Uniqueness and multiplicity of large positive solutions

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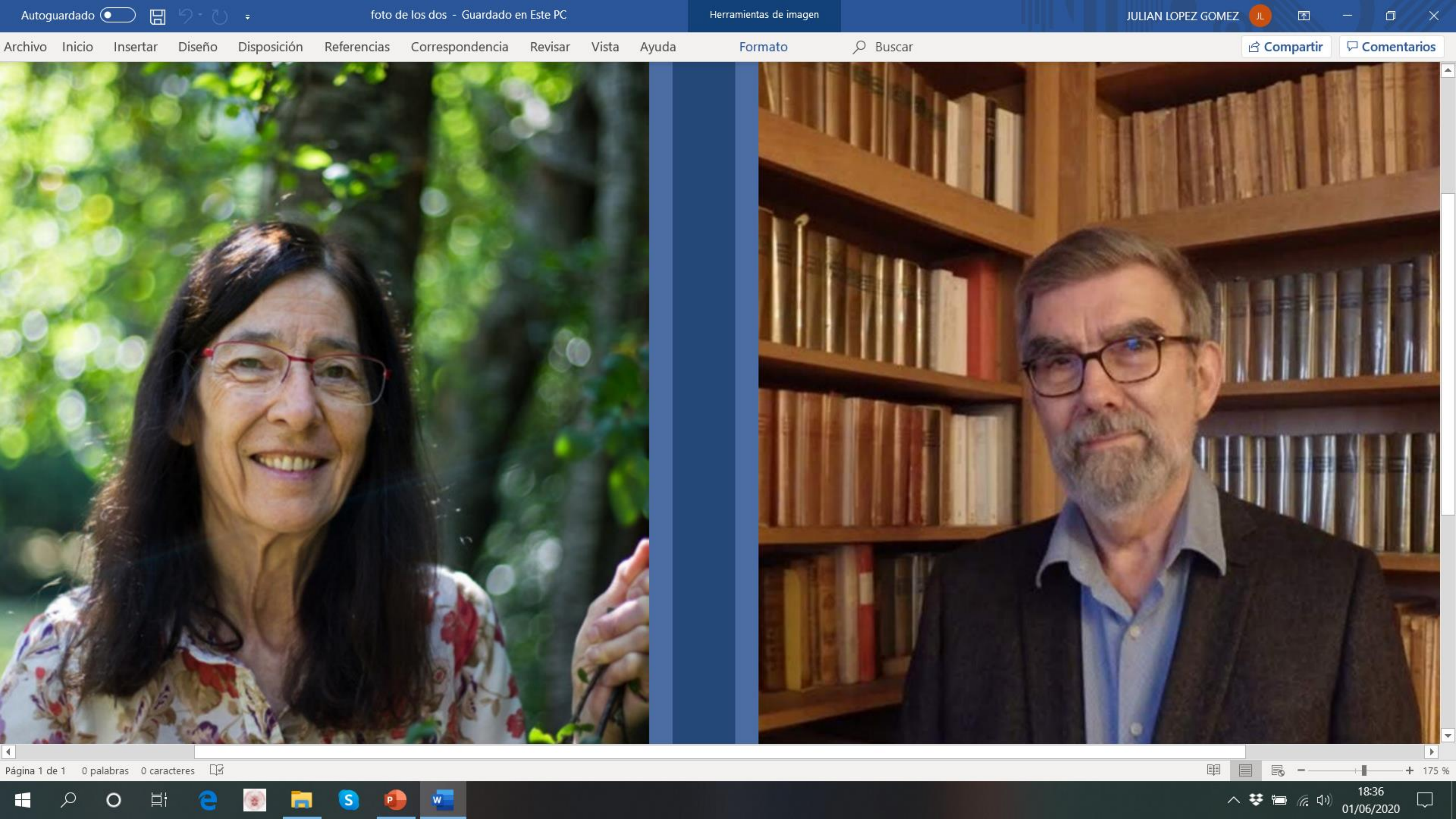
Singular Problems Associated to Quasilinear Equations

Dedié à Marie Françoise et Laurent à l'occasion de leur 70e anniversaire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Avec admiration pour la profondeur de leur travail mathématique!
Con admiración por la profundidad de su trabajo matemático!

彼の数学的研究の深さに感心して

Впечатлен глубиной его математической работы

د هغه د ریاضیاتي کار ژورتیا لپاره د عالي ستاینې سره

General Scheme of the talk

- **Multiplicity** of large positive solutions in a class of one-dimensonal **superlinear indefinite problems**.

- **Multiplicity and uniqueness** of large positive solutions in a class of one-dimensional **sublinear problems**.

- **Uniqueness** of large positive solutions for sublinear problems in a **multidimensional context**.

A superlinear indefinite problem

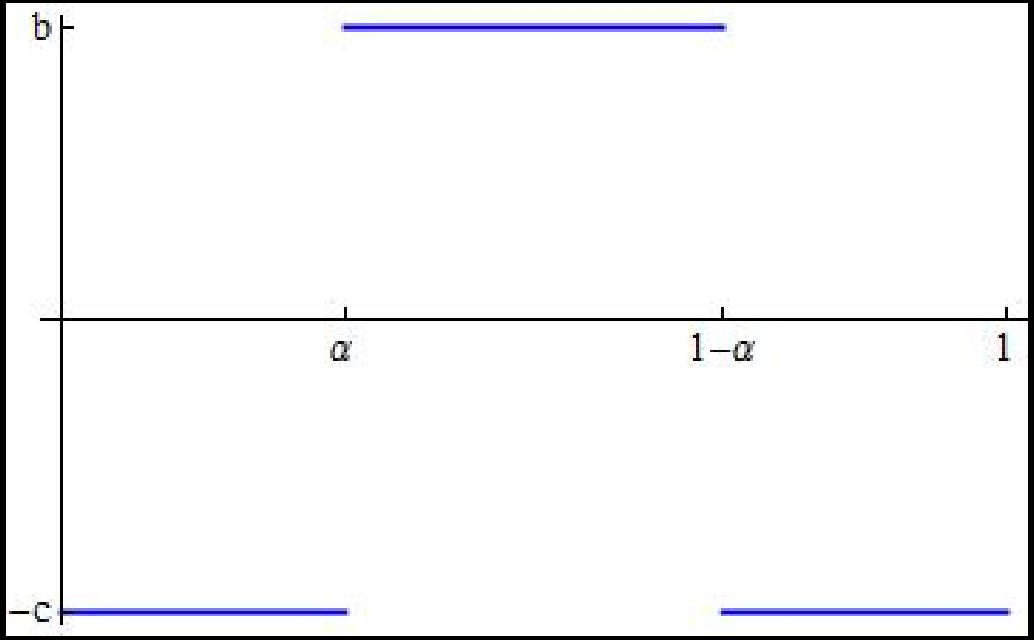
(Molina-Meyer, Tellini, Zanolin and LG)

The problem:

$$-u'' = \lambda u + a(x)u^p \quad in \quad (0,1),$$
$$u(0) = u(1) = \infty,$$

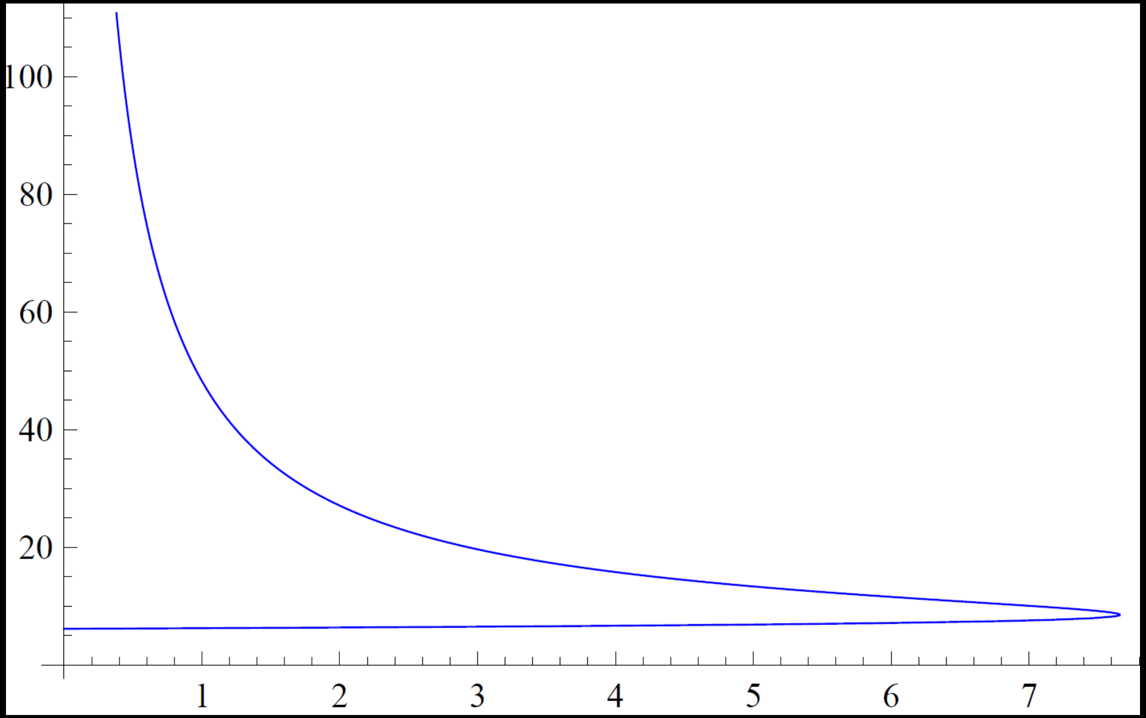
where $p > 1$ is fixed, λ, b are two parameters, and $a(x)$ is the function plotted on the right.

