Some results concerning the nonnegative solutions of nonlinear elliptic equations involving a gradient term

(Joint work with M. Bidaut-Véron and L. Véron)

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Singular problems associated to quasilinear equations,
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In honor of Marie-Françoise Bidaut-Véron and Laurent Véron for their important contributions in partial differential equations and in celebration of their 70th birthday

My personal greetings!

Dear Marie and Laurent, it is a great pleasure for me to participate in this wonderful meeting to celebrate such an important birthday.

And I would like to take this opportunity to express to you my infinite gratitude, not only for your inmense generosity in sharing your mathematics with me, but also for our wonderful friendship.

The purpose of this talk is to give some results concerning a-priori bounds and existence / nonexistence of positive solutions in a domain $\Omega \subseteq \mathbb{R}^N$ of the equations

$$-\Delta u = |u|^{p-1} u\, |\nabla u|^q \tag{1}$$

where $p + q > 1$ and

$$-\Delta u = |u|^{p-1} u + M\, |\nabla u|^q\,, \tag{2}$$

where $p > 1$, $q > 1$ and $M \in \mathbb{R}$.
We address the questions of

- Upper estimates of solutions
- Existence or nonexistence of solutions in all $\mathbb{R}^N$.

These equations have been the subject of many works, starting with the well known Emden-Fowler equation

$$-\Delta u = u^p, \quad p > 1, \tag{3}$$

and then extended to much more general nonlinearities and also to systems, see for example the pioneering works of Gidas-Spruck (1980), Caffarelli (1989), Bidaut-Véron and Véron (1991), Serrin-Zou (2002), Farina (2008) among many others, including Mitidieri, Pohozaev and BV-V for quasilinear equations and systems, where mainly results of Liouville type are obtained when the problem is "sub-critical". The study of equations involving a gradient term is more recent, see for example the works Alarcón, García-Melián Quaas (2013) Filippucci (2008-2011), Burgos, García-Melián, Quaas (2016) among others.
We start with a short listing of known results for the Emden Fowler equation (3) and some more recent results for the equation

$$-\Delta u = |\nabla u|^q, \quad q > 1. \tag{4}$$

The radial study of the Emden Fowler equation (3) involves the existence of two critical values for p, the so called Serrin's exponent, and the so-called Sobolev exponent

$$p_S := \frac{N}{N-2}, \qquad p^* := \frac{N+2}{N-2}.$$

(i) The equation has a particular radial solution $u(x) = C_{N,p}|x|^{-\frac{2}{p-1}}$ if and only if $p > p_S$.

(ii) If $1 < p < p^*$, there are no ground states.

(iii) If $p \geq p^*$, there exists a one parameter family of ground states which are explicit for $p = p^*$:

$$u(x) := u_\lambda(x) = \frac{(N(N-2)\lambda)^{\frac{N-2}{4}}}{(\lambda + |x-a|^2)^{\frac{N-2}{2}}}.$$

(iv) If $1 < p < p^*$, there exists a universal constant $C_{N,p}$ such that in any bounded domain Ω, any positive solution satisfies

$$u(x) \leq C_{N,p}\mathrm{dist}(x, \partial\Omega)^{-\frac{2}{p-1}}.$$

(v) If $1 < p \leq p^*$, all the positive solutions are symmetric with respect to some x_0. (Moving planes method, CG-S 1989).

The proof for the universal estimate follows from Serrin's work (1964-1965) when $1 < p < p_S$ writing

$$-\Delta u = du, \quad d = u^{p-1}$$

$u \in M^{N/(N-2)}(\Omega)$ and then $d \in L^{1+N/2}_{loc}(\Omega)$. The difficult case is when $p_S \leq p < p^*$, done by (Bernstein method) differentiating the equation and look at the equation satisfied by $z = |\nabla v|^2$, where $v = u^\gamma$ for a negative γ, by Gidas & Spruck (1980)[2] and Véron Bidaut-Véron (1991)[4]. It uses the Böchner-Weizenböck formula

$$\frac{1}{2}\Delta z = |D^2 v|^2 + \langle \nabla(\Delta v), \nabla v \rangle \tag{5}$$

which, since $|D^2 v|^2 \geqq \frac{1}{N}(\Delta v)^2$, gives the inequality

$$\frac{1}{2}\Delta z \geqq \frac{1}{N}(\Delta v)^2 + \langle \nabla(\Delta v), \nabla v \rangle. \tag{6}$$

Then, after multiplying by a new power of v, a careful integration by parts in a ball $B(0,R)$ we obtain

$$\int_{B(0,R)} d^\sigma dx = \int_{B(0,R)} u^{\sigma(p-1)} dx \leq C \ R^{N-2\sigma}$$

for some appropriate $\sigma > N/2$.

Problem (4), $q > 1$, is not variational. We summarize our recent results (BV-GH-V, 2014)[2].

- Since constants are solutions, there is no universal estimate for the solutions.
- Any solution in Ω satisfies the universal estimate of the gradient

$$|\nabla u(x)| \le C_{N,q} \mathrm{dist}(x, \partial\Omega)^{-\frac{1}{q-1}}.$$

- Any solution in $\mathbb{R}^N$ is constant.

We prove our estimate by using the Bernstein method: The function $z = |\nabla u(x)|^2$ satisfies the inequality

$$-\Delta z + C z^q \le D\Big(\frac{|\nabla z|^2}{z} + 1\Big),$$

hence a sufficiently large power of z, $v := z^{1+K}$ satisfies

$$-\Delta v + C_0 v^{\frac{q+K}{1+K}} \le D_0$$

and thus v satisfies the Keller Osserman type of estimate

$$v(x) \le C \mathrm{dist}(x, \partial\Omega)^{-\frac{2(1+K)}{q-1}},$$

implying $|\nabla u(x)| \le C \mathrm{dist}(x, \partial\Omega)^{-\frac{1}{q-1}}$.

A first critical case appears when

$$(N-2)p+(N-1)q=N. \tag{7}$$

Indeed, if we look for radial positive solutions of the form $u(x)=\Lambda\,|x|^{-\gamma}$ we find, if $q<2$ and $p+q-1>0$, that $\gamma:=\gamma_{p,q}=\frac{2-q}{p+q-1}$ and

$$\Lambda := \Lambda_{N,p,q} = \gamma_{p,q}^{\frac{1-q}{p+q-1}}\left(N-\frac{2p+q}{p+q-1}\right)^{\frac{1}{p+q-1}}. \tag{8}$$

However, this last quantity exists if and only if the exponents belong to the *supercritical range*, that is when

$$(N-2)p+(N-1)q>N. \tag{9}$$

In the *subcritical range* of exponents i.e. when

$$(N-2)p + (N-1)q < N, \tag{10}$$

we prove that Serrin's classical results (Acta Math. 1964-65)[7, 8] can be applied. We obtain a local Harnack inequality and an *a priori* estimate for positive solutions u in $B_R \setminus \{0\}$ under the form

$$u(x) + |x|\,|\nabla u(x)| \leq c\,|x|^{2-N} \qquad \forall x \text{ s.t. } 0 < |x| \leq \frac{R}{2}, \tag{11}$$

with a constant c depending on u. We have

Theorem A1

Let $\Omega \subset \mathbb{R}^N$ be a domain containing 0, $N \geq 3$, $p \geq 0$, $0 \leq q \leq 2$ and assume (p, q) lies in the sub-critical range (10). If $u \in C^2(\Omega \setminus \{0\})$ is a positive solution of (1) in $\Omega \setminus \{0\}$, then estimate (11) holds in a neighborhood of 0.

