Marie-Françoise Daza, Laurent Ariza: El amor en los tiempos del coronavirus

LUCIO BOCCARDO

(Sapienza Università di Roma - Istituto Lombardo)


Singular Problems Associated to Quasilinear Equations
A WORKSHOP IN CELEBRATION OF MARIE-FRANÇOISE BIDAUT-VÉRON AND LAURENT VÉRON'S 70TH BIRTHDAY.
June 1-3, 2020
The workshop will take place over Zoom.
Speakers
Lucio Boccardo, UNIROMA1, Italy.
Huyuan Chen, JXNU, China.
Julián López Gómez, UCM, Spain.
Manuel Del Pino, Univ. of Bath, UK
Jesús Ildefonso Díaz, UCM, Spain.
Marta García-Huidobro, UC, Chile.
Moshe Marcus, Technion, Israel.
Giuseppe Mingione, UNIPR, Italy.
Vitaly Moroz, Swansea University, UK.
Nguyen Cong Phuc, LSU, USA.
Van Tien Nguyen, NYU, Abu Dhabi.
Alessio Porretta, UNIROMA2, Italy.
Patrizia Pucci, UNIPG, Italy.
Philippe Souplet, LAGA, France
Igor Verbitsky, Univ. of Missouri, USA.
Juan Luis Vázquez, UAM, Spain.
Feng Zhou, ECNU, China.
Organizers:
Quoc-Hung Nguyen, ShanghaiTech University
Phuoc-Tai Nguyen, Masaryk University
The workshop is co-hosted by Institute of Mathematical Sciences, ShanghaiTech University and Department of Mathematics and Statistics, Masaryk University.

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Bonjour

Buon giorno

Good morning

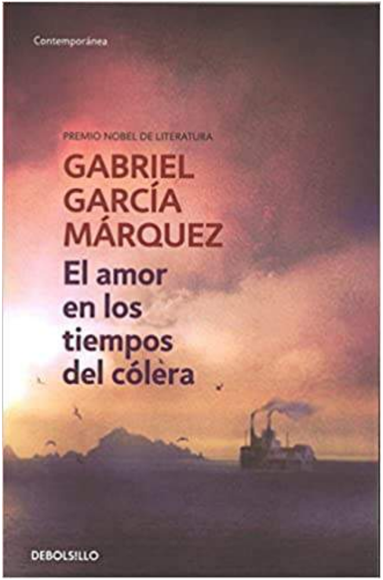
A WORKSHOP IN CELEBRATION OF MARIE-FRANÇOISE BIDAUT-VERON AND LAURENT VERON'S 70TH BIRTHDAY.

Bonjour

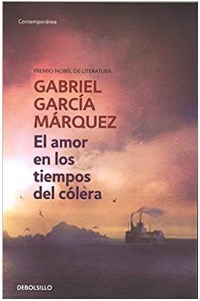
Buon giorno

Good morning

+
thanks to the organizers


Contemporánea
PREMIO NOBEL DE LITERATURA
GABRIEL
GARCÍA
MÁRQUEZ
El amor
en los
tiempos
del cólera
DEBOLSILLO

G.G.M.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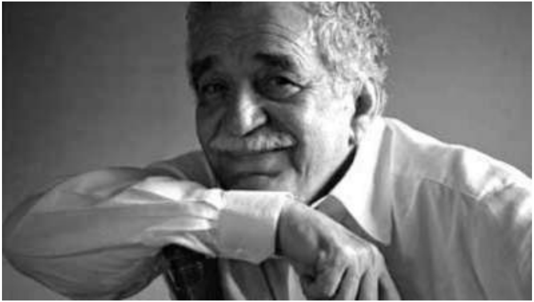

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1 . 6 . 2020

A WORKSHOP IN CELEBRATION OF MARIE-FRANÇOISE BIDAUT-VERON AND LAURENT VERON'S 70TH BIRTHDAY.