The weight function:

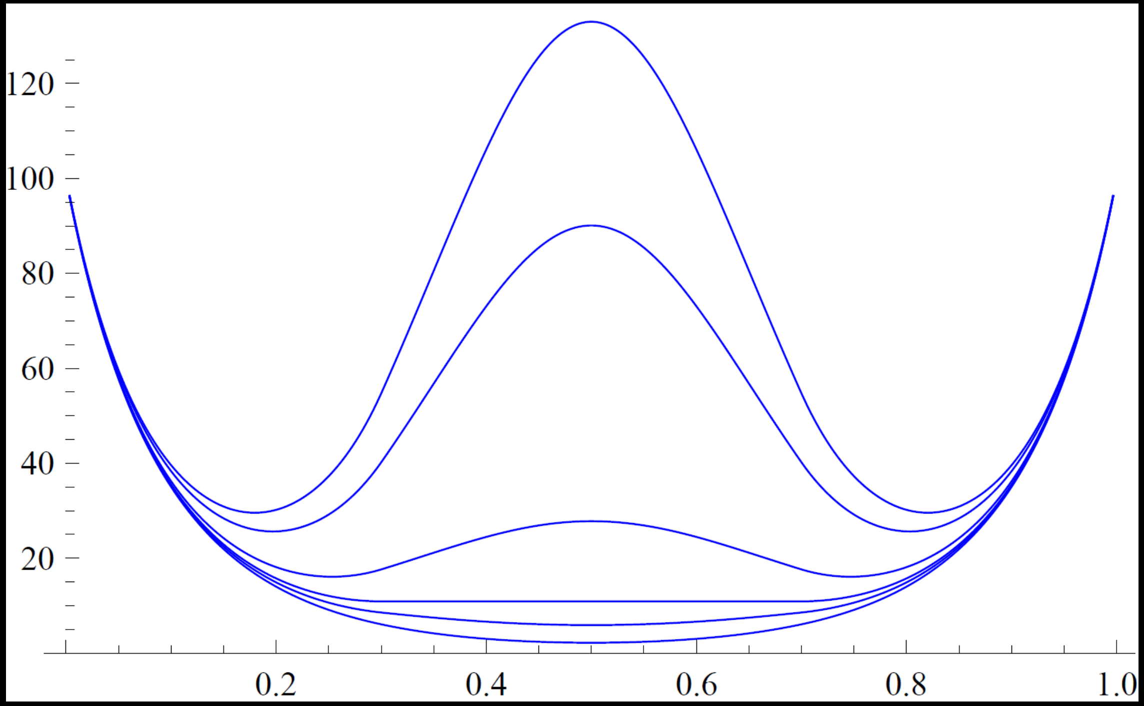


Bifurcation diagram for $\lambda = -70$

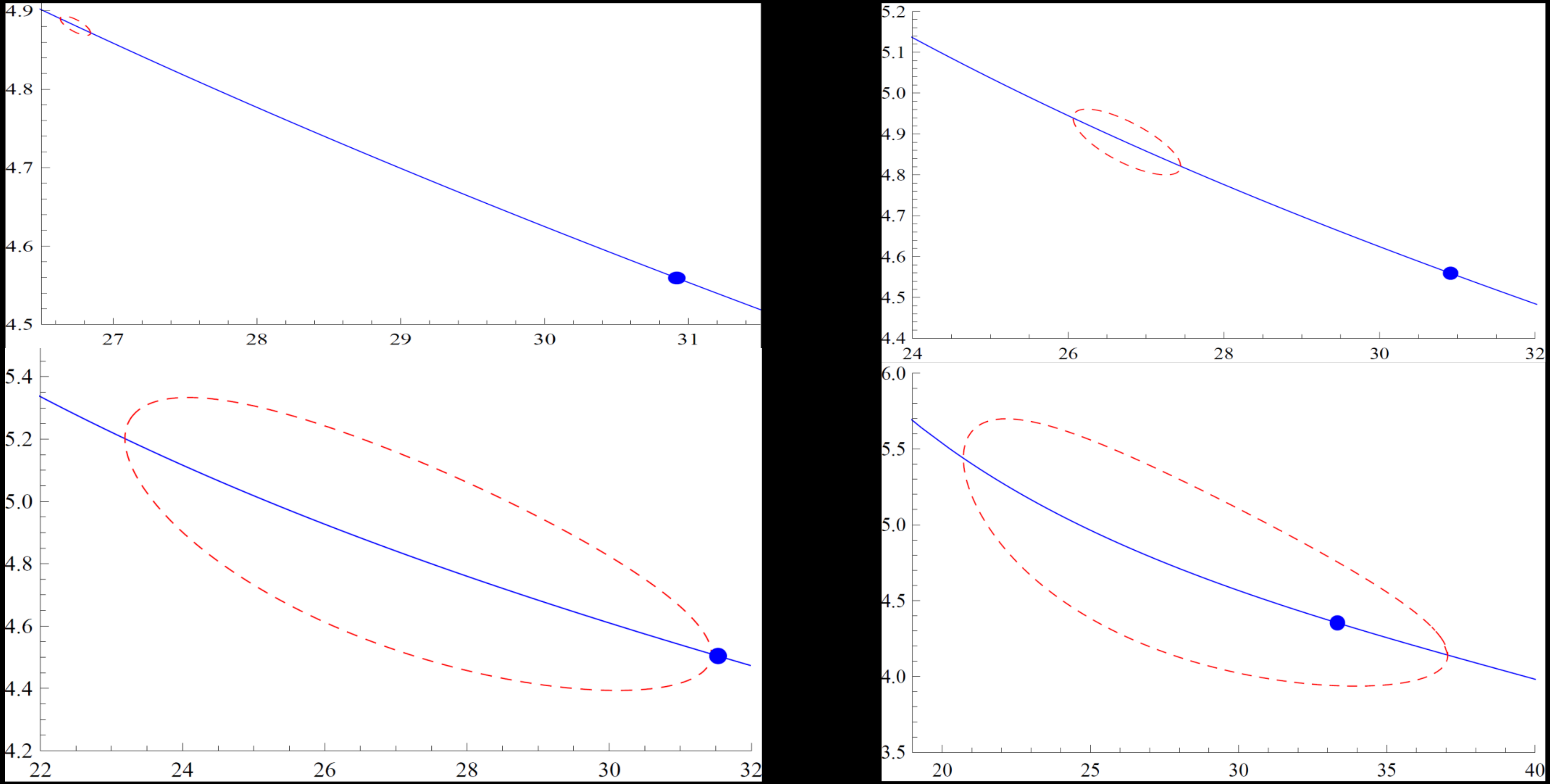
The value of $u(\alpha)$, in ordinates, versus the value of b, in abcisas



Some solutions along the bifurcation diagram on the left.

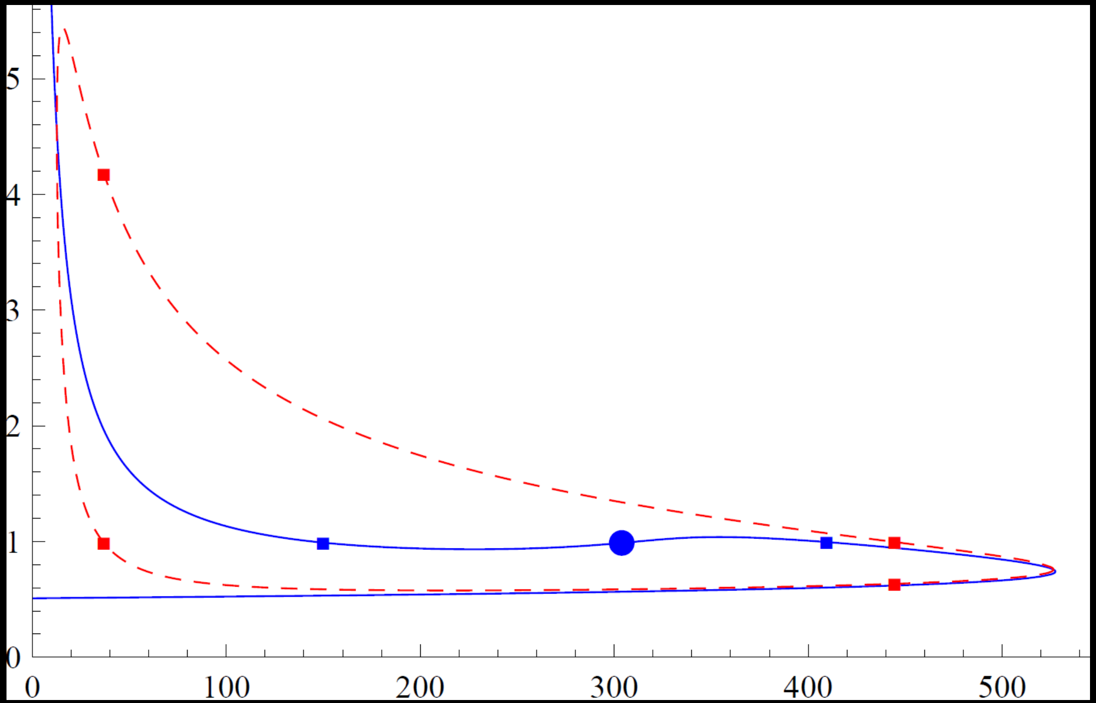


Symmetry breaking of the first loop of solutions. Four pieces of the bifurcation diagram for λ=-140, -141, -142 and -145.

