Nonuniformly elliptic problems

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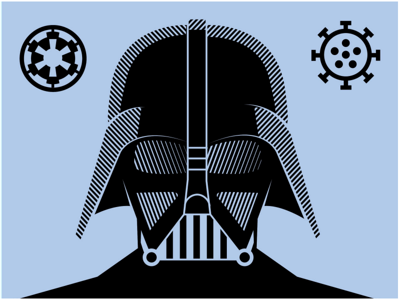

Singular Problems Associated to Quasilinear Equations
ShanghaiTech, June 1, 2020

The uniformly elliptic case

- **We consider nonlinear equations with linear growth**

$$-\text{div } a(Du) = \mu$$

under assumptions

$$\begin{cases} |a(z)| + |\partial a(z)||z| \leq L|z|^{p-1} \\ \nu|z|^{p-2}|\xi|^2 \leq \langle \partial a(z)\xi, \xi \rangle \end{cases}$$

- A typical instance is

$$-\text{div } (|Du|^{p-2}Du) = \mu$$

- Emphasis on Lipschitz estimates.
- We want to consider more general growth and ellipticity assumptions.

Intrinsic rewriting

Theorem

If u solves

$$-\mathrm{div}\, a(Du) = \mu \ ,$$

then

$$|a(Du(x))| \lesssim \int_{B_R(x)} \frac{d|\mu|(y)}{|x-y|^{n-1}} + \fint_{B_R(x)} |a(Du)|\, dy$$

holds

- Duzaar & Min. (JFA 2010) for $2 - 1/n < p < 2$
- Kuusi & Min. (CRAS 2011 + ARMA 2013) for the case $p \geq 2$
- Nguyen & Phuc (Math. Ann. 19) for $\frac{3n-2}{2n-1} < p \leq 2$

A global estimate

Theorem

If u solves

$$-\operatorname{div} a(Du) = \mu \ ,$$

and decays properly at infinity, then

$$|Du(x)|^{p-1} \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}}$$

A classical theorem of Stein

Theorem (Stein, Ann. Math. 1981)

$$Dv \in L(n,1) \Longrightarrow v \text{ is continuous}$$

- Recall that

$$g \in L(n,1) \Longleftrightarrow \int_0^\infty |\{x : |g(x)| > \lambda\}|^{1/n}\, d\lambda < \infty$$

- An example of $L(n,1)$ function is given by

$$\frac{1}{|x| \log^\beta(1/|x|)} \qquad \beta > 1 \quad \text{in the ball } B_{1/2}$$

- $$\triangle u = \mu \in L(n,1) \Longrightarrow Du \text{ is continuous}$$

A nonlinear Stein theorem

- Now notice that

$$\mu \in L(n,1) \Longrightarrow \lim_{R\to 0} \int_{B_R(x)} \frac{d|\mu|(y)}{|x-y|^{n-1}} = 0 \text{ uniformly w.r.t. } x$$

- From the results of Kuusi & Min. it also follows that if

$$\lim_{R\to 0} \int_{B_R(x)} \frac{d|\mu|(y)}{|x-y|^{n-1}} = 0 \text{ uniformly} \Longrightarrow Du \text{ is continuous .}$$

Linear and nonlinear Stein theorems

Theorem (Stein, Ann. Math. 1981)

If u solves the Poisson equation

$$-\operatorname{div} Du = \triangle u = \mu \in L(n,1)$$

then

Du is continuous.

Linear and nonlinear Stein theorems

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Theorem (Kuusi & Min. ARMA 2013)

If u solves the p-Laplacean equation

$$-\text{div}\, a(Du) = \mu \in L(n,1)$$

then

Du is continuous.

Theorem (Kuusi & Min. ARMA 2013)

If u solves the p-Laplacean type equation

$$-\operatorname{div} a(x, Du) = \mu \in L(n,1)$$

with

$$\int_0 \frac{\omega(\varrho)}{\varrho}\, d\varrho < \infty$$

then

Du is continuous.

Here it is

$$\frac{|a(x,z)-a(y,z)|}{|z|^{p-1}} \lesssim \omega(|x-y|)$$

Emphasis on external ingredients μ and $c(\cdot)$

Theorem (Kuusi & Min. ARMA 2013 - Calc. Var. 2014)

If u solves the p-Laplacean equation

$$-\operatorname{div}\left(c(x)a(Du)\right) = \mu$$

with

$$0 < c(\cdot) \text{ is Dini continuous}$$

and

$$\mu \in L(n,1)$$

then

$$Du \text{ is continuous.}$$

A classic from Ladyzhenskaya & Ural'tseva (1970)

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL XXIII, 677–703 (1970)

Local Estimates for Gradients of Solutions of Non-Uniformly Elliptic and Parabolic Equations

O. A. LADYZHENSKAYA AND N. N. URAL'TSEVA
Leningrad University

Various classes of non-uniformly elliptic (and parabolic) equations of second order of the form

$$\sum_{i,j=1}^{n} a_{ij}(x, u, u_x) u_{ij} = a(x, u, u_x), \qquad (1.1)$$

$$a_{ij}(x, u, u_x)\xi_i\xi_j > 0 \quad \text{for} \quad |\xi| = 1,$$

for all solutions $u(x)$ of which $\max_\Omega |u_x|$ can be estimated by $\max_\Omega |u|$ and $\max_{\partial\Omega} |u_x|$, were discussed in [1] (see also [2]).[1] The method used was introduced in [3]. In the same paper a method was suggested for obtaining local estimates of $|u_x|$, i.e., estimates of $\max_{\Omega'} |u_x|$ in terms of $\max_\Omega |u|$ and the distance $d(\Omega', \partial\Omega)$ of $\Omega' \subset \Omega$ from the boundary $\partial\Omega$. In a series of papers (concerning these see [4] and [5]) we have shown that this method is applicable to the whole class of uniformly elliptic and parabolic equations. In the present paper we investigate the possibility of applying it to non-uniformly elliptic and parabolic equations. It turns out that it is applicable, roughly speaking, to those classes of [1] for which the order of nonuniformity of the quadratic form $a_{ij}(x, u, u_x)\xi_i\xi_j$ is less than two. The first part of this paper is devoted to the proof of this assertion.