Indeed, assume $\overline{B}_1 \subset \Omega$. By Brezis-Lions's result, there holds

$$u \in M^{\frac{N}{N-2}}(B_1)\,,\ \nabla u \in M^{\frac{N}{N-1}}(B_1)\,,\ u^p\,|\nabla u|^q \in L^1(B_1), \tag{12}$$

where $M^r = L^{r,\infty}$ denotes the Marcinkiewicz space or Lorentz space of index (r,∞), and there exists $\alpha \geq 0$ such that

$$-\Delta u = u^p\,|\nabla u|^q + \alpha\delta_0 \quad \text{in } \mathcal{D}(B_1). \tag{13}$$

We assume first $pq \neq 0$. Then

$$|u^p\,|\nabla u|^q| \leq |u|^{p\theta} + |\nabla u|^{q\theta'} = c\,|u| + d\,|\nabla u|\,, \tag{14}$$

where $\theta, \theta' \geq 1$, $\frac{1}{\theta} + \frac{1}{\theta'} = 1$, $c = |u|^{p\theta-1}$, $d = |\nabla u|^{q\theta'-1}$. If $\theta > \max\{1, \frac{1}{p}\}$ and $\theta' > \max\{1, \frac{1}{q}\}$, then $c \in M^{\frac{N}{(N-2)(p\theta-1)}}$ and $d \in M^{\frac{N}{(N-1)(q\theta'-1)}}$. We claim that we can choose $\theta > 1$ such that

$$\frac{N}{(N-2)(p\theta-1)} > \frac{N}{2} \quad \text{and} \quad \frac{N}{(N-1)(q\theta'-1)} > N. \tag{15}$$

These inequalities are respectively equivalent to

$$\theta < \frac{N}{p(N-2)} \quad \text{and} \quad \theta' < \frac{N}{q(N-1)}, \tag{16}$$

which is clearly possible from the subcritical assumption (10) by taking $\theta = \frac{N(1-\varepsilon)}{p(N-2)}$ for $\varepsilon > 0$ small enough. Because $M^r(B_1) \hookrightarrow L^{r-\delta}(B_1)$ for any $\delta > 0$, we infer that $c \in L^{\frac{N}{2}+\delta}(B_1)$ and $c \in L^{N+\delta}(B_1)$ and u verifies Harnack inequality in $B \setminus \{0\}$ by Serrin's result. This implies

$$\max_{|x|=r} u(x) \leq K \min_{|x|=r} u(x) \quad \forall r \in (0, \frac{1}{2}] \quad \text{for some } K > 0. \tag{17}$$

The spherical average $\bar{u}$ of u on $\{x : |x| = r\}$ is superharmonic. Hence there exists some $m \geq 0$ such that

$$\bar{u}(r) \leq m r^{2-N}. \tag{18}$$

Combined with (17) it yields $u(x) \leq Km\,|x|^{2-N}$. The estimate on the gradient is standard, see eg [10, Lemma 3.3.2]. $\square$

Our main results deal with the supercritical range. We prove *a priori* estimates of positive solutions of (2) in a punctured domain and existence of ground states in $\mathbb{R}^N$. There are two approaches for obtaining these results. The *direct Bernstein method* and the *integral Bernstein method* popularized by Lions [5] and Gidas and Spruck in [2] respectively. Both methods are based upon differentiating the equation. The direct Bernstein method relies on obtaining pointwise estimates of the gradient through comparison principles via algebraic computations, an intensive use of Young's inequality and without any integration. Our main result in this framework is the following:

Theorem B1

Let $N \geq 2$, $0 \leq q < 2$ and $p \geq 0$ be such that $p + q - 1 > 0$. If u is a positive solution of (1) in B_R and one of the following assumptions is fulfilled,

(i) $p + q - 1 < \frac{4}{N-1}$,

(ii) $0 \leq p < 1$ *and* $p + q - 1 < \frac{(p+1)^2}{p(N-1)}$.

Then there exist positive constants $a = a(N, p, q)$ and $c_1 = c_1(N, p, q)$ such that

$$|\nabla u^a(0)| \leq c_1 R^{-1-a\frac{2-q}{p+q-1}}. \tag{19}$$

The value of the exponent a is not easy to compute, however, in several applications this difficulty can be bypassed.

The proof of this result is long and very technical, and we will use the following result:

Lemma B1

Let $q > 1$ and $a, R > 0$. Assume υ is continuous and nonnegative on $\overline{B}_R$ and C^1 on the set $\mathcal{U}_+ = \{x \in B_R : \upsilon(x) > 0\}$. If υ satisfies

$$-\Delta\upsilon + \upsilon^q \leq a\frac{|\nabla\upsilon|^2}{\upsilon} \tag{20}$$

on each connected component of $\mathcal{U}_+$, there holds

$$\upsilon(0) \leq c_{N,q,a} R^{-\frac{2}{q-1}}. \tag{21}$$

First we set $v = u^{-1/\beta}$ for some parameter $\beta \neq 0$ to be found, and also set $z = |\nabla v|^2$. Then

$$\begin{aligned} \Delta v &= (1+\beta)\frac{|\nabla v|^2}{v} + |\beta|^{q-2}\,\beta v^{1-q-\beta(p+q-1)}\,|\nabla v|^q \\ &= (1+\beta)\frac{z}{v} + |\beta|^{q-2}\,\beta v^s z^{\frac{q}{2}}, \end{aligned} \tag{22}$$

where $s = 1-q-\beta(p+q-1)$. Using now the Weizenböck inequality (6) that we recall

$$\frac{1}{2}\Delta z \geqq \frac{1}{N}(\Delta v)^2 + \langle \nabla(\Delta v), \nabla v\rangle,$$

and by developing it can be seen that

$$\begin{aligned} &-\frac{1}{2}\Delta z + \left(\frac{(1+\beta)^2}{N} - (1+\beta)\right)\frac{z^2}{v^2} + \frac{1}{N}\beta^{2(q-1)}v^{2s}z^q \\ &+\left(\frac{2(1+\beta)}{N} + s\right)|\beta|^{q-2}\,\beta v^{s-1}z^{\frac{q}{2}+1} + (1+\beta)\frac{\langle \nabla z, \nabla v\rangle}{v} + \frac{q}{2}\,|\beta|^{q-2}\,\beta v^s z^{\frac{q}{2}-1}\langle \nabla z, \nabla v\rangle \\ &\leq 0. \end{aligned} \tag{23}$$

Next we set $Y = v^\lambda z$ for some parameter λ, and then, after multiplication by v^λ we get

$$
\begin{aligned}
&-\frac{1}{2}\Delta Y + \left(\frac{(1+\beta)^2}{N} - (1+\beta) - \frac{\lambda}{2}(\lambda+\beta+2) \right) v^{-\lambda-2} Y^2 \\
&\quad + \left(\frac{2(1+\beta)}{N} + s - \frac{\lambda(q-1)}{2} \right) |\beta|^{q-2}\, \beta v^{s-1-\frac{\lambda q}{2}}\, Y^{\frac{q}{2}+1} + \frac{1}{N}\beta^{2(q-1)} v^{2s-\lambda q+\lambda}\, Y^q \\
&\quad \leq -\left(\frac{q}{2}\, |\beta|^{q-2}\, \beta v^{s-\frac{\lambda q}{2}+\lambda}\, Y^{\frac{q}{2}-1} + \frac{\lambda+1}{v} \right) \langle \nabla Y, \nabla v \rangle.
\end{aligned}
$$

As a second step we estimate Y. To this end we let $\varepsilon_0 \in (0,1)$. For any $\varepsilon > 0$ one has

$$
\left| \frac{\langle \nabla Y, \nabla v \rangle}{v} \right| \leq \frac{1}{4\varepsilon} \frac{|\nabla Y|^2}{Y} + \varepsilon v^{-\lambda-2} Y^2.
$$

By choosing ε appropriately we obtain

$$
\begin{aligned}
-\frac{1}{2}\Delta Y &+ \left(\frac{(1+\beta)^2}{N} - (1+\beta) - \frac{\lambda(\beta+\lambda+2-\varepsilon_0)}{2}\right) v^{-\lambda-2} Y^2 \\
&+ \left(\frac{2(1+\beta)}{N} + s - \frac{\lambda(q-1)}{2}\right) |\beta|^{q-2}\beta v^{s-1-\frac{\lambda q}{2}} Y^{1+\frac{q}{2}} \\
&+ \left(\frac{\beta^{2(q-1)}}{N} - \varepsilon_0\right) v^{2s-\lambda q+\lambda} Y^q \leq C(\varepsilon_0)\frac{|\nabla Y|^2}{Y},
\end{aligned} \tag{24}
$$

with $C(\varepsilon_0) = \left(\frac{(\lambda+1)^2}{4} + \frac{q\beta^{2(q-1)}}{16}\right)\frac{1}{\varepsilon_0}$. Next we put H equal to the red expression above so that our inequality reads

$$
-\frac{1}{2}\Delta Y + H \leq C(\varepsilon_0)\frac{|\nabla Y|^2}{Y}.
$$

and consider the trinom (Note $t = v^{s+1+\lambda\frac{2-q}{2}} Y^{\frac{q}{2}-1}$)

$$
\begin{aligned}
\mathbf{T}_{\varepsilon_0}(t) = &\left(\frac{\beta^{2(q-1)}}{N} - \varepsilon_0\right) t^2 + \left(\frac{2(1+\beta)}{N} + s - \frac{\lambda(q-1)}{2}\right) |\beta|^{q-2}\beta t \\
&+ \left(\frac{(1+\beta)^2}{N} - (1+\beta) - \frac{\lambda(\beta+\lambda+2-\varepsilon_0)}{2}\right).
\end{aligned}
$$