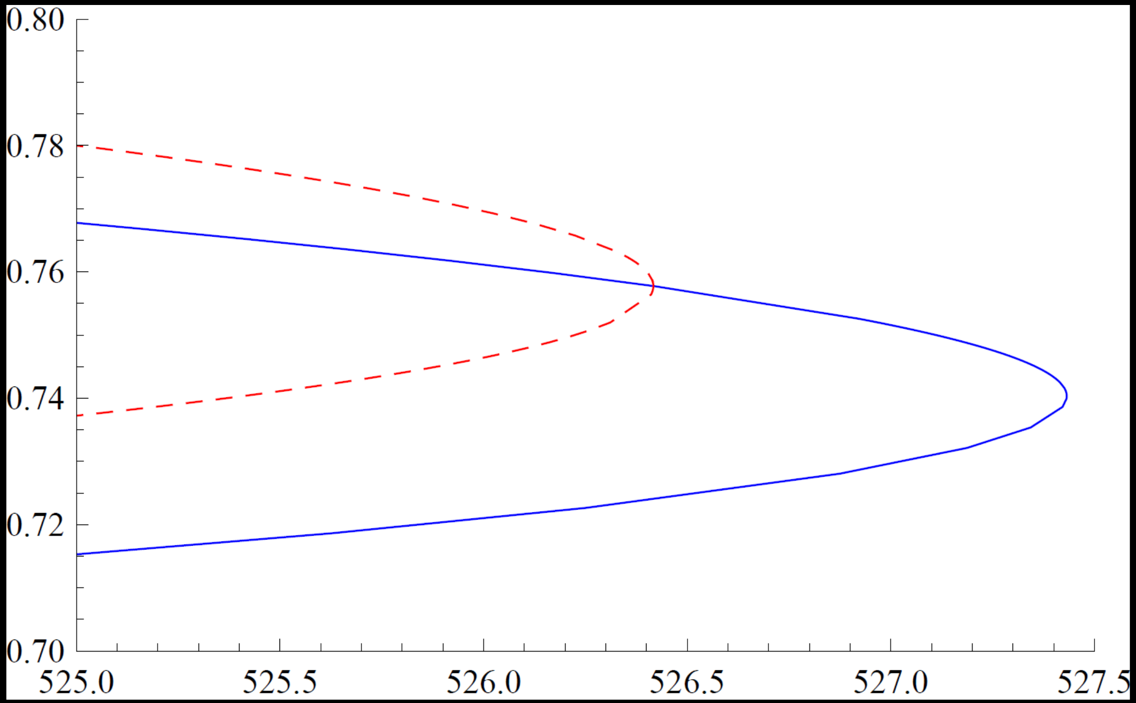


Bifurcation diagram for $\lambda = -300$

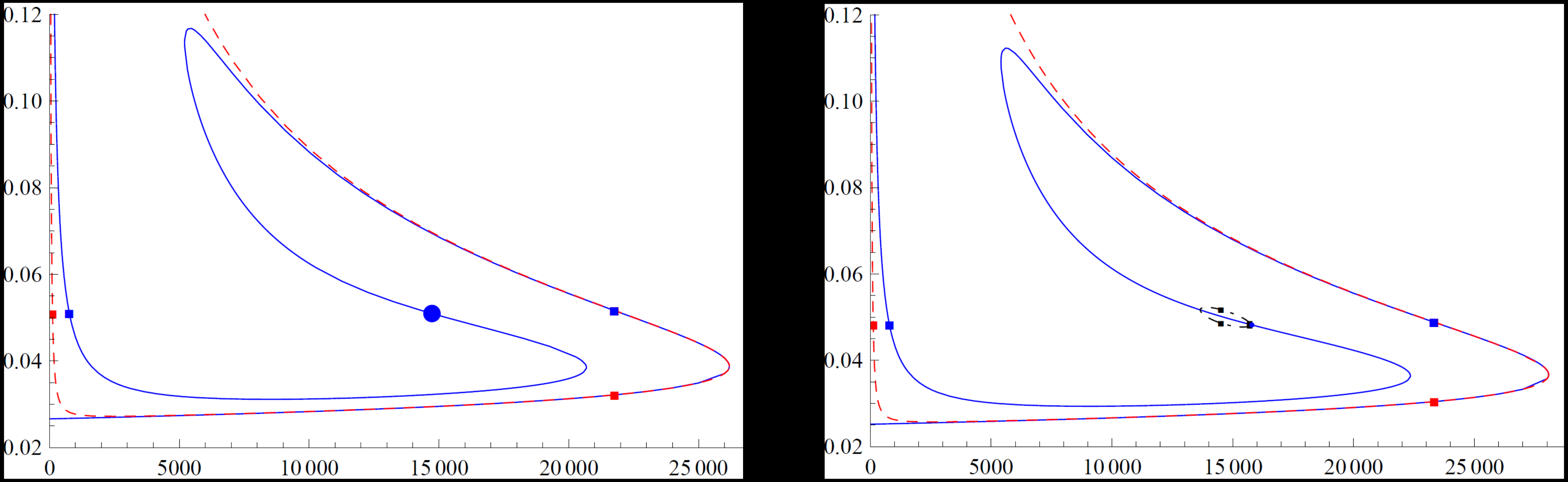
Global bifurcation diagram



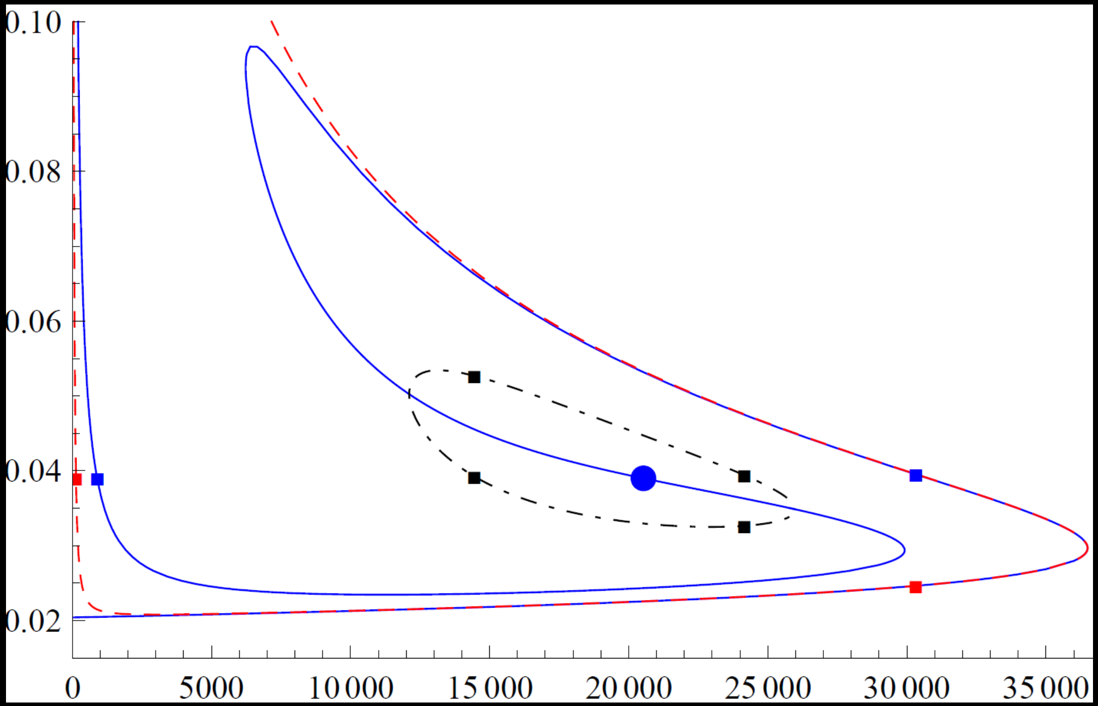
Magnification of a turning point

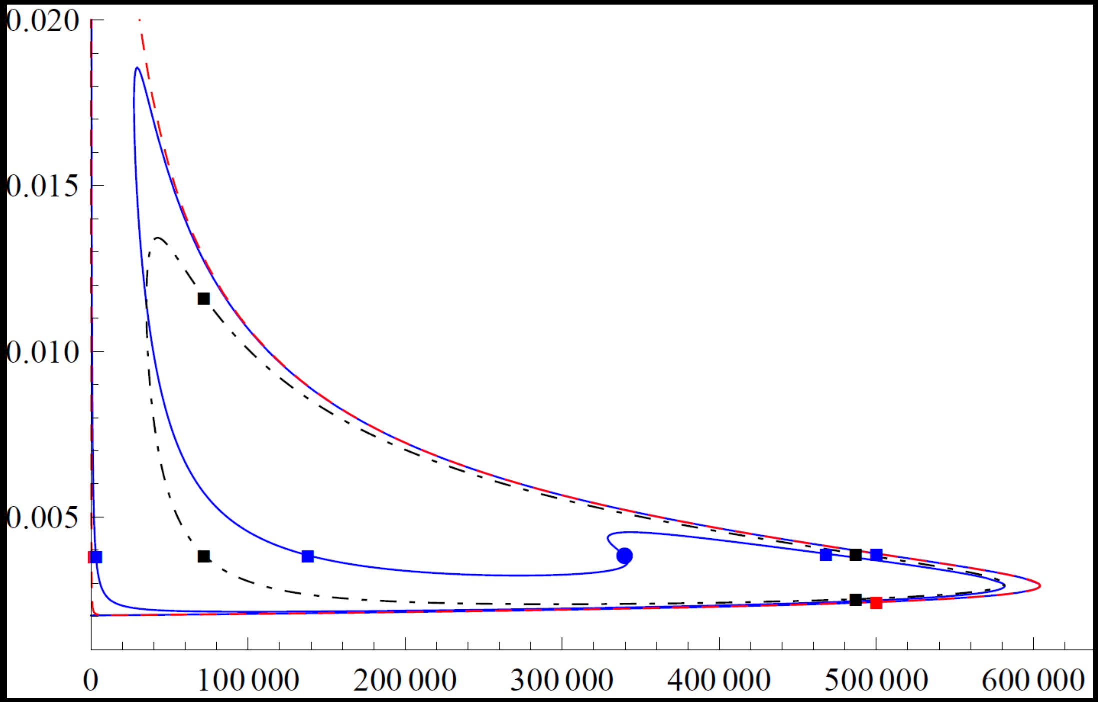


Bifurcation diagrams for $\lambda = -750, -760$



Bifurcation diagrams for $\lambda = -800, -1300$

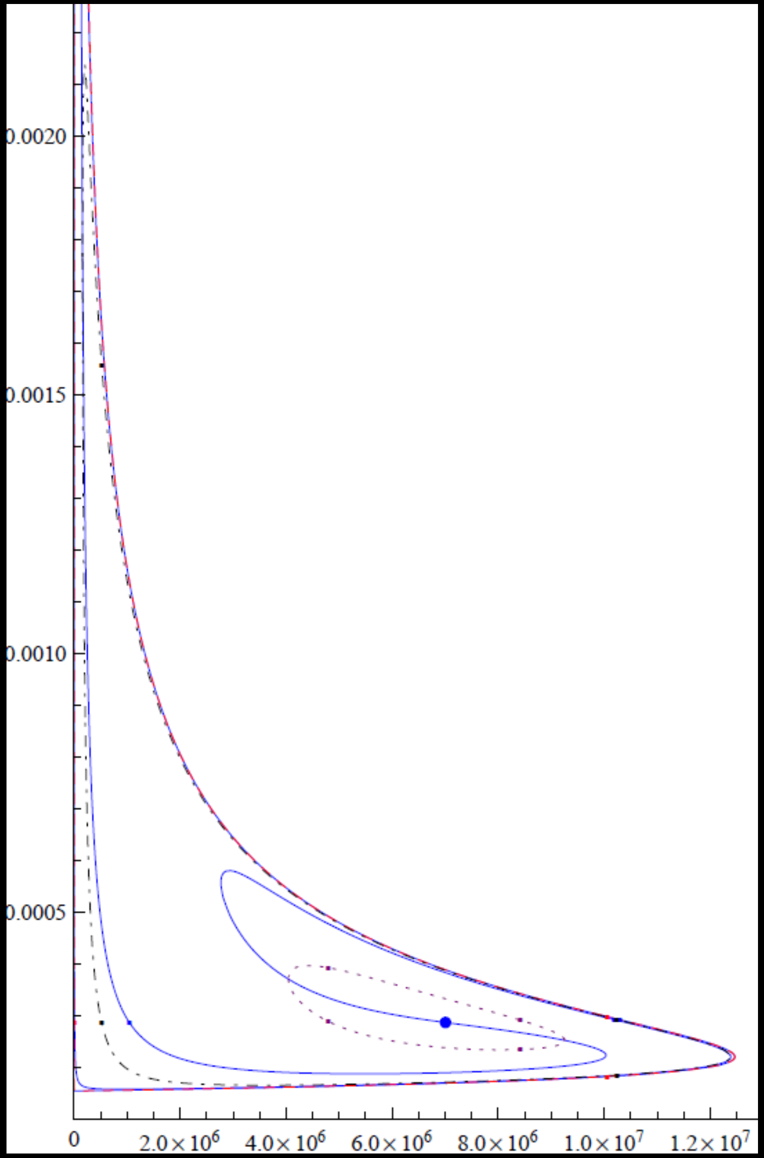




Bifurcation diagram for $\lambda = -2000$.

Every half rotation a new loop of asymmetric solutions emanates from the primary curve; it is persistent for smaller values of λ.

In particular, for some particular values of the parameter b, the number of large positive solutions increases to infinity as $\lambda \downarrow -\infty$.



A one-dimensional sublinear problem

The simplest model:

$$(\mathrm{P})\qquad \begin{cases} \boldsymbol{u'' = f(u)} \\ \boldsymbol{u'(0) = 0} \\ \boldsymbol{u(T) = \infty} \end{cases}$$

where $f \in \mathcal{C}^1[0, \infty)$ satisfies

$$f(0) = 0, \qquad f \gneqq 0.$$

Associated Cauchy problem:

$$\boldsymbol{u'' = f(u)}$$

$$\boldsymbol{u'(0) = 0}$$

$$\boldsymbol{u(0) = x > 0}$$

The solution is (globally) defined in $[0, T_{max}(x))$ for some

$$T_{max}(x) \le \infty.$$

It is explosive if $T_{max}(x) < \infty$.

Theorem (Maire-LG)[JMAA 2018]:

The singular problem (P) has a solution if, and only if, there exists $x \in (0,\infty) \setminus f^{-1}(0)$ such that