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Bon anniversaire

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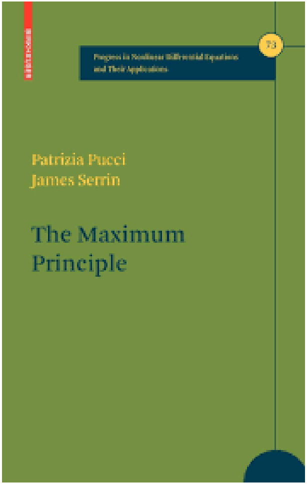
Two maximum principles for two friends



Two maximum principles for two friends

Appl Math Optim 12:191–202 (1984)

Applied Mathematics and Optimization

A Strong Maximum Principle for Some Quasilinear Elliptic Equations

J. L. Vàzquez*

Abstract. In its simplest form the Strong Maximum Principle says that a nonnegative superharmonic continuous function in a domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, is in fact positive everywhere. Here we prove that the same conclusion is true for the weak solutions of $-\Delta u + \beta(u) = f$ with β a nondecreasing function $\mathbb{R} \to \mathbb{R}$, $\beta(0) = 0$, and $f \geqslant 0$ a.e. in Ω if and only if the integral $\int (\beta(s)s)^{-1/2}\,ds$ diverges at $s = 0+$. We extend the result to more general equations, in particular to $-\Delta_p u + \beta(u) = f$ where $\Delta_p(u) = \operatorname{div}(|Du|^{p-2}Du)$, $1 < p < \infty$. Our main result characterizes the nonexistence of a dead core in some reaction-diffusion systems.

1. Introduction

According to the well-known Strong Maximum Principle (SMP) [12] a nonnegative superharmonic continuous function defined in a domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, is in fact positive everywhere in Ω and this result extends to nonnegative solutions of a large class of linear elliptic equations, cf. also [11].

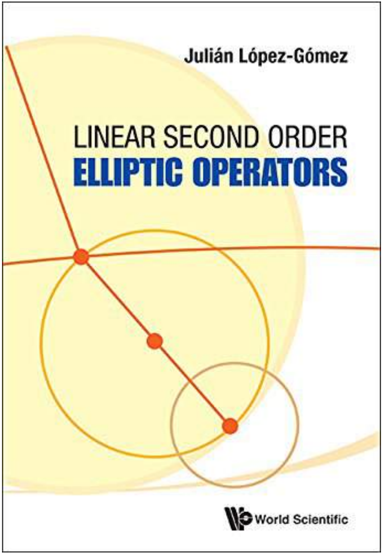
In this paper we investigate the validity of the SMP for certain classes of nonlinear elliptic equations. We begin our investigation with the semilinear equation

$$-\Delta u(x) + \beta(u(x)) = f(x), \qquad x \in \Omega, \tag{1}$$

where Ω is as above a domain in $\mathbb{R}^n$, β is a nondecreasing real function with $\beta(0) = 0$ and $f \geqslant 0$ a.e. in Ω. We show that for a suitable class of solutions of (1) the SMP is true if and only if either $\beta(S) = 0$ for some $S > 0$ or $\beta(S) > 0$ for

*División de Matemáticas, Universidad Autónoma, Madrid-34, Spain. This work was partly done while the author was visiting the University of Minnesota as a Fulbright Scholar.

Two maximum principles for two friends



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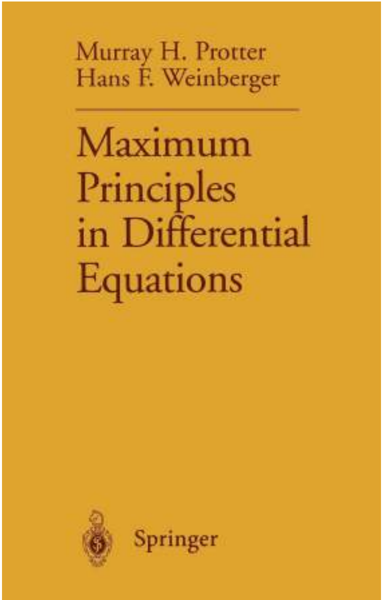

Two maximum principles for two friends

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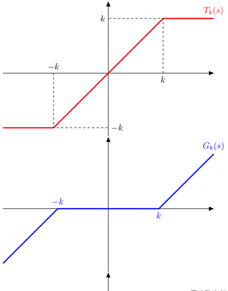

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$$\textit{sketch}: \ \alpha \int_\Omega |\nabla G_k(u_n)|^2 + \int_\Omega a_n(x)|u_n|^\gamma |G_k(u_n)|$$

