, M. Pokorný. *Homogenization of stationary Navier–Stokes–Fourier system in domain with tiny holes*, J. Differential Equations **278** (2021), 463–492.

Control of a floating body system in shallow water

Pei Su
Department of Mathematical Analysis, Charles University, Prague, Czech Republic

This is a joint work with Prof Marius Tucsnak (University of Bordeaux, Bordeaux, France).

We consider the control problem describing the interactions of water waves with a rigid body partially immersed in a bounded container. The body is allowed to move only in the vertical direction. The fluid is modeled by the shallow water equations [1]. The control signal is a vertical force acting on the floating body [2]. The system above represents a class of wave energy converters called the "point absorber device" in engineering [3].

We first derive the full governing equations of the fluid-body system in a water tank and reformulate them as an initial boundary value problem of a first-order evolution system, only defined in a part of the bottom of the domain in the horizontal direction. Then we linearize the equations around the equilibrium state and study its well-posedness. Finally, we focus on the description of the reachable space and stabilizability of the linear system. Our main result asserts that, provided that the floating body is situated in the middle of the tank, any symmetric waves with appropriate regularity can be obtained from the equilibrium state by an appropriate control force. It implies that we can project this system on the subspace of states with symmetry properties to obtain a reduced system which is approximately controllable and strongly stabilizable. We also obtain an explicit non-uniform decay rate in this case. In general, this system is not controllable (even approximately) [4]. Moreover, we discuss the special case when the object floats at one lateral boundary, which is somewhat different in the nonlinear setting [5].

[1] D. Lannes, Modeling shallow water waves, *Nonlinearity*, **33(5)** (2020), R1.

[2] D. Lannes, On the dynamics of floating structures, *Ann. PDE*, **3(1)** (2017), 11.

[3] J. A. Cretel, G. Lightbody, G. P. Thomas, and A. W. Lewis, Maximisation of energy capture by a wave-energy point absorber using model predictive control, *IFAC Proceedings Volumes*, **44(1)** (2011), 3714–3721.

[4] P. Su and M. Tucsnak, Shallow water waves generated by a floating object: a control theoretical perspective, *arXiv:2108.05088* (2021).

[5] P. Su. *Stabilization of the systems describing water waves and their interaction with a floating object*, *Université de Bordeaux* (2021).

Diamond-τ generalized Hukuhara differentiability for fuzzy-valued functions and its application to fuzzy differential equations on time scales

Tri Truong
University of Ostrava, Ostrava, Czech Republic

This is a cooperation work with Baruch Schneider (University of Ostrava, Faculty of Science) and Linh Nguyen (IRAFM, University of Ostrava).

In this talk, based on the generalized Hukuhara difference, we propose the diamond-τ differentiability for fuzzy-valued functions on time scales. We first begin with the definition and important characteristics of the diamond-τ generalized Hukuhara differentiability that are naturally investigated based on the limit of fuzzy-valued functions on time scales. Furthermore, we also study the fuzzy differential equations on time scales by using the proposed conception. Some numerical examples are provided to illustrate the necessity and efficiency of this approach in these problems.

[1] B. Bede, S. G. Gal, et al., Solutions of fuzzy differential equations based on generalized differentiability, Commun. Math. Anal. 9 (2) (2010) 22–41.

[2] J. W. Rogers Jr, Q. Sheng, Notes on the diamond-α dynamic derivative on time scales, J. Math. Anal. Appl. 326 (1) (2007) 228–241.

[3] L. Stefanini, B. Bede, Generalized hukuhara differentiability of interval-valued functions and interval differential equations, Nonlinear Anal. 71 (3-4) (2009) 1311–1328.

[4] M. Shahidi, A. Khastan, Linear fuzzy Volterra integral equations on time scales, Comp. Appl. Math. 39, 172 (2020)

[5] Q. Sheng, M. Fadag, J. Henderson, J. M. Davis, An exploration of combined dynamic derivatives on time scales and their applications, Nonlinear Anal. Real World Appl. 7 (3) (2006) 395–413.

[6] S. Hong, Differentiability of multivalued functions on time scales and applications to multivalued dynamic equations, Nonlinear Anal. 71 (9) (2009) 3622–3637.

[7] V. Lupulescu, Hukuhara differentiability of interval-valued functions and interval differential equations on time scales, Information Sciences 248 (2013) 50–67.

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