Blow-up by aggregation in chemotaxis

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Singular problems associated to quasilinear equations,
In honor of Marie Françoise Bidaut-Véron and Laurent Véron
June 2nd, 2020

The Keller-Segel system in $\mathbb{R}^2$.

$$(KS) \quad \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \mathbb{R}^2 \times (0,\infty), \\ v = (-\Delta)^{-1} u := \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \log \frac{1}{|x-z|} u(z,t)\, dz \\ u(\cdot, 0) = u_0 \geq 0 \quad \text{in } \mathbb{R}^2. \end{cases}$$

is the classical diffusion model for **chemotaxis**, the motion of a population of bacteria driven by standard diffusion and a nonlocal drift given by the gradient of a chemoatractant, a chemical the bacteria produce.

$u(x,t)=$ population density.

$v(x,t)=$ the chemoatractant

Basic properties.

For a regular solution $u(x,t)$ defined up to a time $T > 0$,

$$\begin{cases} u_t = \nabla \cdot (u \nabla (\log u - v)) & \text{in } \mathbb{R}^2 \times (0, T) \\ -\Delta v = u \end{cases}$$

- **Conservation of mass**

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} u(x,t)\, dx &= \lim_{R \to \infty} \int_{\partial B_R} u \nabla (\log u - v) \cdot \nu \, d\sigma \\ &= \lim_{R \to \infty} \int_{\partial B_R} (\nabla u \cdot \nu) - u(\nabla v \cdot \nu)\, d\sigma = 0 \end{aligned}$$

- **The second moment identity**. Let $M = \int_{\mathbb{R}^2} u(x,t)\, dx$, then

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 u(x,t)\, dx = 4M \left(1 - \frac{M}{8\pi}\right)$$

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 u(x,t)\,dx &= \int_{\mathbb{R}^2} |x|^2(\Delta u - \nabla\cdot(u\nabla v))dx \\
&= \int_{\mathbb{R}^2} \Delta(|x|^2) u dx + \int_{\mathbb{R}^2} (2x\cdot\nabla v) u dx \\
&= 4M + 2\int_{\mathbb{R}^2} u(x\cdot\nabla v) dx.
\end{aligned}
$$

From $v(\cdot,t) = \frac{1}{2\pi}\log\frac{1}{|\cdot|} * u(\cdot,t)$ we get

$$
\begin{aligned}
-2\int_{\mathbb{R}^2} u(x\cdot\nabla v)\,dx &= \frac{1}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(x,t)u(y,t)\frac{x\cdot(x-y)}{|x-y|^2}dx\,dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(x,t)u(y,t)\frac{(x-y)\cdot(x-y)}{|x-y|^2}dx\,dy \\
&= \frac{M^2}{2\pi}.
\end{aligned}
$$

Hence

$$
\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 u(x,t)\,dx = 4M - \frac{M^2}{2\pi}
$$

Then, if the initial second moment is finite we have

$$\int_{\mathbb{R}^2} u(x,t)|x|^2 dx = \int_{\mathbb{R}^2} u(x,0)|x|^2 dx + 4M(1 - \frac{M}{8\pi})\, t.$$

As a consequence,

• If $M > 8\pi$ the solution cannot remain smooth beyond some time. $u(x,t)$ **blows-up** in finite time.

• If $M = 8\pi$ The second moment of the solution is preserved in time.

• If $M < 8\pi$ second moment grows linearly in time while mass is preserved (as in heat equation): the solution "diffuses"

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) \\ v = (-\Delta)^{-1} u := \frac{1}{2\pi} \log \frac{1}{|\cdot|} * u \\ u(\cdot, 0) = u_0 \geq 0. \end{cases}$$

• If $M \leq 8\pi$ the solution exists classically at all times $t \in (0, \infty)$.

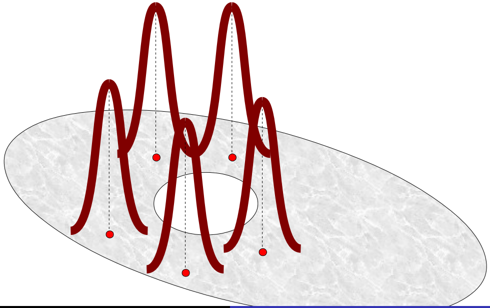
• If $M < 8\pi$ then $u(x, t)$ goes to zero and spreads in self-similar way. (Blanchet-Dolbeault-Perthame (2006); Jäger-Luckhaus (1992).)