If its discriminant is negative there exists $\alpha = \alpha(N, p, q, \beta, \lambda, \varepsilon_0) > 0$ such that $\mathbf{T}_{\varepsilon_0}(t) \geq \alpha(t^2 + 1)$, hence

$$H \geq \alpha \left(v^{-\lambda-2} Y^2 + v^{2s-\lambda q+\lambda} Y^q \right). \tag{25}$$

Assuming $\lambda \neq -2$, we introduce

$$S = \frac{2s - \lambda q + \lambda}{\lambda + 2} = 1 - q - \frac{2\beta(p + q - 1)}{\lambda + 2}, \tag{26}$$

then, if $S > \max\{0, 1 - q\}$, we have $\frac{2S+q}{S+1} > 1$ and

$$Y^{\frac{2S+q}{S+1}} = \left(\frac{Y^2}{v^{\lambda+2}} \right)^{\frac{S}{S+1}} v^{\frac{(\lambda+2)S}{S+1}} Y^{\frac{q}{S+1}} \leq \frac{Y^2}{v^{\lambda+2}} + v^{(\lambda+2)S} Y^q = \frac{Y^2}{v^{\lambda+2}} + v^{2s-\lambda q+\lambda} Y^q,$$

and thus we deduce

$$-\Delta Y + 2\alpha Y^{\frac{2S+q}{S+1}} \leq 2C(\varepsilon_0) \frac{|\nabla Y|^2}{Y}. \tag{27}$$

Using Lemma B1, we derive

$$Y(0) \leq cR^{-\frac{2(S+1)}{S+q-1}} = cR^{-\frac{2(s+1)-\lambda(q-1)}{s+q-1}} = cR^{-2+\frac{(2+\lambda)(2-q)}{\beta(p+q-1)}},$$

from which follows

$$\left|\nabla u^{-\frac{2+\lambda}{2\beta}}(0)\right| \leq \frac{|2+\lambda|\sqrt{c}}{2} R^{-1+\frac{(2+\lambda)(2-q)}{2\beta(p+q-1)}}. \tag{28}$$

Therefore, Theorem B1 will follow with $a = -\frac{\lambda+2}{2\beta}$ and a will be positive if we can choose $\lambda \neq -2$ and $\beta \neq 0$ so that $S > \max\{0, 1-q\}$. After some very lengthy and delicate estimates we can prove that under the assumptions of Theorem B we can always choose such β, λ.

Corollary B1

Under the assumptions on N, p and q of Theorem B1, any positive solution of (2) *in* $\mathbb{R}^N$ *is constant.*

Our next result is an improvement of Theorem B.

Theorem C1

Assume $p \geq 0$, $0 \leq q < 2$ *and define the polynomial* G *by*

$$\begin{aligned} G(p,q) &= \left((N-1)^2 q + N - 2\right) p^2 + b(q)p - Nq^2, \\ \textit{where} \quad b(q) &= N(N-1)q^2 - (N^2+N-1)q - N - 2. \end{aligned} \tag{29}$$

If the couple (p, q) *satisfies the inequality* $G(p, q) < 0$, *then all the positive solutions of* (1) *in* $\mathbb{R}^N$ *are constant.*

In the range of p and q, the condition $G_N(p, q) < 0$ is equivalent to

$$0 \leq p < p_c(q) := \frac{-b(q) + \sqrt{b^2(q) + 4Nq^2\left((N-1)^2 q + N - 2\right)}}{2\left((N-1)^2 q + N - 2\right)}. \tag{30}$$

It is proven by using the integral Bernstein method. The aim of the integral Bernstein method is to obtain estimates of the L^r-norm of the gradient of the solutions in balls for r large enough. Combined with the work of Serrin this leads in Gidas-Spruck to a Harnack inequality. Here we use these integral estimates to prove the non-existence of non-constant global solutions.

We start with the following Weizenböck inequality already used in the proof of Theorem B, but taken here in the weak sense, namely

$$\int_{\mathbb{R}^N} \left(\frac{1}{2} \langle \nabla z, \nabla \phi \rangle + \frac{(\Delta v)^2}{N} \phi - \Delta v (\langle \nabla v, \nabla \phi \rangle + \phi \Delta v) \right) dx \leq 0,$$

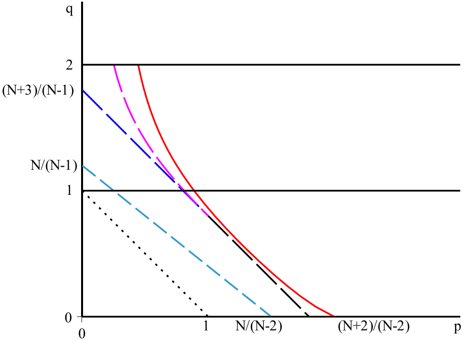
for all $\phi \in C_0^1(\mathbb{R}^N)$, $\phi \geq 0$, hence

$$\int_{\mathbb{R}^N} \left(\frac{1}{2} \langle \nabla z, \nabla \phi \rangle - \frac{N-1}{N} \phi (\Delta v)^2 - \Delta v \langle \nabla v, \nabla \phi \rangle \right) dx \leq 0. \tag{31}$$

Essentially, we choose $\phi = v^{\lambda} z^{e} \eta$ where $\eta \in C_0^3(\mathbb{R}^N)$, $\eta \geq 0$, and derive a series of integral inequalities that will yield, under the assumption that $G(p,q) < 0$, that $|\nabla u(x)| = 0$.
Computations are *very hard*, and the results depend on the choice of e, it works if e is large enough, but not too large, in our result we have set $e = \frac{N-1}{2} q$, which seemed to give the largest region.
Notice that the minimum of p on the curve $G(p,q) = 0$, $0 < q < 2$ is smaller that 1 whenever $N \geq 9$. If $q = 0$, the above reads reads

$$0 \leq p < p_c(0) := \frac{N+2}{N-2}, \tag{32}$$

which is the well known condition obtained by Gidas and Spruck. Furthermore, it can be verified that the domain of (p,q) in which Theorem B applies is included into the set of (p,q) where $G(p,q) < 0$. Our proof is extremely technical and necessitates of many algebraic computations.



The first critical relation $(N-2)p+(N-1)q=N$, see Theorem A

The curve of homogeneity $p+q=1$

The separatrix curve for Theorem B: $(N-1)(p+q-1)=\frac{(p+1)^2}{p}$ and $(N-1)(p+q-1)=4$

The separatrix curve for Theorem C: $G(p,q)=0$

If we just look for radial solutions we obtain an optimal result, namely:

Theorem D1

There exist non-constant radial positive solutions of (2) *in* $\mathbb{R}^N$ *if and only if* $p \geq 0$, $0 \leq q < 1$ *and*

$$p(N-2) + q(N-1) \geq N + \frac{2-q}{1-q}. \tag{33}$$

If equality holds in (33), *there exists an explicit one parameter family of positive radial solutions of* (2) *in* $\mathbb{R}^N$ *under the form*

$$u_c(r) = c\left(Kc^{\frac{(2-q)^2}{(N-2)(1-q)}} + r^{\frac{2-q}{1-q}}\right)^{-\frac{(N-2)(1-q)}{2-q}}, \tag{34}$$

for any $c > 0$ *and some* $K = K(N,q) > 0$.