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$$\Rightarrow posit. + \int_\Omega a_n(x)[|u_n|^\gamma - Q]|G_k(u_n)| \leq 0$$

$$\Rightarrow |u_n| \leq Q^{\frac{1}{\gamma}} \ldots \Rightarrow \ldots \exists\, |u| \leq Q^{\frac{1}{\gamma}}$$

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We discuss the existence properties and of distributional solutions for the boundary value problems (the first with a convection term, the second with a drift term)

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We note that

- **at least formally, if $M(x)$ is symmetric, the two above linear problems are in duality.**

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We note that

- **the differential operators may be not coercive, unless one assumes that either the norm of $|E|$ in $L^N(\Omega)$ is small, or that $\mathrm{div}(\|E\|_N) = 0$: ...**

Papers concerned with this part of the talk

L. Boccardo: Some developments on Dirichlet problems with discontinuous coefficients; *Boll. Unione Mat. Ital*, **2 (2009) 285–297.**
(invited paper in memory of 30-death Stampacchia)

L. Boccardo: Dirichlet problems with singular convection terms and applications; *J. Differential Equations*, **258 (2015) 2290–2314.**

L. Boccardo: Stampacchia-Calderon-Zygmund theory for linear elliptic equations with discontinuous coefficients and singular drift; *ESAIM, Control, Optimization and Calculus of Variations*, **25 (2019), Art. 47, 13 pp.**

Assumptions

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- $f \in L^m(\Omega)$**,** $1 \le m \le \infty$**,**
- $\boxed{E \in (L^N(\Omega))^N}$

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Boundary value problem and Lax-Milgram

u weak

Boundary value problem and Lax-Milgram

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Boundary value problem and Lax-Milgram

u weak /distributional solution of the boundary value problem

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means

$$u \in W_0^{1,2}(\Omega) : \int_\Omega M(x)\nabla u \nabla v = \int_\Omega u\,E(x)\nabla v + \int_\Omega f(x)v,$$

$\forall\ v \in W_0^{1,2}(\Omega)$

Boundary value problem and Lax-Milgram

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Coercivity of $-\mathrm{div}(M(x)\nabla u) + \mathrm{div}(u\,E(x))$

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$$\geq \alpha \int_\Omega |\nabla v|^2 - \left[\int_\Omega |v|^{2^*}\right]^{\frac{1}{2^*}} \left[\int_\Omega |E(x)|^N\right]^{\frac{1}{N}} \left[\int_\Omega |\nabla v|^2\right]^{\frac{1}{2}}$$

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- $E \in L^N$
- $\|E\|_{L^N}$ **not too large**

Our approach hinges on test function method

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The proofs of all the results are very easy

if we assume $\operatorname{div}(E) = 0$

if we assume $\|E\|_N$ small

Stampacchia-Calderon-Zygmund for the two problems

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[2] **paper invitation U.M.I. in memory of 30-Stampacchia**
[3] **ESAIM-COCV 2019**

Stampacchia-Calderon-Zygmund for the two problems

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[2] **paper invitation U.M.I. in memory of 30-Stampacchia**

[3] **ESAIM-COCV 2019**

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- Ω **bounded subset of** $\mathbb{R}^N$,
- **ellipticity:** $0 < \alpha,\ \ \alpha|\xi|^2 \le M(x)\xi\xi,\ \ |M(x)| \le \beta$,
- $f \in L^m(\Omega)$, $1 \le m \le \infty$,
- $\boxed{E \in (L^N(\Omega))^N}$

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and we prove for both the b.v.p. the same Stampacchia-Calderon-Zygmund results of the case

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and we prove for both the b.v.p. the same Stampacchia-Calderon-Zygmund results of the case $E = 0$.