• If $M > 8\pi$ blow-up is expected to take place by **aggregation** which means that at a finite time $u(x,t)$ concentrates and forms a set of Dirac masses with mass at least 8π at a blow-up point.

• Examples of blow-up with precise asymptotics, and mass slightly above 8π were found by

• Herrero-Velázquez (1996), Velázquez (2002, 2006) Raphael and Schweyer (2014).

• Collot-Ghoul-Masmoudi-Nguyen (2019): New method, precise asymptotics and nonradial stability of the blow-up phenomenon.



$$\boxed{u_t = \nabla \cdot (u \nabla(\log u - (-\Delta)^{-1} u))}$$

What special thing happens exactly at the **critical mass** 8π?

$$\boxed{E(u) := \int_{\mathbb{R}^2} u(\log u - (-\Delta)^{-1} u)\, dx}$$

is a Lyapunov functional for (KS). Along a solution $u(x,t)$,

$$\partial_t E(u(\cdot,t)) = -\int_{\mathbb{R}^2} u|\nabla(\log u - (-\Delta)^{-1} u)|^2 dx \leq 0.$$

and this vanishes only at the **steady states** $v = \log u$ or

$$-\Delta v = e^v = u \quad \text{in } \mathbb{R}^2$$

the **Liouville equation**.

$$-\Delta v = e^{v} = u \quad \text{in } \mathbb{R}^2$$

All solutions with finite mass $\int_{\mathbb{R}^2} u < +\infty$ are known:

$$U_{\lambda,\xi}(x) = \lambda^{-2} U_0\left(\frac{x-\xi}{\lambda}\right), \quad U_0(x) = \frac{8}{(1+|x|^2)^2}.$$

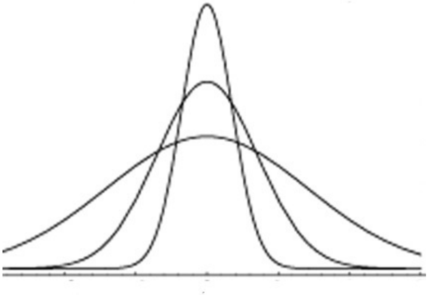
$$\boxed{\int_{\mathbb{R}^2} U_{\lambda,\xi}(x)dx \,=\, 8\pi, \quad E(U_{\lambda,\xi}) = E(U_0) \quad \text{for all} \quad \lambda, \xi},$$

and $U_{\lambda,\xi} \rightharpoonup 8\pi\delta_\xi$ as $\lambda \to 0^+$.

The functions $U_{\lambda,\xi}$ are the extremals for the **log-HLS inequality**

$$\min_{\int_{\mathbb{R}^2} u = 8\pi} E(u) = E(U_0)$$

The functional $E(u)$ loses the P.S. condition along this family, which makes possible the presence of **bubbling phenomena** along the flow. The problem is *critical*.



The critical mass case $\int_{\mathbb{R}^2} u_0 = 8\pi$

• Blanchet-Carlen-Carrillo (2012), Carlen-Figalli (2013): Asymptotic stability of the family of steady states under finite second moment perturbations.

• Lopez Gomez-Nagai-Yamada oscillatory (2014) instabilities.

• Blanchet-Carrillo-Masmoudi (2008) If in addition to critical mass we assume finite second moment

$$\int_{\mathbb{R}^2} |x|^2 u_0(x)\, dx < +\infty$$

then the solution $u(x,t)$ *aggregates in infinite time*: for some $\lambda(t) \to 0$ and some point q we have that (near q)

$$u(x,t) \approx \frac{1}{\lambda(t)^2} U_0\left(\frac{x-q}{\lambda(t)}\right) \quad \text{as } t \to +\infty$$

no information about the rate.

• Chavanis-Sire (2006), Campos (2012) formal analysis to derive the rate $\lambda(t)$.

• Ghoul-Masmoudi (2019) Construction of a radial solution with this profile that confirms formal rate

$$\lambda(t) \sim \frac{1}{\sqrt{\log t}} \quad \text{as } t \to +\infty.$$

Stability of the phenomenon inside the radial class is found. Full stability left as an open problem.