$$T = T_{max}(x) \equiv \frac{1}{\sqrt{2}} \int_x^{\infty} \frac{d\theta}{\sqrt{\int_x^{\theta} f}} < \infty.$$

Thus, impossing the Keller-Ossermann condition, KO, is impossing that the maximal existence time of the solution, $T_{max}(x)$, is finite.

Theorem (Maire-LG)[JMAA 2018]: Suppose that $f(0)=0, f(u)$ is increasing and $T_{max}(x_0)<\infty$ for some $x_0>0$. Then, the problem (P) has a unique positive solution for every $T>0$.

By continuous dependence, $T_{max}(x)\uparrow\infty$ as $x\downarrow 0$. Moreover, T_{max} is strictly decreasing, and $T_{max}(x)\downarrow 0$ as $x\uparrow\infty$, by a result of Dumont, Dupaigne, Goubet & Radulescu [ANS 2007]

Since T_{max} decays strictly, one can construct examples of non-increasing functions, $f_n(u)$, for which the problem (P) possesses a unique positive large solution.

Thus, the strict monotonicity of $f(u)$ is far from necessary for the uniqueness.

J. López-Gómez and L. Maire / Nonlinear Analysis: Real World Applications 47 (2019) 291–305

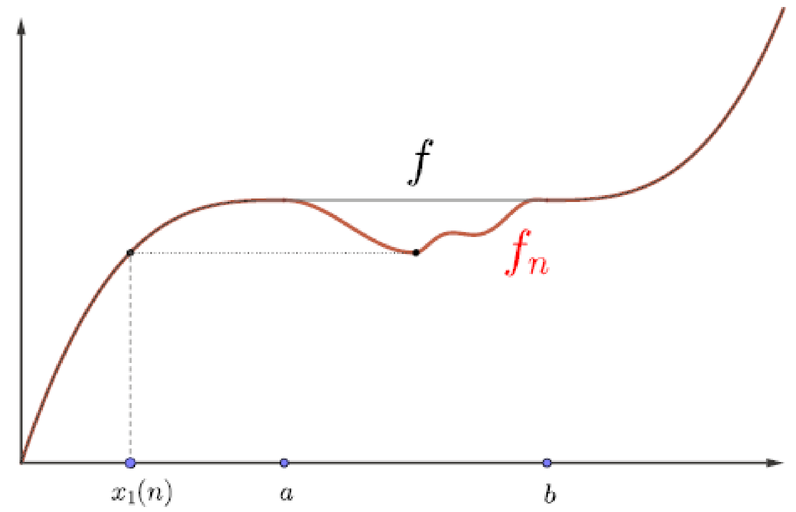


Fig. 3.1. The graphs of f and f_n, and the construction of $x_1(n)$.

Theorem (Maire-LG)[JMAA 2018]: Suppose that $f \in C^1[0,\infty)$, $f^{-1}(0) = \{x_0 = 0, x_1, \ldots, x_p\}$, $f(u) > 0$ for all $u \in (0,\infty) \setminus \{x_1, \ldots, x_p\}$, and $T_{max}(y_j) < \infty$ for some $y_j \in (x_j, x_{j+1}), j \in \{0, 1, \ldots, p\}$ $(x_{p+1} = \infty)$.

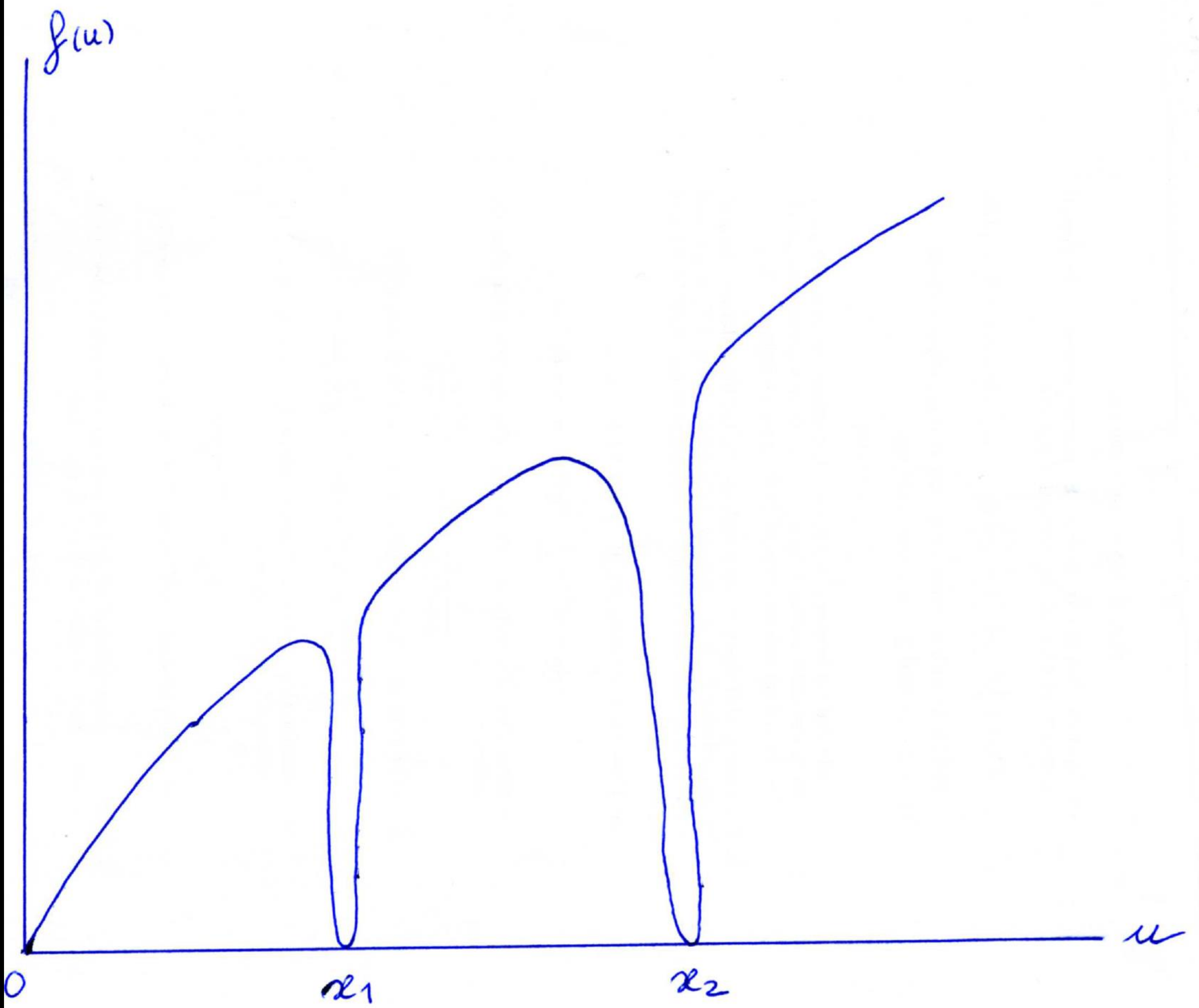
Then, there exist $T_* < T^*$ such that:

a) (P) has at least $2p+1$ **positive solutions** for every $T > T^*$

b) (P) has a **unique positive solution** for every $T < T_*$

A paradigmatic example is provided by the function

$$f(u) = u \prod_{j=1}^{p} (u - x_j)^2$$



f(u)
u
0
x1
x2

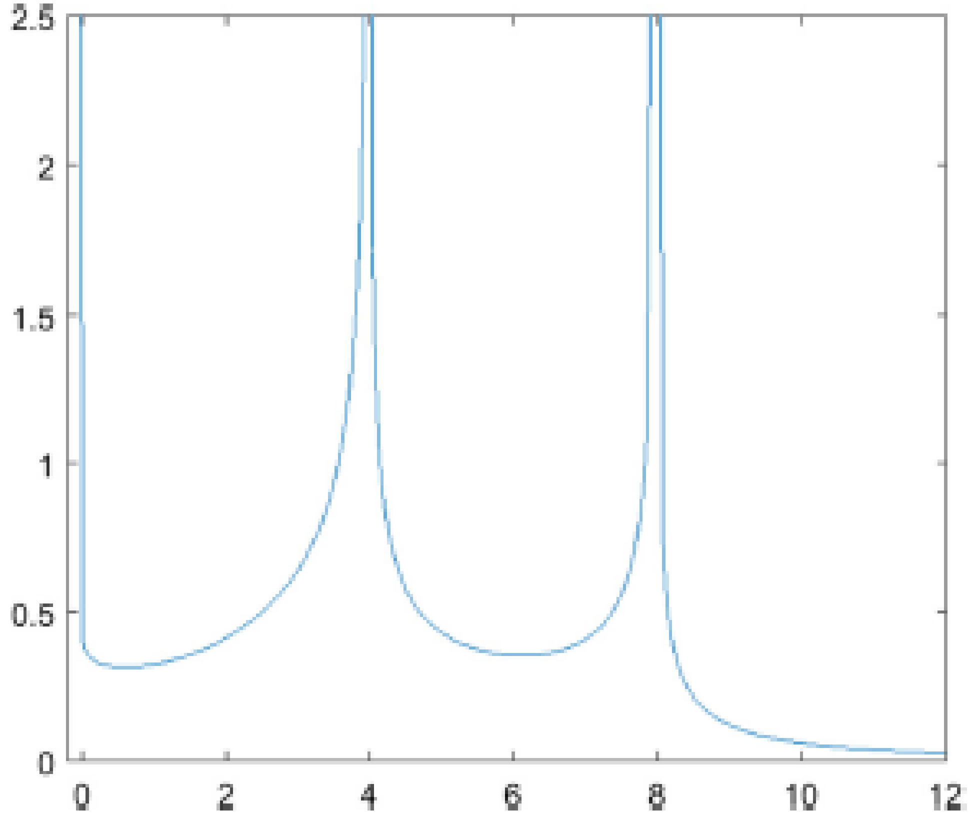


Fig. 3.1. The time map $\mathcal{T}(x)$ for $f(u) = u(u-4)^2(u-8)^2$.

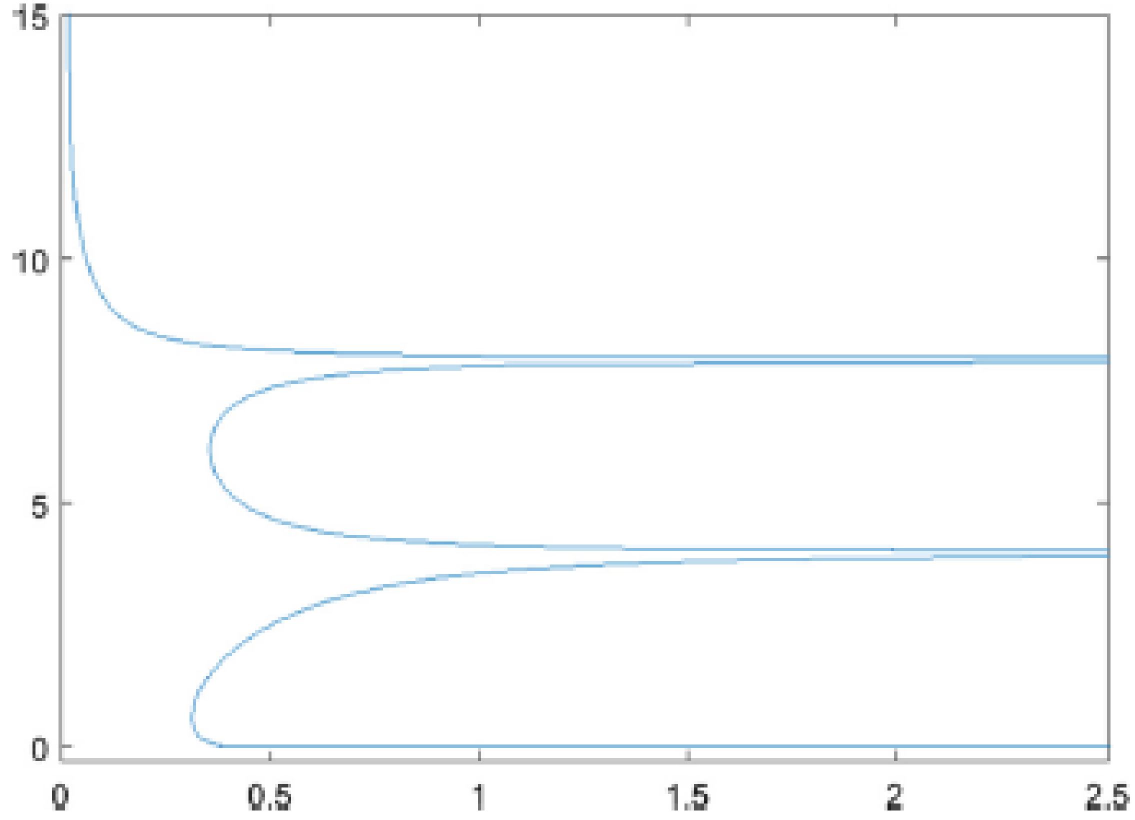
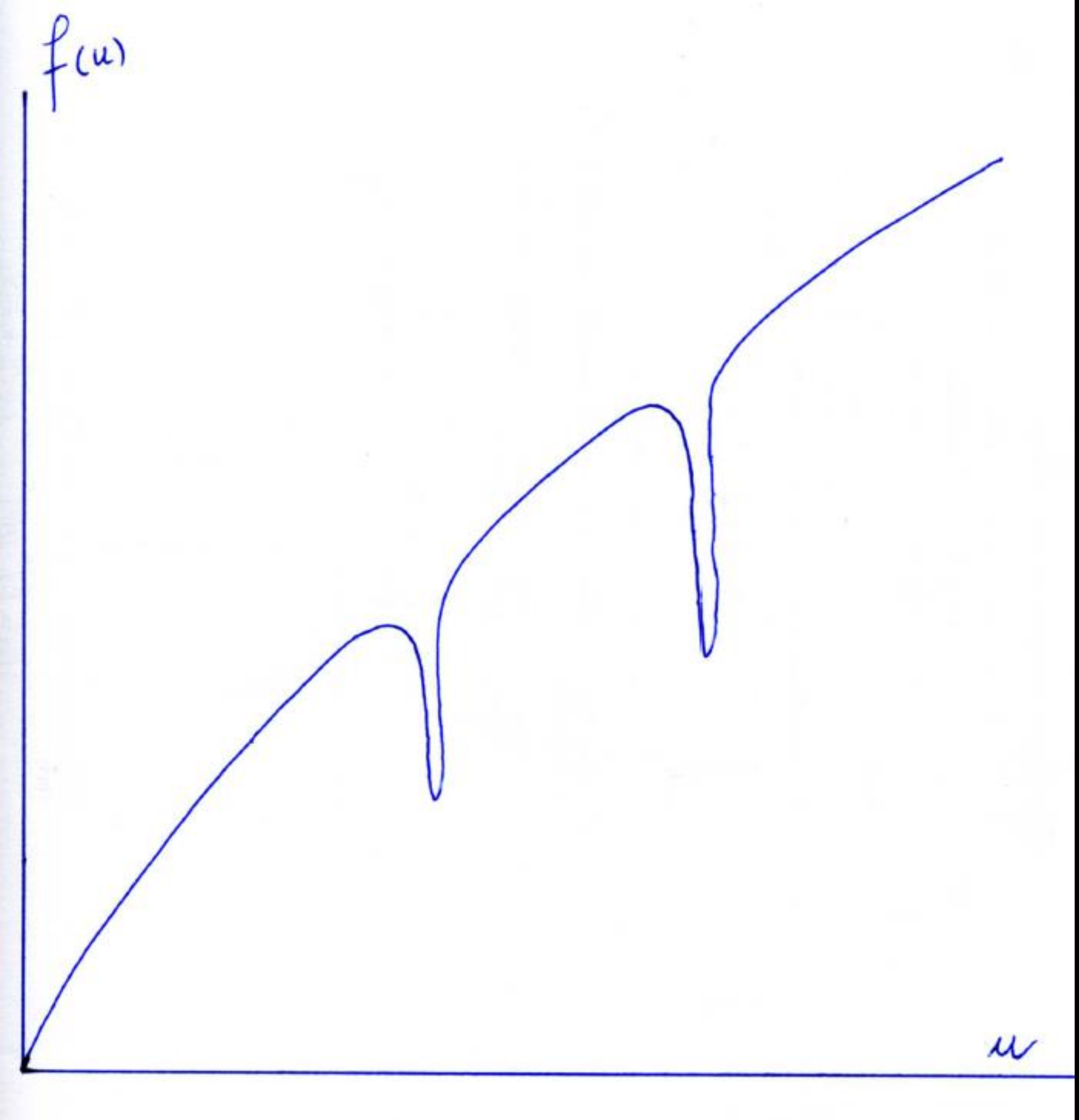