Our equations takes the form

$$-\left(u'' + \frac{N-1}{r}u'\right) = u^p|u'|^q, \qquad u'(0) = 0. \tag{35}$$

From this equation we see that u can be written under the form

$$u(r) = u(0) + \int_0^r s^{1-N} \int_0^s u^p(t)\left|u'(t)\right|^q t^{N-1} dt \qquad \forall r > 0. \tag{36}$$

If $q \geq 1$, the solution satisfying $u(0) = a > 0$ is the unique fixed point of the mapping $v \mapsto \mathcal{T}[v]$ defined in the set of functions in $C([0, r_0])$ with value a for $r = 0$ by

$$\mathcal{T}[v](r) := a + \int_0^r s^{1-N} \int_0^s v^p(t)\left|v'(t)\right|^q t^{N-1} dt \qquad \forall r > 0.$$

Clearly $\mathcal{T}$ is a strict contraction if $r_0 > 0$ is small enough. Since $u \equiv a$ is a solution in $\mathbb{R}^N$, it is the unique one.

Let now $q \in [0,1)$. We search for nonconstant solutions by dividing equation (35) by $|u'|^q$, which gives

$$-\left((|u'|^{m-2}u')' + \frac{\nu-1}{r}|u'|^{m-2}u'\right) = (1-q)u^p$$

where $m = 2-q$ and $\nu = N-(N-1)q$, that means $v = (1-q)u$ is a radial solution of

$$-\Delta_m v = \mathrm{div}(|\nabla v|^{m-2}\nabla v) = v^p$$

in dimension ν, where $m > 1$ for $q < 1$. We now apply results by Bidaut-Véron in (1989): for this problem the critical Sobolev exponent is (recall $\dfrac{N(p-1)+p}{N-p}$)

$$p^* = \frac{\nu(m-1)+m}{\nu-m},$$

hence a second critical case appears when

$$(N-2)p + (N-1)q = N + \frac{2-q}{1-q}, \tag{37}$$

and (33) and (34) follow.

Following the work of Giacomini-Bidaut-Véron in 2010, equation (35) can be reduced to a quadratic system (2010)): set

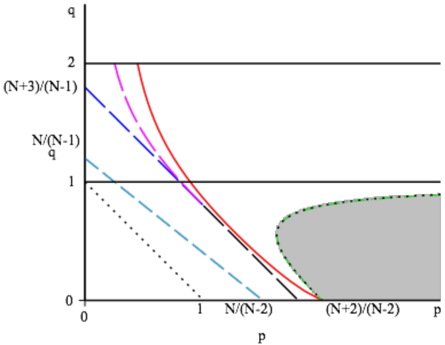
$$t = \ln r, \qquad X(t) = -r\frac{u'}{u}, \qquad Z(t) = -ru^p|u'|^{q-2}u'$$

then

$$\begin{cases} X_t = X(X - (N-2) + Z) \\ Z_t = Z(N - (N-1)q - pX + (q-1)Z \end{cases}$$

where a new value $\nu = N - (N-1)q$ appears playing the role of dimension. Then we can make a complete study by phase-plane analysis.

Summary picture for $N = 6$



The first critical relation (N-2)p+(N-1)q=N, see Theorem A

The curve of homogeneity p+q=1

The separatrix curve for Theorem B: (N-1)(p+q-1)= $\dfrac{(p+1)^2}{p}$ and (N-1)(p+q-1) =4

The separatrix curve for Theorem C: G(p,q)=0

The second critical relation in the radial case : p(N-2)+q(N-1)=N+(2-q)/(1-q), see Theorem D

Equation (2) has been the subject of many works in the radial case when $M < 0$, where the two terms $|u|^{p-1}u$ and $M|\nabla u|^q$ are in competition. The first work in that case is due to Chipot and Weissler [15] who, in particular, solved completely the case $N = 1$. Then Serrin and Zou [20] performed a very detailed analysis. Very little is known in the case $M > 0$. Under the scaling transformation T_k defined for $k > 0$ by

$$u_k := T_k[u](x) = k^{\frac{2}{p-1}} u(kx), \tag{38}$$

(2) becomes

$$-\Delta u_k = |u_k|^{p-1}u_k + k^{\frac{2p-q(p+1)}{p-1}} M|\nabla u_k|^q, \tag{39}$$

and thus, if $q \neq \frac{2p}{p+1}$, (2) can be reduced to

$$-\Delta u = |u|^{p-1}u \pm |\nabla u|^q. \tag{40}$$

Observe that when $q < \frac{2p}{p+1}$, the limit equation of (39) when $k \to 0$ is the Lane-Emden equation

$$-\Delta u = |u|^{p-1}u, \tag{41}$$

and thus the exponent p is dominant.

The other scaling transformation

$$v_k := S_k[u](x) = k^{\frac{2-q}{q-1}} u(kx), \tag{42}$$

transforms (2) into

$$-\Delta v_k = k^{\frac{(p+1)q-2p}{q-1}} |v_k|^{p-1} v_k + M |\nabla v_k|^q , \tag{43}$$

and if $q > \frac{2p}{p+1}$, the limit equation of (43) when $k \to 0$ is the Riccati equation

$$-\Delta v = M |\nabla v|^q , \tag{44}$$

thus the exponent q is dominant. In [15] and [20] most of the study deals with the case $q \neq \frac{2p}{p+1}$.
In the *critical case* i.e. when

$$q = \frac{2p}{p+1}, \tag{45}$$

then not only the sign of M but also its value plays a fundamental role, with a delicate interaction with the exponent p. Notice that an equivalent form of (2) is

$$-\Delta v = \lambda |v|^{p-1} v \pm |\nabla v|^q \tag{46}$$

with $\lambda > 0$. The case $N \geq 2$ was left open by Serrin and Zou [20] and the first partial results are due to Fila and Quittner [17] and Voirol [23, 24]. The case $M > 0$ was not considered.

The techniques we used to prove the following results are based upon a delicate extension of the ones already introduced above in the case of the product. Our first nonradial result dealing with the case $q > \frac{2p}{p+1}$ is the following:

Theorem A2

Let $N \geq 1$, $p > 1$ and $q > \frac{2p}{p+1}$. Then for any $M > 0$, any solution of (2) in a domain $\Omega \subset \mathbb{R}^N$ satisfies

$$|\nabla u(x)| \leq c_{N,p,q} \left(M^{-\frac{p+1}{(p+1)q-2p}} + (M\mathrm{dist}\,(x, \partial\Omega))^{-\frac{1}{q-1}} \right) \quad \text{for all } x \in \Omega. \tag{47}$$

As a consequence, any ground state has at most a linear growth at infinity:

$$|\nabla u(x)| \leq c_{N,p,q} M^{-\frac{p+1}{(p+1)q-2p}} \qquad \text{for all } x \in \mathbb{R}^N. \tag{48}$$

Our proof relies on a direct Bernstein method combined with Keller-Osserman's estimate applied to $|\nabla u|^2$. It is important to notice that the result holds for any $p > 1$, showing that, in some sense, the presence of the gradient term has a regularizing effect.

In the case $q < \frac{2p}{p+1}$ we prove a non-existence result:

Theorem A'2

Let $N \geq 1$, $p > 1$, $1 < q < \frac{2p}{p+1}$ and $M > 0$. Then there exists a constant $c_{N,p,q} > 0$ such that there is no positive solution of (2) *in* $\mathbb{R}^N$ *satisfying*

$$u(x) \leq c_{N,p,q} M^{\frac{2}{2p-(p+1)q}} \qquad \text{for all } x \in \mathbb{R}^N. \tag{49}$$

This is done by reducing by scaling to the case $M = 1$: We set $u = \alpha^{2/(p-1)} v(\alpha x)$ with $\alpha = M^{\frac{p-1}{2p-(p+1)q}}$ and using Lemma B1 applied to $z = |\nabla v|^2$. Hence a radial ground state has to start large at 0.

When q is critical with respect to p the situation is more delicate since the value of M plays a fundamental role. Our first statement is a particular case of a more general result given by Alarcón, García-Melián and Quaas.