Theorem (Dávila, del Pino, Dolbeault, Musso, Wei, Arxiv 2019)

There exists a function $u_0^*(x)$ with

$$\int_{\mathbb{R}^2} u_0^*(x)dx = 8\pi, \quad \int_{\mathbb{R}^2} |x|^2 u_0^*(x)dx < +\infty$$

such that for any initial condition in (KS) that is a small perturbation of u_0^* and has mass 8π, the solution has the form

$$\begin{aligned} u(x,t) &= \frac{1}{\lambda(t)^2} U_0\left(\frac{x-q}{\lambda(t)}\right) + o(1) \\ \lambda(t) &= \frac{1}{\sqrt{\log t}}(1+o(1)), \end{aligned}$$

as $t \to +\infty$.

Let us explain the mechanism in Theorem 1. We look for a solution of

$$S(u) := -\,u_t + \nabla \cdot (u \nabla(\log u - (-\Delta)^{-1} u)) = 0$$

which is close to $\frac{1}{\lambda^2} U_0(y)$, $y = \frac{x}{\lambda}$ where $0 < \lambda(t) \to 0$ is a parameter function to be determined. Let

$$U(x,t) = \frac{\alpha}{\lambda^2} U_0(y) \chi, \quad y = \frac{x}{\lambda}.$$

Here $\chi(x,t) = \chi_0(|x|/\sqrt{t})$, where χ_0 is smooth with $\chi_0(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$ and $\alpha(t) = 1 + O\big(\frac{\lambda^2}{t}\big)$ is such that $\int_{\mathbb{R}^2} U\, dx = 8\pi$.
We look for a local correction of the form $u = U + \varphi$ where

$$\varphi(x,t) = \frac{1}{\lambda^2} \phi\,(y,t)\,, \quad y = \frac{x}{\lambda}.$$

We compute for $\varphi(x,t) = \frac{1}{\lambda^2}\phi(y,t)$,

$$S(U+\varphi) = S(U) + \mathcal{L}_U[\varphi] - \varphi_t + O(\|\varphi\|^2)$$

$$\mathcal{L}_U[\varphi] = \Delta\varphi - \nabla V \cdot \nabla\varphi - \nabla U \cdot \nabla(-\Delta)^{-1}\varphi \approx \lambda^{-4} L_0[\phi]$$

and for $|x| \ll \sqrt{t}$ we have ($V_0 = \log U_0$)

$$L_0[\phi] = \Delta_y\phi - \nabla V_0 \cdot \nabla\phi - \nabla U_0 \cdot \nabla(-\Delta)^{-1}\phi$$

We will have obtained an improvement of the approximation if we solve

$$L_0[\phi] + \lambda^4 S(U) = 0, \quad \phi(y,t) = O(|y|^{-4-\sigma})$$

Let us consider the elliptic problem

$$L_0[\phi] = E(y) = O(|y|^{-6-\sigma}) \quad \text{in } \mathbb{R}^2$$

Which can be written as

$$\nabla \cdot (U_0 \nabla g) = E(y), \quad g = \frac{\phi}{U_0} - (-\Delta)^{-1}\phi$$

Assume E radial $E = E(|y|)$ and $\int_{\mathbb{R}^2} E = 0$. We solve as

$$g(y) = \int_{|y|}^{\infty} \frac{d\rho}{\rho U_0(\rho)} \int_{\rho}^{\infty} E(r) r dr \;=\; O(|y|^{-\sigma}).$$

Now we solve, setting $\psi = (-\Delta)^{-1}\phi$,

$$\Delta\psi + U_0\psi = -U_0 g \quad \text{in } \mathbb{R}^2.$$

It can be solved for $\psi = O(|y|^{-2-\sigma})$ (Fredholm alternative) iff

$$\int_{\mathbb{R}^2} g Z_0 = 0,$$

where $Z_0 = (y \cdot \nabla U_0 + 2U_0)$. Now,

$$\int_{\mathbb{R}^2} |y|^2 E(y) dy = \int_{\mathbb{R}^2} \nabla \cdot (U_0 \nabla |y|^2) g dy = 2 \int_{\mathbb{R}^2} Z_0 g dy$$