[2] paper invitation U.M.I. in memory of 30-Stampacchia

[3] ESAIM-COCV 2019

$|E| \in L^N(\Omega)$

$\boxed{\textbf{as } E = 0}$

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$$\begin{cases} -\mathrm{div}(M(x)\nabla u)) = -\mathrm{div}(u\,E(x)) + f(x) & : \Omega, \\ u = 0 & : \partial\Omega \end{cases}$$

1. $\frac{2N}{N+2} \leq m < \frac{N}{2} \Rightarrow \; u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$;

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Theorem (70-Brezis)

$E = 0$, $m > \frac{N}{2}$, it is false that $u \in W_0^{1,m^*}(\Omega)$

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Remark

$E = 0$, $\frac{2N}{N+2} + \delta_{Meyers} < m < \frac{N}{2}$, $u \in ?$

Same results for the drift problem

[4] **ESAIM-COCV 2019**

Same results for the drift problem

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[4] **ESAIM-COCV 2019**

"Nonlinear"

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"Nonlinear" approach to a linear problem

$$-\mathrm{div}(M(x)\nabla u_n) = -\mathrm{div}\left(\frac{u_n}{1+\frac{1}{n}|u_n|} E(x)\right) + f(x)$$

Other recent papers

L. Boccardo, S. Buccheri, G.R. Cirmi: Two linear noncoercive Dirichlet problems in duality; Milan J. Math. 86 (2018), 97–104.

L. Boccardo, S. Buccheri, R.G. Cirmi: Calderon-Zygmund theory for infinite energy solutions of nonlinear elliptic equations with singular drift; NODEA, to appear.

L. Boccardo, S. Buccheri: A nonlinear homotopy between two linear Dirichlet problems; Rev. Mat. Complutense, to appear.

L. Boccardo: Two semilinear Dirichlet problems "almost" in duality; Boll. Unione Mat. Ital. 12 (2019), 349–356.

L. Boccardo, L. Orsina, A. Porretta: Some noncoercive parabolic equations with lower order terms in divergence form. Dedicated to Philippe Bénilan. J. Evol. Equ. 3 (2003), 407–418.

$\cancel{E \in (L^N(\Omega))^N}$

If $E \notin (L^N(\Omega))^N$, even for nothing, as in

$\boxed{|E| \leq \frac{|A|}{|x|}, \quad A \in \mathbb{R}}$, $0 \in \Omega$,

[5] **JDE 2015; +Orsina, Nonlin.Anal. 2019**

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the framework changes completely:

$u \in W_0^{1,2}(\Omega)$ or $u \in W_0^{1,q}(\Omega)$ depends on the size of A. [5]

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1) **if** $|A| < \frac{\alpha(N-2m)}{m}$, **and** $\frac{2N}{N+2} \le m < \frac{N}{2}$, **then** $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$;
2) **if** $|A| < \frac{\alpha(N-2m)}{m}$, **and** $1 < m < \frac{2N}{N+2}$, **then** $u \in W_0^{1,m^*}(\Omega)$;
3) **if** $|A| < \alpha(N-2)$, **and** $m = 1$, **then** $\nabla u \in (M^{\frac{N}{N-1}}(\Omega))^N$ **and** $u \in W_0^{1,q}(\Omega)$, **for every** $q < \frac{N}{N-1}$;
4) **if** $\alpha(N-2) \le |A| < \alpha(N-1)$, **then** $u \in W_0^{1,q}(\Omega)$, **for every** $q < \frac{N\alpha}{|A|+\alpha}$

[5] JDE 2015; +Orsina, Nonlin.Anal. 2019

Radial ex.

$\cancel{E \in (L^N(\Omega))^N}$

$$E \in (L^2(\Omega))^N$$

[6] **JDE 2015**

$E \in (L^2(\Omega))^N$

$$\begin{cases} \textit{definition of solution;} \\ \textit{existence of solution.} \end{cases}$$ [6]

[6] **JDE 2015**

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If we add the zero order term "$+u$", the framework changes completely.

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Simpler proofs

PhD

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Simpler proofs

PhD course, UCM, November 2019

By duality: problems with very singular drifts

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7 **DIE 2019**

8 **only** L^2

By duality: problems with very singular drifts

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[7] **DIE 2019**

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- **application to the existence in some Hamilton-J. eq. with lower order term having q-dependence w.r.t. gradient,** $q < 2$.