Theorem B2

Let $N \geq 2$, $p > 1$ if $N = 2$ or $1 < p \leq \frac{N}{N-2}$ if $N = 3$, $q = \frac{2p}{p+1}$ and $M > -\mu^*$ where

$$\mu^* := \mu^*(N) = (p+1)\left(\frac{N-(N-2)p}{2p}\right)^{\frac{p}{p+1}}. \tag{50}$$

Then there exists no nontrivial nonnegative supersolution of (2) *in an exterior domain.*

Concerning ground states, we prove their nonexistence for any $p > 1$ provided $M > 0$ is large enough: indeed

Theorem C2

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a domain, $p > 1$, $q = \frac{2p}{p+1}$. For any

$$M > M_\dagger := \left(\frac{p-1}{p+1}\right)^{\frac{p-1}{p+1}} \left(\frac{N(p+1)^2}{4p}\right)^{\frac{p}{p+1}}, \tag{51}$$

and any $\nu > 0$ such that $(1-\nu)M > M_\dagger$, there exists a positive constant $c_{N,p,\nu}$ such that any solution u in Ω satisfies

$$|\nabla u(x)| \leq c_{N,p,\nu} \left((1-\nu)M - M_\dagger\right)^{-\frac{p+1}{p-1}} \left(\mathrm{dist}\,(x, \partial\Omega)\right)^{-\frac{p+1}{p-1}} \quad \text{for all } x \in \Omega. \tag{52}$$

Consequently there exists no nontrivial solution of (2) in $\mathbb{R}^N$.

The next result, based upon an elaborate Bernstein method, complements Theorem C under a less restrictive assumption on M but a more restrictive assumption on p.

Theorem D2

Let $1 < p < \frac{N+3}{N-1}$, $N \geq 2$, $1 < q < \frac{N+2}{N}$ *and* $\Omega \subset \mathbb{R}^N$ *be a domain. Then there exist* $a > 0$ *and* $c_{N,p,q} > 0$ *such that for any* $M > 0$, *any positive solution* u *in* Ω *satisfies*

$$|\nabla u^a(x)| \leq c_{N,p,q} \left(\mathrm{dist}\,(x, \partial\Omega)\right)^{-\frac{2a}{p-1}-1} \quad \textit{for all } x \in \Omega. \tag{53}$$

Hence there exists no nontrivial nonnegative solution of (2) *in* $\mathbb{R}^N$.

It is remarkable that the constants a and $c_{N,p,q}$ do not depend on $M > 0$, a fact which is clear when $q \neq \frac{2p}{p+1}$ by using the transformation T_k, but much more delicate to highlight when $q = \frac{2p}{p+1}$ since (2) is invariant.

When $|M|$ is small, we use an integral method to obtain the following result which contains, as a particular case, the previous estimates by Gidas and Spruck and ours. The key point of this method is to prove that the solutions in a punctured domain satisfy a local Harnack inequality.

Theorem E2

Let $N \geq 3$, $1 < p < \frac{N+2}{N-2}$, $q = \frac{2p}{p+1}$. Then there exists $\epsilon_0 > 0$ depending on N and p such that for any M satisfying $|M| \leq \epsilon_0$, any positive solution u in $B_R \setminus \{0\}$ satisfies

$$u(x) \leq c_{N,p} |x|^{-\frac{2}{p-1}} \quad \text{for all } x \in B_{\frac{R}{2}} \setminus \{0\}. \tag{54}$$

As a consequence there exists no positive solution of (2) in $\mathbb{R}^N$, and any positive solution u in a domain Ω satisfies

$$u(x) + |\nabla u(x)|^{\frac{2}{p+1}} \leq c'_{N,p} \left(\operatorname{dist}(x, \partial\Omega)\right)^{-\frac{2}{p-1}} \quad \text{for all } x \in \Omega. \tag{55}$$

Note that under the assumptions of Theorem E, there exist ground states for $|M|$ large enough when $1 < p < \frac{N}{N-2}$, or any $p > 1$ if $N = 1, 2$.

The radial solutions of (2) are functions $r \mapsto u(r)$ defined in $(0, \infty)$ where they satisfy

$$- u_{rr} - \frac{N-1}{r} u_r = |u|^{p-1} u + M \, |u_r|^{\frac{2p}{p+1}} . \tag{56}$$

Because of the invariance of (56) under the transformation T_k there exists an autonomous variant of (2) obtained by setting

$$u(r) = r^{-\frac{2}{p-1}} x(t) \quad \text{with} \quad t = \ln r. \tag{57}$$

Then

$$x_{tt} + L x_t - \frac{2K}{p-1} x + |x|^{p-1} x + M \left| \frac{2K}{p-1} x - x_t \right|^{\frac{2p}{p+1}} = 0 \tag{58}$$

with

$$K = \frac{(N-2)p - N}{p-1} \quad \text{and} \quad L = \frac{(N-2)p - (N+2)}{p-1} = K - \frac{2}{p-1}. \tag{59}$$

Setting $y(t) = -r^{\frac{p+1}{p-1}} u_r(r)$, then $(x(t), y(t))$ satisfies the system

$$\begin{aligned} x_t &= H_1(x,y) = \frac{2}{p-1} x - y \\ y_t &= H_2(x,y) = -Ky + |x|^{p-1} x + M \, |y|^{\frac{2p}{p+1}} \end{aligned} \tag{60}$$

and we denote by $\mathbf{H}$ the vector field of $\mathbb{R}^2$ with components H_1 and H_2.

We are mainly interested in the trajectories of the system which remain in the first quadrant $\mathbf{Q}=\{(x,y)\in\mathbb{R}^2 : x>0, y>0\}$. Among these trajectories, the ones corresponding to *ground states*, i.e. positive C^2 solutions u of (56) are defined on $[0,\infty)$. They verify $u_r(0)=0$ and actually they are C^∞ on $(0,\infty)$. Using the invariance of the equation under T_k all the ground states can be derived by scaling from a unique one which satisfies $u(0)=1$. Since it is easy to prove that such a solution u is decreasing, in the variables (x,y), a ground state is a trajectory of (60) in $\mathbf{Q}$, defined on $\mathbb{R}$ and satisfying $\lim_{t\to-\infty}\frac{y(t)}{x(t)}=0$. The corresponding trajectory is denoted by $\mathbf{T}_{reg}$.

Contrarily to the case of the Lane-Emden equation (41), there exists no natural Lyapunov function when $M\neq 0$. This makes the study much more delicate and ours is based upon a phase plane analysis. The solutions of (56) invariant under T_k for any $k>0$ correspond to constant solutions of (58) and have the form

$$U(r)=Xr^{-\frac{2}{p-1}} \quad \text{for all } r>0, \tag{61}$$

where X is a positive root of

$$X^{p-1}+M\left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}}X^{\frac{p-1}{p+1}}-\frac{2K}{p-1}=0. \tag{62}$$

This equation, which correspons to finding the fixed points of the system, plays a fundamental role in the description of the set of solutions of (56). The following constant, defined for $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p \leq \frac{N}{N-2}$ has an important role in the description of the set roots of (62),

$$\mu^* := \mu^*(N) = (p+1)\left(\frac{N-(N-2)p}{2p}\right)^{\frac{p}{p+1}}. \tag{63}$$

This set is described in the following proposition.

Proposition 1 *1- If $M \geq 0$, equation (62) admits a unique positive root if and only if $N \geq 3$ and $p > \frac{N}{N-2}$.*

2- If $M < 0$ and $p \geq \frac{N}{N-2}$, equation (62) admits a unique positive root X_M.

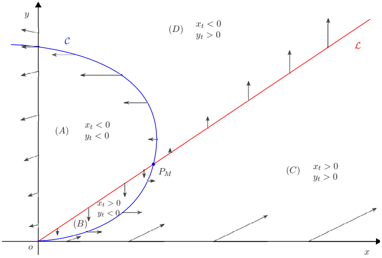
3- If $M < 0$ and either $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p \leq \frac{N}{N-2}$, there exists no positive root of (62) if $-\mu^* < M < 0$, a unique positive root if $M = -\mu^* < 0$ and two positive roots $X_{1,M} < X_{2,M}$ if $M < -\mu^*$.

We also set $Y_M = \frac{2}{p-1}X_M$ and $P_M = (X_M, Y_M)$ (resp. $Y_{j,M} = \frac{2}{p-1}X_{j,M}$ and $P_{j,M} = (X_{j,M}, Y_{j,M})$, for j=1,2) and define the corresponding singular solutions $U_M(r) = X_M r^{-\frac{2}{p-1}}$ (resp. $U_{j,M}(r) = X_{j,M} r^{-\frac{2}{p-1}}$).