Hence we can solve as desired ($\phi = O(|y|^{-4-\sigma})$ if

$$\int_{\mathbb{R}^2} |y|^2 E(y)dy = 0 = \int_{\mathbb{R}^2} E(y)dy$$

Now, the equation we need to solve is

$$L_0[\phi] + \lambda^4 S(U) = 0, \quad \phi(y,t) = O(|y|^{-4-\sigma})$$

So we need

$$\int_{\mathbb{R}^2} S(U)|x|^2 dx = \int_{\mathbb{R}^2} S(U)dx = 0.$$

$$S(U) = -U_t + \nabla \cdot (U\nabla(\log U - (-\Delta)^{-1}U) = S_1 + S_2$$

We clearly have $\int_{\mathbb{R}^2} S(U) = 0$.
A direct computation (that uses $\int_{\mathbb{R}^2} U = 8\pi$) gives

$$\int_{\mathbb{R}^2} |x|^2 S_2 = 0.$$

Finally

$$\int_{\mathbb{R}^2} S_1|x|^2 = -\int_{\mathbb{R}^2} U_t|x|^2 = \partial_t(\int_{\mathbb{R}^2} U|x|^2) = 0$$

if and only if $\int_{\mathbb{R}^2} U|x|^2 = \mathit{constant}$.

We have

$$\int_{\mathbb{R}^2} U(x,t)|x|^2 dx = \int_0^\infty \chi_0(s/\sqrt{t})U(\rho/\lambda)\lambda^{-2}\rho^3 d\rho$$
$$\sim \int_0^{\frac{\sqrt{t}}{\lambda}} U(r)r^3 dr \sim \lambda^2 \log(\sqrt{t}/\lambda)$$

Thus the requirement is $\lambda^2 \log(\sqrt{t}/\lambda) = c^2$, and we get

$$\boxed{\lambda(t) = \frac{c}{\sqrt{\log t}} + O\left(\frac{\log(\log t)}{\log t}\right)}$$

For the actual proof:

We let $\lambda(t),\ \alpha(t)$ be parameter functions with

$$\lambda(t) = \frac{1}{\sqrt{\log t}}(1+o(1)), \quad \alpha(t) = 1 + o(t^{-1}).$$

The function $U = \alpha\lambda^{-2}U_0\left(\frac{x}{\lambda}\right)\chi$ is defined as before. We look for a solution of the form

$$u(x,t) \ = \ U[\lambda,\alpha] + \varphi$$

$$\varphi(x,t) \ = \ \eta\frac{1}{\lambda^2}\phi^{in}\left(\frac{x}{\lambda},t\right) + \phi^{out}(x,t)$$

where $\eta(x,t) = \chi_0\left(\frac{|x|}{\sqrt{t}}\right)$.

The pair (ϕ^{in},ϕ^{out}) is imposed to solve a coupled system, *the inner-outer gluing system* that leads to $u(x,t)$ be a solution

The system involves the main part of the linear operator near the core and far away from it.

$$\mathcal{L}_U[\varphi] = \Delta_x \varphi - \nabla_x V \cdot \nabla \varphi - \nabla_x U \cdot \nabla (-\Delta)^{-1} \varphi$$

Near 0, $\mathcal{L}_U[\varphi] \approx \lambda^{-4} L_0[\phi]$ for $\varphi = \lambda^{-2}\phi(y,t), \ y = \frac{x}{\lambda}$. Away:

$$-\nabla_x V \sim \frac{4x}{|x|^2}, \quad \nabla_x U \sim \frac{4\lambda^2 x}{|x|^5}$$

So, setting $r = |x|$, far away from the core the operator looks like

$$\mathcal{L}_U[\varphi] \approx \Delta_x \varphi + \frac{4}{r}\partial_r \varphi$$

(for radial functions $\varphi(r)$, $\mathcal{L}_U[\varphi] \approx \varphi'' + \frac{5}{r}\varphi'$, a $6d$-Laplacian).