Fig. 3.2. Bifurcation diagram for $f(u) = u(u-4)^2(u-8)^2$.



$$f(u) = u\prod_{j=1}^{p}\left[(u - x_j)^2 + \epsilon\right]$$

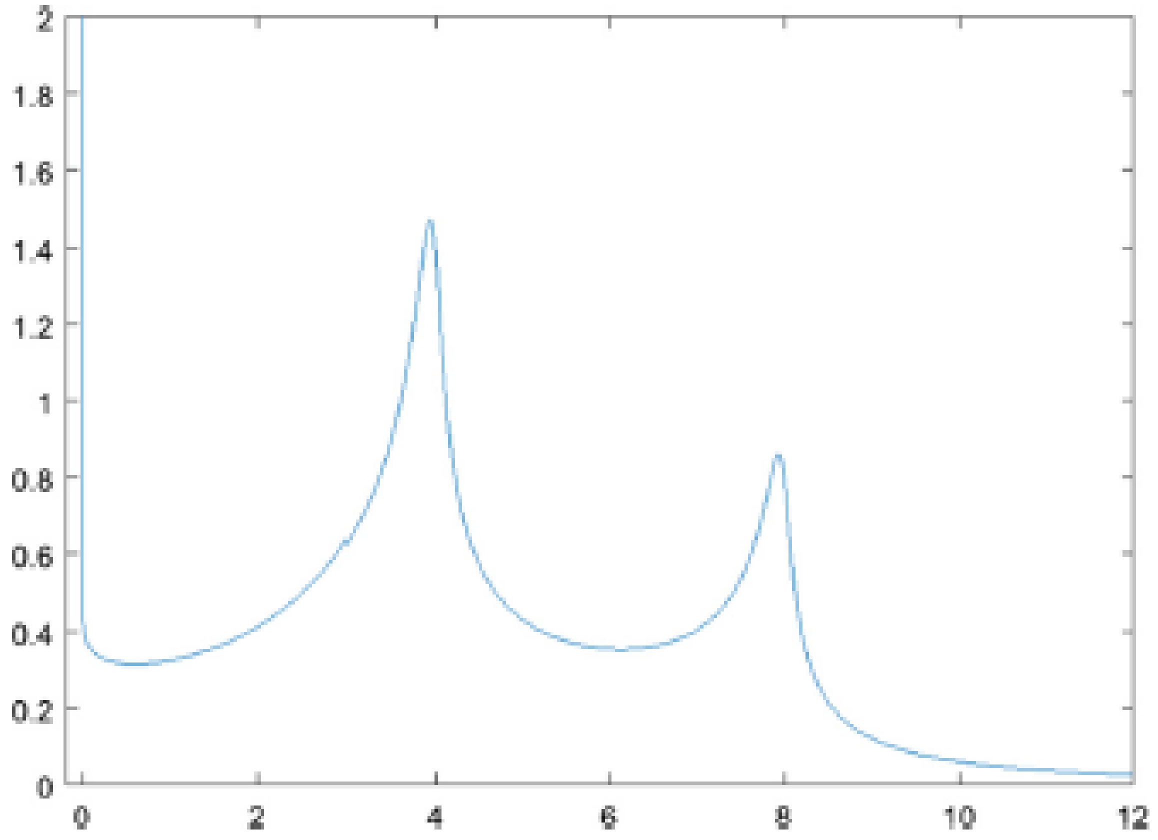


Fig. 3.3. The time map $\mathcal{T}(x)$ for $f(u) = u((u-4)^2 + 0.01)((u-8)^2 + 0.01)$.

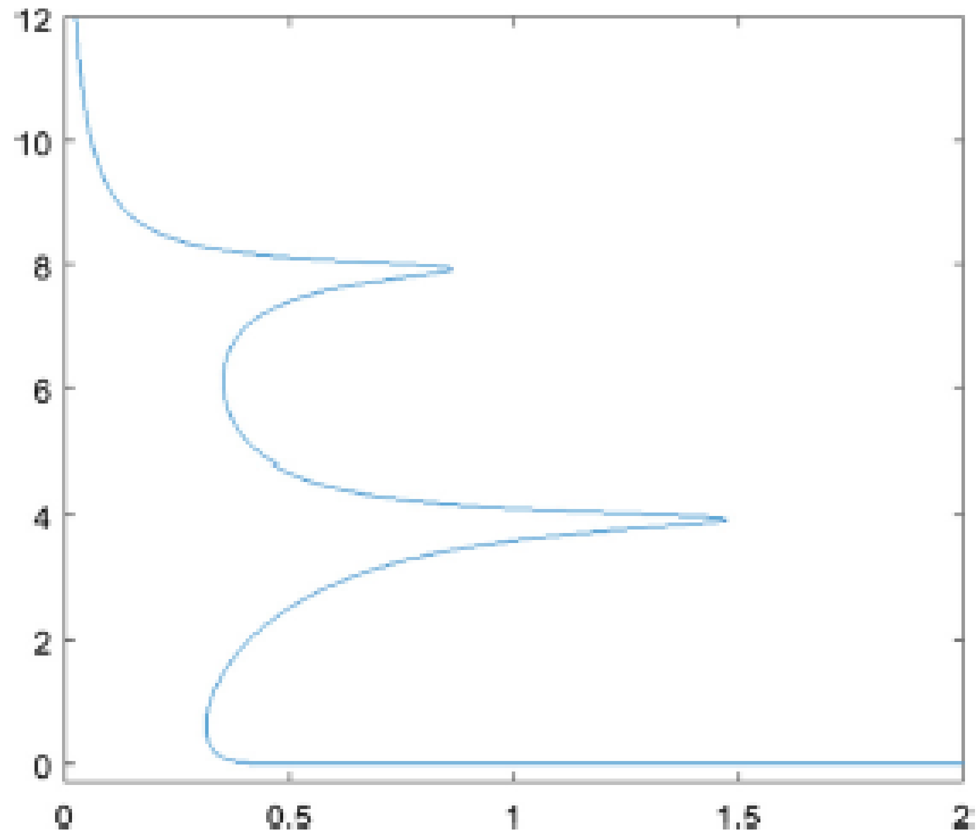


Fig. 3.4. Bifurcation diagram for $f(u) = u((u-4)^2 + 0.01)((u-8)^2 + 0.01)$.

The multidimensional problem

Now, Ω stands for a bounded sub-domain of $\mathbb{R}^N$, $N \geq 1$, with **Lipschitz boundary,** $\partial\Omega$, $f \in C^1[0,\infty)$ is an **increasing** function such that $f(0) = 0$, satisfying the **Keller-Osserman** condition, and $a \in C(\bar{\Omega})$ is a non-negative function which is **positive on a neigbourhood of $\partial\Omega$.**

Open problem: Should be unique the positive solution of

$$\begin{cases} \Delta u = a(x)\, f(u) \quad \text{in} \quad \Omega, \\ u = \infty \quad \text{on} \quad \partial\Omega. \end{cases}$$

Theorem LG [JDE-2006]
Cano-Casanova & LG [JMAA-2009]

Suppose that:

a) Ω is a **ball** or an **annulus**,

b) $H = \lim_{u\uparrow\infty} \frac{f(u)}{u^p} > 0$ for some $p > 1$,

c) the function $A(t) = \int_t^\eta \left(\int_0^s a^{\frac{1}{p+1}}\right)^{-\frac{p+1}{p-1}} \mathrm{ds},\ \mathrm{t} \in (0,\eta]$, satisfies

$$\lim_{t\downarrow 0} \frac{A(t)A''(t)}{(A'(t))^2} = I_0 > 0.$$

Then,

$$\lim_{d(x)\downarrow 0} \frac{L(x)}{A(d(x))} = (I_0^p H)^{-\frac{1}{p-1}} \left(\frac{p+1}{p-1}\right)^{\frac{p+1}{p-1}} \Rightarrow \textbf{Uniqueness}$$

Localizing on $\partial\Omega \Rightarrow$ Uniqueness on C^2 $\Omega' s$

Some special conditions on $f(u)$

Superhomogeneous of degree $p > 1$ LG [DCDS 2007]

$$f(\gamma u) \geq \gamma^p f(u) \text{ for all } \gamma > 1 \text{ and } u > 0.$$

It implies that $f(u)$ satisfies KO and $\dfrac{f(u)}{u}$ is increasing.