The vector field is inward in (resp. outward of) $\mathbf{Q}$ on the axis $\{(x,y) : x>0, y=0\}$ (resp. $\{(x,y) : x=0, y>0\}$). We set

$$\mathcal{L} := \left\{(x,y)\in\mathbf{Q} : y = \frac{2x}{p-1}\right\} \quad \text{and} \quad \mathcal{C} := \left\{(x,y)\in\mathbf{Q} : x = \left(Ky - My^{\frac{2p}{p+1}}\right)^{\frac{1}{p}}\right\} \tag{64}$$

and $\psi(y) = \left(Ky - My^{\frac{2p}{p+1}}\right)^{\frac{1}{p}}$. Then $x_t = 0$ on $\mathcal{L}$ and $y_t = 0$ on $\mathcal{C}$. The curves $\mathcal{L}$ and $\mathcal{C}$ have zero, one or two intersections in $\mathbf{Q}$ according the value of $K = \dfrac{(N-2)p-N}{p-1}$ and M.



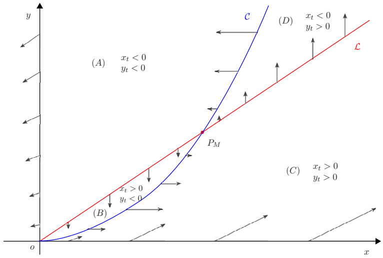


Figure: $M > 0$, $p > \dfrac{N}{N-2}$,$(0,0)$ is a saddle point $\quad$ $M < 0$, $p \geq \dfrac{N}{N-2}$, $(0,0)$ is a source.

In these two cases, whenever u is a regular solution, it either changes sign, and if not, either $u \sim r^{2-N}$ or $u \sim U_M(r)$ at ∞, or u has an ω-limit cycle surrounding P_M.

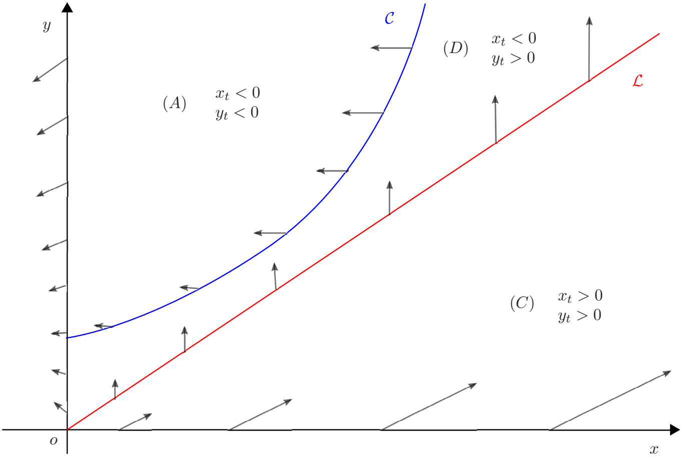


Figure: $-\mu^* < M < 0$, $1 < p < \dfrac{N}{N-2}$.

There are no ground states and no positive singular solutions

Graphic representation of the vector field **H**

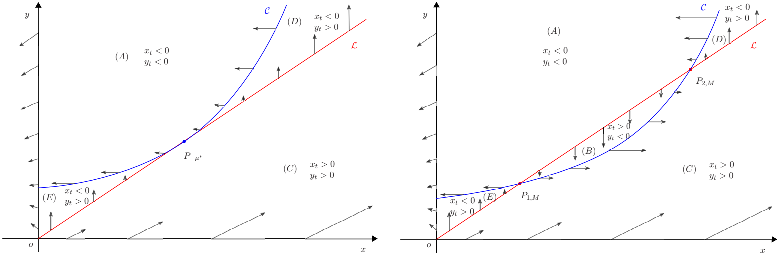


Figure: $M = -\mu^*$, $1 < p < \dfrac{N}{N-2}$, $\qquad M < -\mu^*$, $1 < p < \dfrac{N}{N-2}$.

In the last case $P_{1,M}$ is a saddle point, if $\overline{M} < M < -\mu^*$ then $P_{2,M}$ is a source and if $M < \overline{M}$, then $P_{2,M}$ is a sink. $P_{2,\overline{M}}$ is a weak source.

Recall briefly the description of the positive solutions of the Lane-Emden equation (41), i.e. $M = 0$: there exists radial ground states if and only if $N \geq 3$ and $p \geq \frac{N+2}{N-2}$. If $p = \frac{N+2}{N-2}$ these ground states are explicit and they satisfy $\lim_{r \to \infty} r^{N-2} u(r) = c > 0$. There exist infinitely many singular solutions u ondulating around U_M. Note that a ground state corresponds to a homoclinic orbit at 0 for system (60) and these singular solutions are cycles surrounding P_M. We recall that an orbit of (60) which connects two different equilibria (resp. the same equilibrium) when $t \in \mathbb{R}$ is called *heteroclinic* (resp. *homoclinic*).

We describe next the ground states and the singular global solutions of (56) in $\mathbb{R}^N \setminus \{0\}$. We present first the results for $M > 0$. The following value of M appears when we linearize the system (60) at the equilibrium P_M,

$$\overline{M} = \overline{M}(N,p) = \frac{(p+1)\left((N-2)p - (N+2)\right)}{(4p)^{\frac{p}{p+1}}\left((N-2)(p-1)^2 + 4\right)^{\frac{1}{p+1}}}. \tag{65}$$

Then $\overline{M}$ is positive (resp. negative) if $p > \frac{N+2}{N-2}$ (resp. $p < \frac{N+2}{N-2}$). It is easy to see that if $M = \overline{M}$ then the characteristic values of the linearized operator at P_M are purely imaginary.

Theorem A2r

Let $N \geq 1$, $p > 1$ and $M > 0$.

1- If $1 < p \leq \frac{N+2}{N-2}$ if $N \geq 3$, and any $p > 1$ if $N = 1, 2$, then equation (56) admits no ground state.

2- If $N \geq 3$ and $p > \frac{N+2}{N-2}$, there exist constants $\tilde{M}_{min}, \tilde{M}_{max}$ verifying

$$0 < \overline{M} < \tilde{M}_{min} \leq \tilde{M}_{max} \quad \text{such that}$$

- if $0 < M < \tilde{M}_{min}$ there exist ground states u satisfying $u(r) \sim U_M(r)$ when $r \to \infty$.
- if $M = \tilde{M}_{min}$ or $M = \tilde{M}_{max}$ there exists a ground state u minimal at infinity, that is satisfying $\lim_{r\to\infty} r^{N-2}u(r) = c > 0$.
- for $M > \tilde{M}_{max}$ there exists no radial ground state.

The values of $\tilde{M}_{min}$ and $\tilde{M}_{max}$ appear as transition values for which the ground state still exists but it is smaller than the others at infinity; it is of order r^{2-N} instead of $r^{-\frac{2}{p-1}}$. They are not explicit but they can be estimated as functions of N and p. It is a numerical evidence that $\tilde{M}_{min} = \tilde{M}_{max}$ in the phase plane analysis of system (60) and we conjecture that this is true. When $M = \tilde{M}_{min}$ or $\tilde{M}_{max}$, the system (60) admits homoclinic trajectories.

When M is negative, the precise description of the trajectories of (60) depends also on the value of p with respect to $\frac{N}{N-2}$. It is proved in [8, Th. B, E] that for $N \geq 3$ and $1 < p < \frac{N+2}{N-2}$ there exists $\epsilon_0 > 0$ such that if $|M| \leq \epsilon_0$ equation (2) admits no positive solution in $\mathbb{R}^N$. The same conclusion holds if $N \geq 3$, $1 < p \leq \frac{N}{N-2}$ (or $N = 2$ and $p > 1$) and $M > -\mu^*$. We first consider the case $p \geq \frac{N}{N-2}$ for which there exists a unique explicit singular solution U_M, and the results present some similarity with the ones of Theorem A.