The inner-outer gluing system is, up to lower order terms,

$$\boxed{\phi_t^{out} = \Delta_x \phi^{out} \; + \; \frac{4}{r}\partial_r \phi^{out} + G(\lambda, \alpha, \phi^{in})}$$

$$G(\lambda, \phi^{in}) = (1-\eta)\frac{\lambda\dot{\lambda}}{r^4} + 2\nabla\eta\nabla_x \frac{\phi^{in}}{\lambda^2} + (\Delta_x \eta + \frac{4}{r}\partial_r \eta)\frac{\phi^{in}}{\lambda^2} + \cdots$$

$$\boxed{\lambda^2 \partial_t \phi^{in} = L_0[\phi^{in}] + H(\lambda, \alpha, \phi^{out}) \quad \text{in } \mathbb{R}^2}$$

$$H(\lambda, \alpha, \phi^{out}) = \lambda\dot{\lambda}(2U + y \cdot \nabla_y U(y)) - \lambda^2 \nabla_y U_0 \cdot \nabla_y (-\Delta_x)^{-1} \phi^{out}.$$

where $L_0[\phi] = \Delta_y \phi - \nabla_y V_0 \cdot \nabla_y \phi - \nabla_y U_0 \cdot \nabla_y (-\Delta_y)^{-1} \phi$. We couple this system with the "solvability" conditions

$$\boxed{\int_{\mathbb{R}^2} |y|^2 H(\lambda, \alpha, \phi^{out})(y,t)\, dy = \int_{\mathbb{R}^2} H(\lambda, \alpha, \phi^{out})(y,t)\, dy 0}$$

for all $t > 0$.

Key step: Building up a linear operator that nicely inverts

$$\lambda^2\phi_t = L_0[\phi] + h(y,t) \quad \text{in } \mathbb{R}^2 \times (0,\infty),$$

$|h(y,t)| \lesssim \gamma(t)|y|^{-5-\sigma}$, $\gamma = |\lambda_0\dot{\lambda}_0|$ under the conditions

$$\int_{\mathbb{R}^2} |y|^2 h(y,t)\,dy = 0 = \int_{\mathbb{R}^2} h(y,t)\,dy$$

producing a "rapidly decaying solution"

$$|\phi(y,t)| \lesssim \gamma(t)|y|^{-3-\sigma}.$$

The decay makes the system essentially decoupled.

Blow-up in a finite time $T > 0$

Theorem 2 Given points $q_1, \dots q_k \in \mathbb{R}^2$, there exists an initial condition $u_0(x)$ with

$$\int_{\mathbb{R}^2} u_0(x) dx > 8k\pi,$$

such that the solution of (KS) satisfies for some $T > 0$

$$\begin{aligned} u(x,t) &= \sum_{j=1}^{k} \frac{1}{\lambda_j(t)^2} U_0\left(\frac{x-q_j}{\lambda_j(t)}\right) + O(1) \\ \lambda_j(t) &= \beta_j (T-t)^{\frac{1}{2}} e^{-\frac{1}{2}\sqrt{|\log(T-t)|}} (1+o(1)). \end{aligned}$$

as $t \to T$.

Previous results: Velazquez 2004, Raphael-Schweyer, 2014, Collot-Ghoul-Nguyen-Masmoudi Arxiv 2019.

In the previous result, as $t \to T$,

$$u(x,t) \rightharpoonup \sum_{j=1}^{k} 8\pi\delta_{q_j} + \text{ a small function}$$

Multiple Blow-up at a single point:

Theorem

There exists a solution to (KS) such that as $t \to T$,

$$u(x,t) \rightharpoonup 8k\pi\delta_0(x) + \textit{ a small function}$$

The profile looks at main order for some $\alpha, \beta > 0$

$$u(x,t) = \sum_{j=1}^{k} \frac{1}{\lambda_j(t)^2} U_0\left(\frac{x - a_j\sqrt{T-t}}{\lambda(t)}\right) + O(1)$$

$$\lambda(t) = \beta(T-t)^{\frac{1}{2}} e^{-\alpha\sqrt{|\log(T-t)|}}(1+o(1)),$$

a_j's are vertices of a k-regular polygon, such that

$$\frac{1}{2}a_j = 4\sum_{i\neq j} \frac{a_i - a_j}{|a_i - a_j|^2}$$

$$a_j = 2\sqrt{k-1}e^{2\pi i \frac{j}{k}}$$

Formal-numerical asymptotics for this solutions were previously found by Seki-Sujiyama-Velázquez (2013)