Superaditive with constant $C \geq 0$ Marcus -Véron [JEE 2004]

$$f(u+v) \geq f(u) + f(v) - C \text{ for all } u, v \geq 0.$$

It is weaker than the superhomogeneity. It is extremelly sharp.

Theorem Maire & LG [ZAMP 2017]

Suppose that:

a) Ω **is starshaped with respect to** $x_0 \in \Omega$,

b) $f(u)$ **is superhomogeneous of degree** $p > 1$,

c) $a(x)$ **is nonincreasing near** $\partial\Omega$ **along rays from** x_0.

Then, the singular problem has a unique positive solution.

It is also valid for general domains obtained by substracting finitely many star-shaped disjoint domains to a given star-shaped domain of $\mathbb{R}^N$.

The proofs are based on the SMP. They adapt the old proof of the uniqueness in the radially symmetric case by

LG [DCDS-2007]

Impossing the regularity of the domain
one can relax the requirements on $f(u)$

Theorem Maire-LG [ZAMP 2017]

Suppose that:

a) Ω is a bounded domain of class C^2,
b) $f(u)$ satisfies KO and it is superaditive of constant $C \geq 0$,
c) $a(x)$ decays along the normal directions on $\partial\Omega$.

Then, $\Delta u = a(x) f(u)$ has a unique large positive solution.

The sharpest multidimensional results

Associated to the function f, according to **Maire-Véron-LG [ZAMP 2020],** one can also consider the function

$$\boldsymbol{g}(\boldsymbol{x},\ell) \stackrel{\text{def}}{=} \mathbf{inf}\{\boldsymbol{f}(\boldsymbol{x},\ell+\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}):\ \boldsymbol{u} \geq \mathbf{0}\}, \qquad (\boldsymbol{x},\ell) \in \overline{\boldsymbol{\Omega}} \times [\mathbf{0},\infty).$$

There always holds $g \leq f$ and $g(x,\cdot)$ is monotone non-decreasing as $f(x,\cdot)$ is. Thus, if g satisfies the KO-condition, then also f satisfies it, thought the converse can be false.

[If $f(x,\cdot)$ is convex for all $x \in \overline{\Omega}$, then $f = g$]

Following **Marcus-Véron [CPAM 2003],** it is said that $\boldsymbol{\Delta u = g(x,u)}$ possesses a **strong barrier at some** $\boldsymbol{z \in \partial\Omega}$ if for sufficiently small $r > 0$ there exists a positive supersolution $u_{r,z}$ of $\Delta u = g(x,u)$ in $\Omega \cap B_r(z)$ such that $u_{r,z} \in \mathcal{C}(\overline{\Omega} \cap B_r(z))$ and

$$\lim_{\substack{y \to x \\ y \in \overline{\Omega} \cap B_r(z)}} u_{r,z}(y) = \infty \quad \text{for all } x \in \Omega \cap \partial B_r(z).$$

The function $u_{r,z}$ also is a supersolution of $\Delta u = f(x,u)$ because $g \leq f$.
By a result of **Marcus-Véron** [CPAM 2003], this condition holds at every $z \in \partial\Omega$ provided $\partial\Omega$ is $\mathcal{C}^2$ and, for some $\alpha > 0$ and every (x,u)

$$g(x,u) \geq d^{\alpha}(x) u^{p}.$$

It also holds when $\partial\Omega$ satisfies the local graph condition and $g(x,u) = a(x)G(u)$ with $a > 0$ on $\partial\Omega$, and $G(u)$ satisfies KO.

Theorem Maire-Véron-LG [ZAMP 2020]

Suppose that **Ω is Lipschitz continuous** and $f \in \mathcal{C}(\bar{\Omega} \times \mathbb{R})$ satisfies $f(x,0)=0$, $u \mapsto f(x,u)$ **is nondecreasing** for all $x \in \bar{\Omega}$, $f(\cdot,u)$ decays nearby $\partial\Omega$, **and the associated function** $g \in \mathcal{C}(\bar{\Omega} \times \mathbb{R})$ is positive on a neighborhood, $\mathcal{U}$, of $\partial\Omega$, and, for every compact subset $K \subset \mathcal{U}$ there exists a continuous nondecreasing function $h_K : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(x,u) \geq h_K(u) \geq 0$ for all $x \in K$ and $u \geq 0$, where h_K **satisfies the KO condition. If the equation $\Delta u = g(x,u)$ possesses a strong barrier at every $z \in \partial\Omega$, then $\Delta u = f(x,u)$ admits a unique large positive solution.**

This result relaxes the superaditivity condition of Marcus-Véron. Therefore, it generalizes the previous results of Marcus-Véron and Maire-LG. Indeed, by the superaditvity,

$$f(x,u+\ell) \geq f(x,u) + f(x,\ell) - C \quad \text{for all } x \in \Omega \text{ and } u,\ell \geq 0.$$

As this condition implies that

$$g(x,\ell) = inf\big(f(x,u+\ell) - f(x,u)\big) \geq f(x,\ell) - C,$$

it is apparent that $g(x,u)$ satisfies KO-loc if $f(x,u)$ does it.

Sketch of the proof:

- **Step 1:** **Under the assumptions of the theorem, the next problem has, at least, one positive solution**

$$\begin{aligned} \Delta\ell &= g(x,\ell) && in \quad \Theta_0 \\ \ell &= 0 && on \quad \Gamma_{0,0} \\ \ell &= \infty && on \quad \Gamma_{\infty,0} \end{aligned}$$

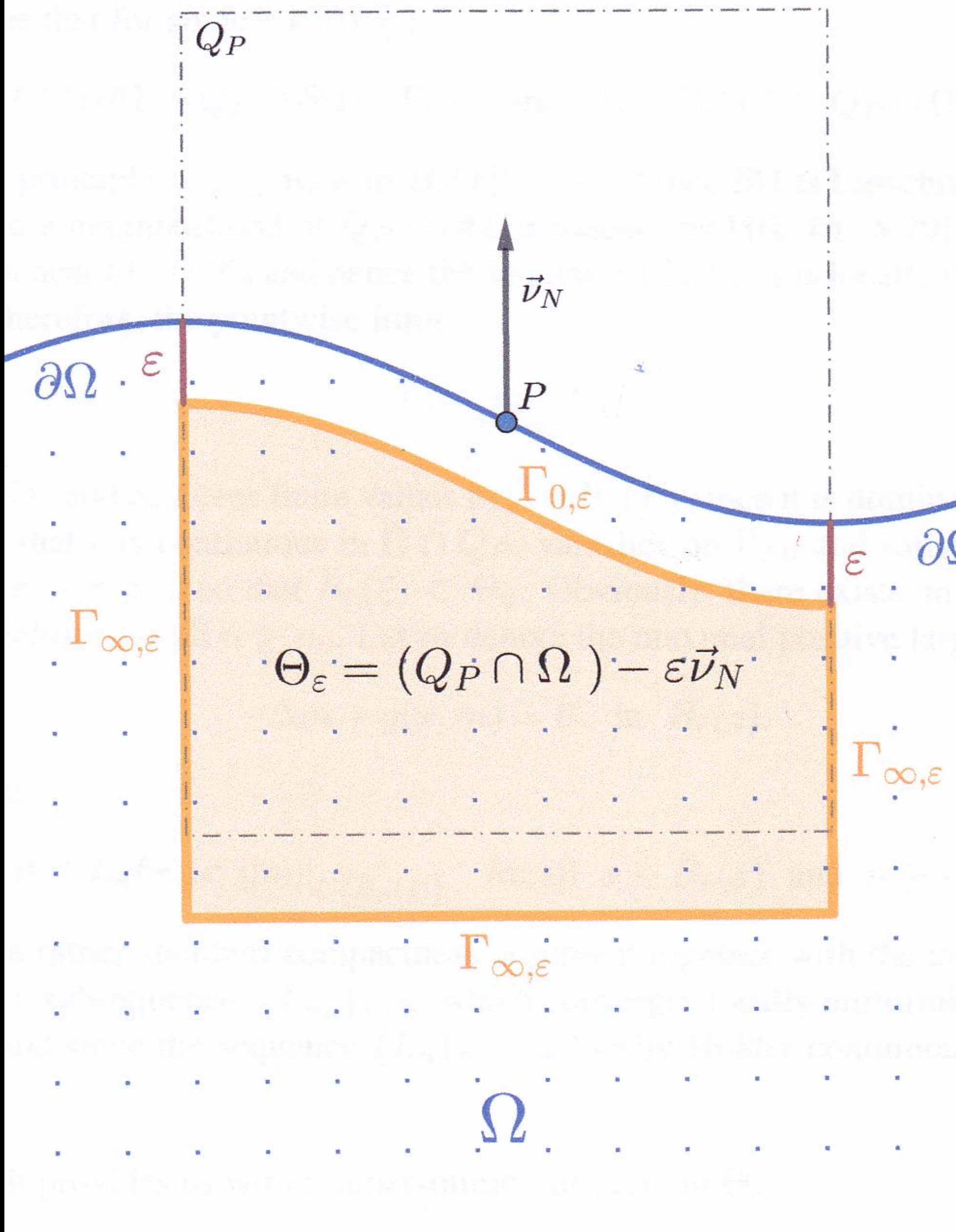


Figure 2.2: The domain Θ_ϵ

Step 2: **For sufficiently small $\epsilon > 0$, the function**

$$\overline{u_\epsilon}(x) = u_{min}(x + \epsilon\nu_N) + \ell(x + \epsilon\nu_N) \quad \text{for all } x \in \Theta_\epsilon$$

is a supersolution of $\Delta u = f(x, u)$ **in** Θ_ϵ **such that**

$$\overline{u_\epsilon} = \infty \quad on \quad \partial\Theta_\epsilon$$

Proof of step 2:

- $-\Delta\overline{u_\epsilon}(x) = -\Delta \boldsymbol{u}_{\boldsymbol{min}}(\boldsymbol{x}+\boldsymbol{\epsilon}\boldsymbol{\nu}_N) - \Delta\ell(\boldsymbol{x}+\boldsymbol{\epsilon}\boldsymbol{\nu}_N)$
- $= -f\big(x+\epsilon\nu_N, \boldsymbol{u}_{\boldsymbol{min}}(\boldsymbol{x}+\boldsymbol{\epsilon}\boldsymbol{\nu}_N)\big) - g(x+\epsilon\nu_N, \ell(\boldsymbol{x}+\boldsymbol{\epsilon}\boldsymbol{\nu}_N))$
- $\geq -f\big(x, \boldsymbol{u}_{\boldsymbol{min}}(\boldsymbol{x}+\boldsymbol{\epsilon}\boldsymbol{\nu}_N)\big) - g(x, \ell(\boldsymbol{x}+\boldsymbol{\epsilon}\boldsymbol{\nu}_N))$
- $\geq -f\big(x, \boldsymbol{u}_{\boldsymbol{min}}(\boldsymbol{x}+\boldsymbol{\epsilon}\boldsymbol{\nu}_N)\big) - f\big(x, \boldsymbol{u}_{\boldsymbol{min}}(\boldsymbol{x}+\boldsymbol{\epsilon}\boldsymbol{\nu}_N) + \ell(\boldsymbol{x}+\boldsymbol{\epsilon}\boldsymbol{\nu}_N)\big)$
- $+ f\big(x, \boldsymbol{u}_{\boldsymbol{min}}(\boldsymbol{x}+\boldsymbol{\epsilon}\boldsymbol{\nu}_N)\big)$
- $= -f(x, \overline{u_\epsilon}(x)),$

which ends the proof of Step 2.

Step 3: Ending the proof:

Since $\overline{u_\epsilon}(x)$ is bounded on $\partial\Theta_\epsilon$, by the maximum principle,

$$u_{max}(x) \leq \overline{u_\epsilon}(x) = u_{min}(x + \epsilon\nu_N) + \ell(x + \epsilon\nu_N), \quad x \in \Theta_\epsilon.$$

Thus, letting $\epsilon \downarrow 0$, yields to

$$0 \leq u_{max}(x) - u_{min}(x) \leq \ell(x), \quad x \in \Theta_0.$$

Therefore,

$$\lim_{x \to \Gamma_{0,0}} (u_{max}(x) - u_{min}(x)) = 0.$$

As this holds in a neighborhood of each $P \in \partial\Omega$, we find that

$$\lim_{d(x) \downarrow 0} (u_{max}(x) - u_{min}(x)) = 0.$$

Set $L \equiv u_{min} - u_{max} \leq 0.$ **By the monotonicity of** $f(x,u)$, **it is apparent that**

$$-\Delta L = f(x, u_{max}) - f(x, u_{min}) \geq 0 \quad \textbf{in} \quad \Omega.$$

Thus, since $L = 0$ **on** $\partial\Omega$, **we can infer that** $L = 0$ **in** Ω. **Equivalently,**

$$u_{max} = u_{min}$$

⊞

Our second result, valid under less regularity on Ω, **requires an additional condition on** $f(x,u)$.

Theorem Maire-Véron-LG [ZAMP 2020]

Assume that Ω satisfies the local graph condition and $f(x,u)$ and $g(x,u)$ satisfy the conditions of the previous theorem. Suppose, in addition, that there is $\phi \in C^2(\mathbb{R}_+)$ such that $\phi(0) = 0$, $\phi(r) > 0$ for $r > 0$,

$$\phi'(r) \geq 0 \quad and \quad \phi''(r) \leq 0 \quad for\,all\; r \geq 0,$$

and, for sufficiently small $\epsilon > 0$,

$$\frac{f(x, r + \epsilon\phi(r))}{f(x,r)} \geq 1 + \epsilon\phi'(r) \quad for\,all \;\; r \geq 0 \;\; and \;\; x \in \bar{\Omega}.$$

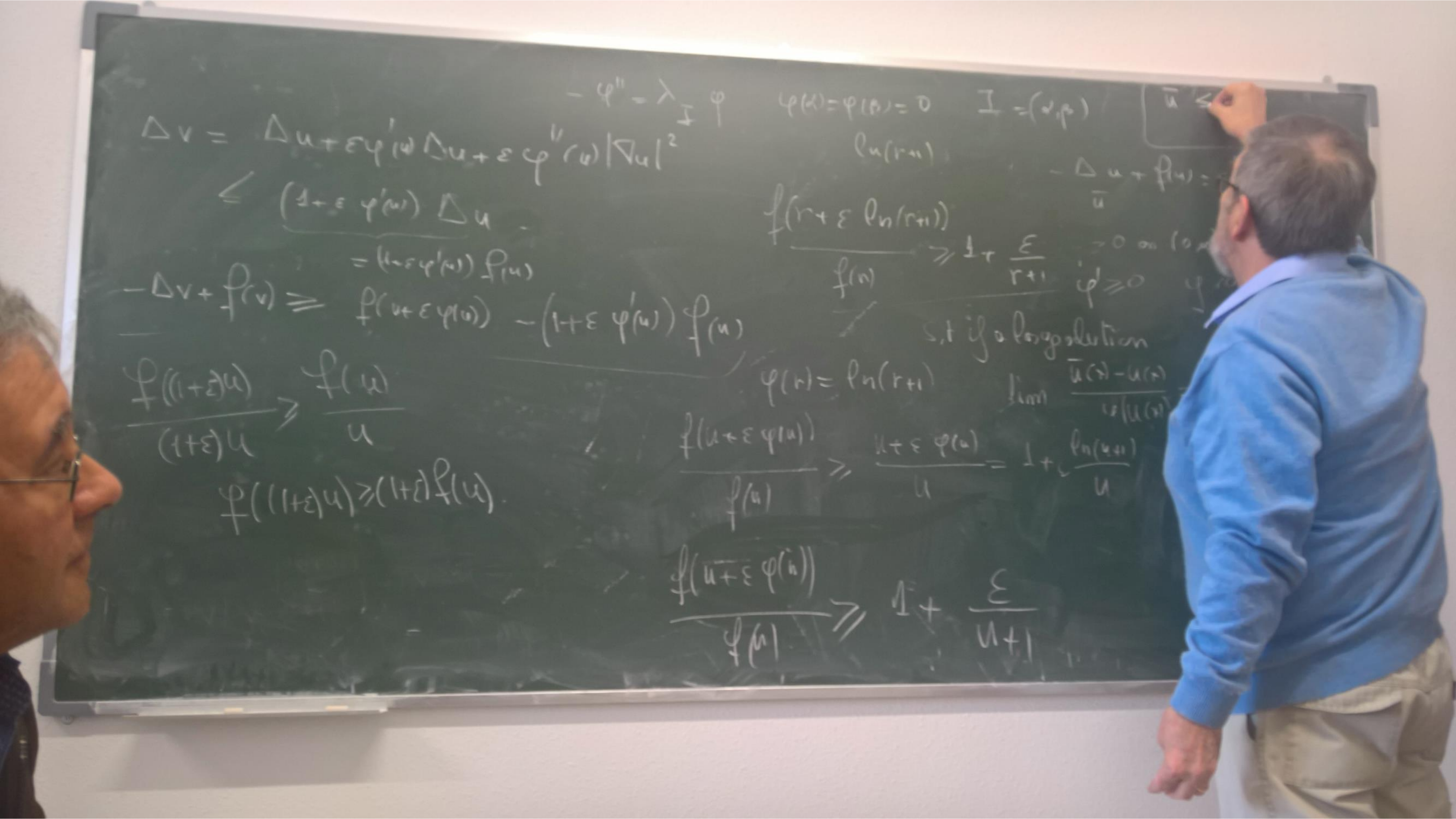
Then, $\Delta u = f(x,u)$ has, at most, a unique large positive solution.

When $\phi(r) = r$, **the condition on** $f(x,u)$ **becomes**

$$u \longmapsto \frac{f(x,u)}{u} \quad \textbf{is nondecreasing on} \quad (0,\infty).$$

Being much stronger than the one of our theorem, it is a rather usual condition imposed by many authors to get uniqueness of large positive solutions in a number of settings.

For the choice $\phi(r) = Log\,(1+r)$, **our condition is weaker!**


$-\varphi'' = \lambda_I \varphi$
$\varphi(\alpha) = \varphi(\beta) = 0$
$I = (\alpha, \beta)$
$\Delta v = \Delta u + \varepsilon \varphi'(u) \Delta u + \varepsilon \varphi''(u) |\nabla u|^2$
$\ln(r+1)$
$\leq (1 + \varepsilon \varphi'(u)) \Delta u$
$= (1 + \varepsilon \varphi'(u)) f(u)$
$-\Delta \bar{u} + f(u) =$
$\frac{f(r + \varepsilon \ln(r+1))}{f(r)} \geq 1 + \frac{\varepsilon}{r+1}$
$\varphi' \geq 0$
$-\Delta v + f(v) \geq f(u + \varepsilon \varphi(u)) - (1 + \varepsilon \varphi'(u)) f(u)$
s.t. if a large solution
$\frac{f((1+\varepsilon)u)}{(1+\varepsilon)u} \geq \frac{f(u)}{u}$
$\varphi(r) = \ln(r+1)$
$\lim \frac{\bar{u}(x) - u(x)}{\psi(u(x))}$
$\frac{f(u + \varepsilon \varphi(u))}{f(u)} \geq \frac{u + \varepsilon \varphi(u)}{u} = 1 + \frac{\ln(u+1)}{u}$
$f((1+\varepsilon)u) \geq (1+\varepsilon) f(u)$.
$\frac{f(u + \varepsilon \varphi(u))}{f(u)} \geq 1 + \frac{\varepsilon}{u+1}$

Happy seventies !!