Theorem B2r

Let $N \geq 3$, $p \geq \frac{N}{N-2}$ and $M < 0$. Then

1- If $p \geq \frac{N+2}{N-2}$, then equation (56) admits ground states u. Moreover they satisfy $u(r) \sim U_M(r)$ as $r \to \infty$.
2- If $\frac{N}{N-2} \leq p < \frac{N+2}{N-2}$, there exist numbers $\tilde{\mu}_{min}$ and $\tilde{\mu}_{max}$ verifying

$$0 < |\overline{M}| < \tilde{\mu}_{min} \leq \tilde{\mu}_{max} < \mu^*(1) \quad \text{such that}$$

(i) for $M < -\tilde{\mu}_{max}$ there exist ground states u such that $u(r) \sim U_M(r)$ when $r \to \infty$.
(ii) for $M = -\tilde{\mu}_{min}$ or for $M = -\tilde{\mu}_{max}$ there exist ground states minimal at infinity in the sense that $u(r) \sim cr^{2-N}$ when $r \to \infty$, $c > 0$.
(iii) for $-\tilde{\mu}_{min} < M < 0$ there exists no radial ground state.

Here also the value of $\tilde{\mu}_{min}$, $\tilde{\mu}_{max}$ are not explicit and we conjecture that they coincide.

The situation is more complicated when $1 < p < \frac{N}{N-2}$ and $M < -\mu^*$ because *there exist two explicit singular solutions* $U_{1,M}$ *and* $U_{2,M}$ which coincide when $M = -\mu^*$.

Theorem C2r

Let $M < 0$, $N \geq 3$ and $1 < p < \frac{N}{N-2}$, or $N = 2$ and $p > 1$. Then there exist two constants $\tilde{\mu}_{min}$ and $\tilde{\mu}_{max}$ verifying

$$\mu^* \leq |\overline{M}| < \tilde{\mu}_{min} \leq \tilde{\mu}_{max} < \mu^*(1),$$

such that

1- If $M < -\tilde{\mu}_{max}$ then equation (56) admits ground states u either ondulating around $U_{2,M}$ or such that $u(r) \sim U_{2,M}(r)$ as $r \to \infty$.

2- If $M = -\tilde{\mu}_{min}$ or $M = -\tilde{\mu}_{max}$ there exists a ground state u such that $u(r) \sim U_{1,M}(r)$ as $r \to \infty$.

3- If $-\tilde{\mu}_{min} < M < 0$ there exists no radial ground state.

Here again $\tilde{\mu}_{min}$ and $\tilde{\mu}_{max}$ appear as transition values for which the ground state still exists but it is smaller than the others at infinity: it behaves like $U_{1,M}$ instead of $U_{2,M}$.

The proof of this theorem is very elaborate in particular in the case $N = 2$. In the case $N = 1$ the result is already proved by Chipot-Weissler in [15]. The nonexistence of ground state, not necessarily radial for $M > -\mu^*$ is proved by Alarcón et al in [1] and independently in [8] with a different method. In the radial case it was obtained much before in the case $N = 1$ in [15] and then by Fila and Quittner [17] who raised the question whether the condition $-\tilde{\mu}_{min} < M < 0$ is optimal for the non-existence of radial ground state. This question received a negative answer in the work of Voirol [23] who extended the domain of non-existence to $-\mu^* - \epsilon < M \leq -\mu^*$.

Map for the ground state solutions.

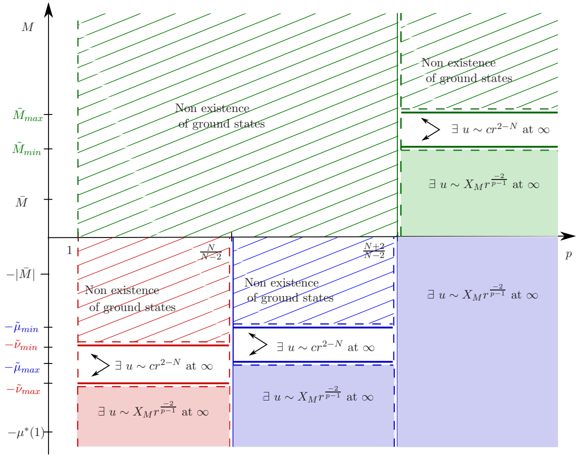


Figure: Theorems A2r, B2r and C2r.

We prove that the system (60) admits a Hopf bifurcation when $M = \overline{M}$. When $p > \frac{N+2}{N-2}$ we also prove the existence of different types of positive singular solutions.

Theorem A'2r

Let $N \geq 3$.

1- If $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$ for any $M > 0$ there exists a unique (always up to scaling transformation) positive singular solution u of (56) satisfying $\lim_{r\to 0} r^{\frac{2}{p-1}} u(r) = X_M$ and $\lim_{r\to\infty} r^{N-2}u(r) = c > 0$.

2- If $p > \frac{N+2}{N-2}$, then
(i) If $M > \tilde{M}_{max}$, there exists a unique singular solution u of (56) with the same behaviour as in 1.
(ii) If $\overline{M} < M < \tilde{M}_{min}$ there exist positive singular solutions u ondulating around U_M on $\mathbb{R}$.

In terms of the system (60) the 1- and 2-(i) correspond to the existence of a heteroclinic orbit in $\mathbf{Q}$ connecting P_M to $(0,0)$ and (ii) to the existence of a cycle in $\mathbf{Q}$ surrounding P_M.

The next result is the counterpart of Theorem B for singular soulutions.

Theorem B'2r

Let $N \geq 3$ and $\frac{N}{N-2} < p < \frac{N+2}{N-2}$.

(i) If $\overline{M} < M < 0$ there exists a unique (up to scaling) positive singular solution u of (56), such that $u(r) \sim U_M(r)$ when $r \to 0$ and $u(r) \sim cr^{2-N}$ when $r \to \infty$ for some $c > 0$.

(ii) If $-\tilde{\mu}_{min} < M < 0$ there exist positive singular solutions u ondulating around U_M on $[0, \infty)$ and singular solutions ondulating around U_M in a neighbourhood of 0 and satisfying $u(r) \sim cr^{2-N}$ for some $c > 0$ when $r \to \infty$.

In terms of the system (60), (i) corresponds to a heteroclinic orbit connecting P_M and $(0, 0)$, while (ii) to the existence of a periodic solution in $\mathbf{Q}$ around P_M, and the existence of a solution in $\mathbf{Q}$ converging to $(0, 0)$ at ∞ and having a limit cycle at $t = -\infty$ which is a periodic orbit around P_M.

Positive singular solutions for $M < 0, 1 < p < \frac{N}{N-2}$, $N \geq 3$.

The next result is the counterpart of Theorem C when dealing with singular solutions.

Theorem C'2r

Let $M < 0$, $N \geq 3$ and $1 < p < \frac{N}{N-2}$. Then there exist positive real numbers $|\overline{M}| < \hat{\mu}_{min} \leq \hat{\mu}_{max} < \tilde{\mu}_{min} \leq \tilde{\mu}_{max} < \mu^*(1)$ with the following properties

(i) If $M < -\mu^*$ there exist positive singular solutions u such that $u(r) \sim cr^{2-N}$ with $c > 0$ when $r \to 0$ and $u(r) \sim U_{1,M}(r)$ as $r \to \infty$.

(ii) If $\overline{M} \leq M < -\mu^*$ there exists a unique up to scaling positive singular solution u, such that $u(r) \sim U_{2,M}(r)$ as $r \to 0$ and $u(r) \sim U_{1,M}(r)$ as $r \to \infty$. Furthermore $u(r) > U_{1,M}(r)$ for all $r > 0$.

(iii) If $-\hat{\mu}_{min} < M < -|\overline{M}|$ there exist positive singular solutions u ondulating around $U_{2,M}$ at 0 and such that $u(r) \sim U_{1,M}(r)$ as $r \to \infty$, and positive singular solutions u ondulating around $U_{2,M}$ on $\mathbb{R}$.

(iv) If $M = -\tilde{\mu}_{min}$ or $M = -\hat{\mu}_{max}$ there exists a positive singular solution u different from $U_{1,M}$ such that $u(r) \sim U_{1,M}(r)$ when $r \to 0$ and $r \to \infty$.