A related problem: The harmonic map flow $\mathbb{R}^2 \mapsto S^2$

The harmonic map flow from $\mathbb{R}^2$ into S^2.
$u : \mathbb{R}^2 \times [0, T) \to S^2$:

$$(HMF) \qquad \begin{cases} u_t = \Delta u + |\nabla u|^2 u & \text{in } \mathbb{R}^2 \times (0, T) \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^2 \end{cases}$$

- We have that $|u_0| \equiv 1 \implies |u| \equiv 1$.
- (HMF) is the L^2-gradient flow of the Dirichlet energy:

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^2} |\nabla u(\cdot, t)|^2 = -2 \int_{\mathbb{R}^2} u_t^2$$

Finite energy harmonic maps $\mathbb{R}^2 \to S^2$: critical points of Dirichlet energy. Solutions of

$$\Delta u + |\nabla u|^2 u = 0 \quad \text{in } \mathbb{R}^2, \quad |u| \equiv 1, \quad \int_{\mathbb{R}^2} |\nabla u|^2 < +\infty$$

Example:

$$U(y) = \begin{pmatrix} \frac{2y}{1+|y|^2} \\ \frac{|y|^2-1}{1+|y|^2} \end{pmatrix}, \quad y \in \mathbb{R}^2.$$

the canonical **1-corrotational harmonic map**.

Known :

• Blow-up must be type II, it can only take place at isolated points, by bubbling of finite-energy harmonic maps (Struwe, Tian, F.H. Lin, Topping 1985-2008).

•Continuation after blow-up, uniqueness: Struwe, Topping, Freire, Rupflin.

• Examples known: in the radial 1-corrotational class only. Chang-Ding-Ye 1991, Raphael-Schweyer 2013.

Our main result: For any given finite set of points of Ω and suitable initial and boundary values, then a solution with a simultaneous blow-up at those points exists, with a profile resembling a translation, scaling and rotation of U around each bubbling point. Single point blow-up is **codimension-1 stable**.

The functions

$$U_{\lambda,q,\alpha}(x) \ := \ Q_\alpha U\left(\frac{x-q}{\lambda}\right).$$

with $\lambda > 0$, $q \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ and

$$Q_\alpha \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \ = \ \begin{bmatrix} e^{i\alpha}(y_1 + iy_2) \\ y_3 \end{bmatrix},$$

is the α-rotation around the third axis.
All these are least energy harmonic maps:

$$\int_{\mathbb{R}^2} |\nabla U_{\lambda,q,\alpha}|^2 = 4\pi.$$

Theorem (Dávila, del Pino, Wei, 2020)

Let us fix points $q_1, \dots, q_k \in \mathbb{R}^2$. *Given a sufficiently small* $T > 0$, *there exists an initial condition* u_0 *such the solution* $u(x,t)$ *of (HMF) blows-up as* $t \uparrow T$ *in the form*

$$u_q(x,t) = \sum_{j=1}^{k} Q_{\alpha_i(t)}\, U\left(\frac{x - q_i}{\lambda_i(t)}\right) + u_*(x) + o(1)$$

in the energy and uniform senses where u_* *is a regular function,*

$$\lambda_i(t) = \frac{\kappa_i(T-t)}{|\log(T-t)|^2}.$$

$$|\nabla u_q(\cdot,t)|^2 \rightharpoonup |\nabla u_*|^2 + 4\pi \sum_{j=1}^{k} \delta_{q_j}$$

Construction of a bubbling solution $k = 1$

Given a $T > 0$, $q \in \Omega$, we want

$$S(u) := -u_t + \Delta u + |\nabla u|^2 u = 0 \quad \text{in } \Omega \times (0, T)$$

$$u(x,t) \approx U(x,t) := U_{\lambda(t),\xi(t),\alpha(t)}(x) = Q_{\omega(t)}\, U\left(\frac{x - \xi(t)}{\lambda(t)}\right)$$

for certain functions $\xi(t)$, $\lambda(t)$ and $\omega(t)$ of class $C^1[0, T]$ such that

$$\xi(T) = q, \quad \lambda(T) = 0,$$

so that $u(x,t)$ blows-up at time T and the point q. We want to find values for these functions so that for a small remainder v we have that $u = U + v$ solves the problem.