[7] DIE 2019

[8] only L^2

An elliptic system connected with the mathematical study of PDE models for chemotaxis

[9] **JDE 2015**
[10] **Comm.PDE + L. Orsina**

An elliptic system connected with the mathematical study of PDE models for chemotaxis

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + u = -\mathrm{div}(u\, M(x)\nabla \psi) + f(x)\,, \\ -\mathrm{div}(M(x)\nabla \psi) = u^{\theta}\,. \end{cases}$$ [9][10]

[9] JDE 2015

[10] Comm.PDE + L. Orsina

$$\begin{cases} -\mathrm{div}(M(x)\nabla u)) = -\mathrm{div}(u\,E(x)) + f(x) & : \Omega, \\ u = 0 & : \partial\Omega \end{cases}$$

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Theorem

$E \in (L^N(\Omega))^N$, $f \in L^m(\Omega)$ with $m \geq \frac{2N}{N+2}$ and $f(x) \geq 0$ (of course not zero a.e.). Then the solution $u \in W_0^{1,2}(\Omega)$ is positive and it is zero at most on a set of zero Lebesgue measure [a].

[a] weak max. pr.

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Remark

In the proof we only need $E \in (L^2(\Omega))^N$**: blue and red assumptions.**

$$\begin{cases} -\mathrm{div}(M(x)\nabla\psi)) = -\mathrm{div}(\psi\, E(x)) + g(x) & : \ \Omega, \\ \psi = 0 & : \ \partial\Omega \end{cases}$$

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with Alberto Farina

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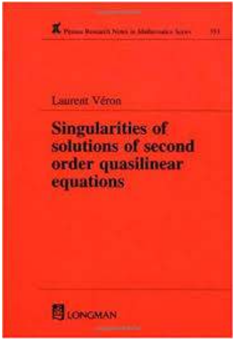
books

books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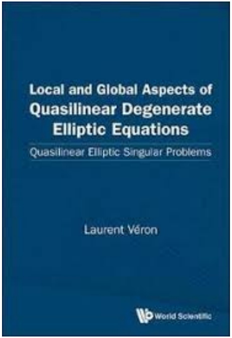
Local and Global Aspects of Quasilinear Degenerate Elliptic Equations
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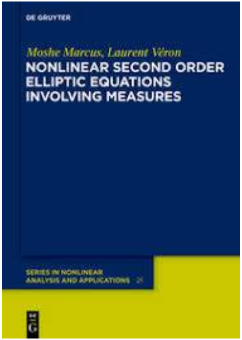
DE GRUYTER
Moshe Marcus, Laurent Véron
NONLINEAR SECOND ORDER ELLIPTIC EQUATIONS INVOLVING MEASURES
SERIES IN NONLINEAR ANALYSIS AND APPLICATIONS 21

in a conference in Cortona (organizers Juan Luis and Lucio)

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Calculus of Variations (in the study of integral functionals)

Recall this large and important class of integral functionals

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The Euler-Lagrange equation for J is (at least formally) the quasilinear elliptic problem

$$\begin{cases} -\mathrm{div}(A(x,u)\nabla u) + \frac{1}{2}A'(x,u)|\nabla u|^2 + \lambda\, u = f & \textbf{in } \Omega, \\ u = 0 & \textbf{on } \partial\Omega. \end{cases}$$

Consider the Dirichlet problem

$$u \in W_0^{1,2}(\Omega) : -\mathrm{div}([a(x)+|u|^q]\nabla u)+\lambda\, u+b(x)\, u|u|^{p-1}|\nabla u|^2 = f(x)$$

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Background

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Theorem (Weak Maximum Principle / easy)

If $f \geq 0$, then the weak solution u is such that $u \geq 0$ almost everywhere in Ω.

Consider the Dirichlet problem

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$$u \in W_0^{1,2}(\Omega) : -\mathrm{div}([a(x)+|u|^q]\nabla u)+\lambda\, u+b(x)\, u|u|^{p-1}|\nabla u|^2 = f(x)$$

even with f(x) very singular

Theorem (Strong Maximum Principle)

If $f \geq 0$ (and not almost everywhere equal to zero), then for every set $\omega \subset\subset \Omega$ there exists $m_\omega > 0$ such that $u(x) \geq m_\omega$ almost everywhere in ω.

to MARIE-F. and LAURENT

je vous souhaite tous le bien

je vous souhaite tous le bien

without

without

Ciao

Ciao