(v) If $-\tilde{\mu}_{min} < M < -\hat{\mu}_{max}$ there exists a positive singular solution u such that $\lim_{r \to 0} r^{N-2} u(r) = c > 0$ and either ondulating around $U_{2,M}$ or such that $u(r) \sim U_{2,M}(r)$ when $r \to \infty$.
(vi) If $N \geq 3$ and $M = -\mu^*$, there exist positive singular solution u satisfying $\lim_{r \to 0} r^{N-2} u(r) = c > 0$ and $u(r) \sim U_{-\mu^*}(r)$ as $r \to \infty$.

Map for the singular positive solutions.

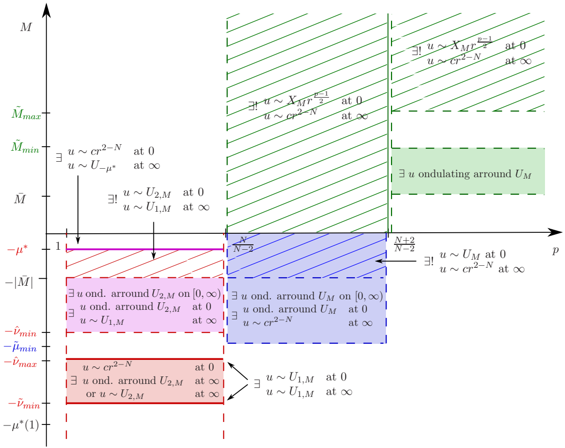


Figure: Theorems A'2r, B'2r and C'2r.

HAPPY BIRTHDAY!!

THANK YOU VERY MUCH FOR YOUR ATTENTION

- Bidaut-Véron M. F. *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type*. Arch. Rat. Mech. Anal. **107** (1989), 293-324.
- Bidaut-Véron M. F., Garcia-Huidobro M., Véron L. *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type*. Calc. Var. Part. Diff. Equ. **54** (2015), 3471-3515.
- Bidaut-Véron M. F., Pohozaev S. *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type*. Journal d'Analyse Mathématique **84** (2001), 1-49.
- Bidaut-Véron M.F., Véron L. *Nonlinear elliptic equations on Compact Riemannian manifolds and asymptotics of Emden equations*. Invent. Math. **106** (1991), 489-539.
- Brezis H., Lions P. L. *A note on isolated singularities for linear elliptc equations*. Math. Anal. Appl. Adv., Suppl. Stud. **7A** (1981), 263-266.
- Burgos-Pérez M., Garcia-Mellían J., Quass A. *Classification of supersolutions and Liouville theorems for some nonlinear elliptic problems*. Disc. Cont. Dyn. Sys. **36** (2016), 4703-4721.
- Caristi G., Mitidieri E. *Nonexistence of positive solutions of quasilinear equations*. Adv. Diff. Equ. **2** (1997), 317-359.

Fillipucci R., *Nonexistence of positive weak solutions of elliptic inequalities.* Nonlinear Anal. **70** (2009), 2903-2916.

Gidas B., Spruck J., *Global and local behaviour of positive solutions of nonlinear elliptic equations*. Comm. Pure Appl. Math. **34** (1981), 525-598.

Lieberman G., *T*he natural generalizationj of the natural conditions of Ladyzhenskaya and Uralt́seva for elliptic equations, Comm. Part. Diff. Equ. **16** (1991), 311-361.

Licois J. R., Véron L., *A class of nonlinear conservative elliptic equations in cylinders*. Ann. Sc. Norm. Sup. Pisa, Cl. Sci. **26** (1998), 249-283.

Lions P. L., *Résolution de problèmes elliptiques quasilinéaires.* Arch. Rational Mech. Anal. **74** (1980), 335ÃŘ353.

Mitidieri E., Pohozaev S. *A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities*, Trudy Matematicheskogo Instituta Imeni VA Steklova **234** (2001), pp. 1-383.

Serrin J., *Local behavior of solutions of quasi-linear equations*. Acta Math. **111** (1964), 247-302.

Serrin J., *Singularities of solutions of quasi-linear equations*. Acta Math. **113** (1965), 219-240.

- Smoller J., *Schock Waves and Reaction Diffusion Equations*. Grundlehren der Matematischen Wissenschaften **258** 2nd ed. 68 (1994), xxi+pp.1-632.
- Véron L., *Local and global aspects of quasilinear degenerate elliptic equations. Quasilinear elliptic singular problems*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2017). xv+ pp. 1-457.

- Alarcón S., García-Melián J., Quass A., *Nonexistence of positive supersolutions to some nonlinear elliptic problems*. J. Math. Pures Appl. **99** (2013), 618-634.
- Alarcón S., García-Melián J., Quass A., *Existence and non-existence of solutions to elliptic equations with a general convection term*. Proc. Royal Soc. Edinburgh A **144** (2014), 225-239.
- Anderson L.R., Leighton W., *Liapunov functions for autonomous systems of second order*, J. Math. Anal. Appl. **23** (1968), 645-664.
- Bidaut-Véron M.F., *Local and global behaviour of solutions of quasilinear elliptic equations of Emden-Fowler type*, Arch. Rat. Mech. Anal. **107** (1989), 3293-324.
- Bidaut-Véron M.F., *Self-similar solutions of the p-Laplace heat equation: the fast diffusion case*, Pacific Journal of Maths **227** (2006), 201-269.
- Bidaut-Véron M.F., Garcia-Huidobro M., Véron L., *Local and global properties of solutions of quasilinear Hamilton-Jacobi equations*. J. Funct. Anal. **267** (2014), 3294-3331.
- Bidaut-Véron M.F., Garcia-Huidobro M., Véron L, *Estimates of solutions of elliptic equations with a source reaction term involving the product of the function and its gradient*, Duke Math. J., to appear.
- Bidaut-Véron M.F., Garcia-Huidobro M., Véron L, *A priori estimates for elliptic equations with reaction terms involving the function and its gradient*, soumis.

- Bidaut-Véron M.F., Grillot P., *Asymptotic behaviour of elliptic systems with mixed absorption and source terms*, Asymptotic Anal. **19** (1999), 117-147.
- Bidaut-Véron M.F., Giacomini H., *A new dynamical approach of Emden-Fowler equations and systems*. Adv. Differential Equations **15** (2010), 1033-1082.
- Bidaut-Véron M. F., Pohozaev S. *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type*. Journal d'Analyse Mathématique **84** (2001), 1-49.
- Bidaut-Véron M. F., Raoux Th. *Asymptotic of solutions of some nonlinear elliptic systems*. Comm. Part. Diff. Equ. **21** 1035-1086 (1996).
- Chicone C., Tian J. H. *On general properties of quadratic systems*, Amer. Math. Monthly **89** (1982), 167-178.
- Chipot M., *On a class of nonlinear elliptic equations*, P.D.E. Banach Center Pub. Polish Acad. Sci. Varszawa **27** (1992), 76-79.
- Chipot M., Weissler, F., *Some blow-up results for a nonlinear parabolic equation with a gradient term*, S.I.A.M. J. of Num. Anal. **20** (1989), 886-907
- Fila M., *Remarks on blow-up for a nonlinear parabolic equation with a gradient term*, Proc. A.M.S. **111** (1991), 785-801.

- Fila M., Quittner, *Radial positive solutions for a semilinear elliptic equation with a gradient term*, Adv. Math. Sci. Appl **2** (1993), 39-45
- Hubbard J. H., West B. H. *Differential Equations: A Dynamical Systems Approach: Ordinary Differential Equations*, Texts in Applied Mathematics **5**, Springer-Verlag Berlin Heidelberg (1991).
- Kuznetsov Y., *Elements of Applied Bifurcation Theory*, Applied Math Sci. **110**, Springer-Verlag (2004).
- Serrin J., Zou H., *Existence and non-existence results for ground states of quasilinear elliptic equations*. Arch. Rat. Mech. Anal. **121** (1992), 101-130.
- Souplet Ph., *Recent results and open problems on parabolic equations with gradient nonlinearities*, E.J.D.E., **20** (2001), 1-19
- Vazquez J. L., Véron L., *Singularities of elliptic equations with an exponential nonlinearity*, Math. Ann. **269** (1984), 119-135.
- Voirol F. X., Thèse de Doctorat, Université de Metz, (1994).
- Voirol F. X., *Coexistence of singular and regular solutions for the equation of Chipot and Weissler,* Acta Math. Univ-Comenianae **65** (1996), 53-64.