We want, for $y = \frac{x-\xi(t)}{\lambda(t)}$,

$$\boxed{u \approx U(x,t) + \eta Q_\omega \phi(y,t) + \Pi_{U^\perp}[\Phi^0(x,t) + \Psi^*(x,t)]},$$

for a function $\phi(y,t)$ with $\phi(\cdot,t) \cdot W \equiv 0$, and that vanishes as $t \to T$ and that has space decay in y. η is a cut-off function concentrated near the blow-up point,

$$\Pi_{U^\perp}[Z] := Z - (Z \cdot U)U.$$

- Φ^0 is a function that depends on the parameters and basically eliminates at main order the error far away.
- $\Psi^*(x,t)$ is close to a fixed function $Z_0^*(x)$ that we specify below.

We require for $x = \xi + re^{i\theta}$, $p(t) = \lambda(t)e^{i\omega(t)}$,

$$\Phi^0_t - \Delta_x \Phi^0 - \frac{2}{r}\begin{bmatrix} \dot{p}(t)e^{i\theta} \\ 0 \end{bmatrix} = 0.$$

$$\Phi^0[\omega, \lambda, \xi] := \begin{bmatrix} \varphi^0(r,t)e^{i\theta} \\ 0 \end{bmatrix}$$

$$\varphi^0(r,t) = \int_0^t p(s) r k(r, t-s)\, ds, \quad k(r,t) = 2\frac{1 - e^{-\frac{r^2}{4t}}}{r^2}.$$

We take

$$Z_0^*(x) = \begin{bmatrix} z_0^*(x) \\ z_{03}^*(x) \end{bmatrix}, \quad z_0^*(x) = z_{01}^*(x) + i z_{02}^*(x).$$

$$Z_0^*(q) = 0, \quad \operatorname{div} z_0^*(q) + i \operatorname{curl} z_0^*(q) \neq 0$$

For the ansatz

$$u \approx U(x,t) + \eta Q_\omega \phi(y,t) + \Pi_{U^\perp}[\Phi^0(x,t) + \Psi^*(x,t)], \quad y = \frac{x - \xi(t)}{\lambda(t)}$$

with $\Psi^*(x,0)$ close to $Z_0^*(x)$ we need the parameters to satisfy specific relations.

In fact $\phi(y,t)$ should approximately satisfy an equation of the form

$$\begin{cases} \Delta_y \phi + |\nabla U(y)|^2 \phi + 2(\nabla U \cdot \nabla \phi) U(y) = E(y,t) = O(|y|^{-3}) \\ \phi(y,t) \cdot U(y) = 0 \quad \text{in } \mathbb{R}^2, \\ \phi(y,t) \to 0 \quad \text{as } |y| \to \infty \end{cases}$$

where $E(y,t)$ is the error of approximation.

- $E(y,t)$ depends *non-locally* on $p(t) = \lambda e^{i\omega}$ through $\Phi_0(x,t)$.
- We need for solvability conditions of the form

$$\int_{\mathbb{R}^2} E(y,t) \cdot Z_\ell(y)\, dy \;=\; 0$$

where $Z_\ell(y)$ are generators of invariances under λ- dilatons and ω-rotations for the harmonic map problem

$$\Delta U + |\nabla U|^2 U = 0$$

These conditions lead to $p(t) = \lambda(t)e^{i\omega(t)}$ approximately satisfying

$$\boxed{\int_0^{t-\lambda(t)^2} \frac{\dot{p}(s)}{t-s} ds = 2(\mathrm{div}\, z_0^*(q) + i\mathrm{curl}\, z_0^*(q)) =: a_0^*.}$$

We recall that $a_0^* \neq 0$ This implies that

$$a_0^* = -|a_0^*|e^{i\omega_*}$$

for a unique $\omega_* \in (-\pi, \pi)$. It turns out that the following function is an accurate approximate solution:

$$\boxed{\omega(t) \equiv \omega_*, \quad \dot{\lambda}(t) = -|\mathrm{div}\, z_0^*(q) + i\mathrm{curl}\, z_0^*(q)| \frac{|\log T|}{\log^2(T-t)}}$$

Happy Birthday Marie Françoise and Laurent