In the second part we analyze a different method of obtaining local estimates for $|u_x|$ which is applicable to elliptic equations of the form

$$-\sum_{i=1}^{n} \frac{d}{dx_i} a_i(x, u, u_x) + a(x, u, u_x) = 0, \qquad (1.2)$$

and embraces such interesting cases as equations for the mean curvature of a

[1] We shall use the notation

$$u_{x_i} = u_i, \qquad u_x = (u_1, \cdots, u_n), \qquad u_{x_i x_j} = u_{ij},$$

$$u_x^2 = \sum_{i=1}^{n} u_i^2, \qquad |u_x| = \sqrt{u_x^2}, \qquad u_{xx}^2 = \sum_{i,j=1}^{n} u_{ij}^2, \qquad |u_{xx}| = \sqrt{u_{xx}^2}.$$

A classic from Trudinger (1971)

On the Regularity of Generalized Solutions
of Linear, Non-Uniformly Elliptic Equations

NEIL S. TRUDINGER

Communicated by J. C. C. NITSCHE

1. Introduction

We consider in this paper the simplest form of a second order, linear, divergence structure equation in n variables, namely

$$\frac{\partial}{\partial x_i}\left(a^{ij}(x)\frac{\partial u}{\partial x_j}\right)=0 \tag{1.1}$$

where the coefficients a^{ij}, $1\leqq i, j\leqq n$, are measurable functions on a domain Ω in Euclidean n space E^n. Following the usual summation convention, repeated indices will indicate summation from 1 to n. We assume always that $n\geqq 2$.

Equation (1.1) is *elliptic* in Ω if the coefficient matrix $\mathscr{A}(x)=[a^{ij}(x)]$ is positive almost everywhere in Ω. Let $\lambda(x)$ denote the minimum eigenvalue of $\mathscr{A}(x)$ and set

$$\mu(x)=\sup_{1\leqq i,\, j\leqq n}|a^{ij}(x)| \tag{1.2}$$

so that

$$\lambda(x)|\xi|^2\leqq a^{ij}(x)\,\xi_i\,\xi_j\leqq n^2\mu(x)|\xi|^2 \tag{1.3}$$

for all $\xi\in E^n$, $x\in\Omega$. We will say that equation (1.1) is *uniformly elliptic* in Ω if the function $\gamma(x)=\mu(x)/\lambda(x)$ is essentially bounded in Ω. If γ is not necessarily bounded, then equation (1.1) is referred to as *non-uniformly elliptic*. We note here that uniformly elliptic equations for which λ^{-1} is unbounded have sometimes been referred to as *degenerate elliptic* [9].

Uniformly elliptic equations of the form (1.1), with bounded λ^{-1} and μ, have been extensively studied in the literature, two of the major results being a *Hölder estimate* for generalized solutions, due to DEGIORGI [1] and NASH [11], and a *Harnack inequality*, due to MOSER [7]. The purpose of this paper is to extend these results to a class of non-uniformly elliptic equations. In order to accomplish this, our methods differ substantially from those previously proposed and hence may be considered as new proofs of the original results. Various features of our proofs do coincide, however, with techniques in MOSER's two papers [6], [7]. An essential difference is that in order to obtain the stronger results we need to extract more information from the equation.

A classic from Leon Simon (1976)

Interior Gradient Bounds for Non-uniformly Elliptic Equations

LEON SIMON

In [1] Bombieri, De Giorgi and Miranda were able to derive a local interior gradient bound for solutions of the minimal surface equation with n independent variables, $n \geqq 2$, thus extending the result previously established by Finn [2] for the case $n = 2$. Their method was to use test function arguments together with a Sobolev inequality on the graph of the solution (Lemma 1 of [1]). A much simplified proof of their result was later given by Trudinger in [12].

Since the essential features of the test function arguments given in [1] generalized without much difficulty to many other non-uniformly elliptic equations, it was apparent that interior gradient bounds could be obtained for these other equations provided appropriate analogues for the Sobolev inequality of [1] could be established. Ladyzhenskaya and Ural'tseva obtained such inequalities ([4], Lemma 1) for a rather large class of equations, including the minimal surface equation as a special case. They were thus able to obtain gradient bounds for this class of equations.

In §2 of [5] a general Sobolev inequality was established on certain generalized submanifolds of Euclidean space. In the special case of nonparametric hypersurfaces in $\mathbf{R}^{n+1}$ of the form $x_{n+1} = u(x)$, where u is a C^2 function defined on an open subset $\Omega \subset \mathbf{R}^n$, the inequality of [5] implies

$$(1) \qquad \left\{\int_\Omega h^{n/(n-1)} v\,dx\right\}^{(n-1)/n} \leqq c \int_\Omega \left[\left\{\sum_{i,j=1}^{n} g^{ij} h_{x_i} h_{x_j}\right\}^{1/2} + h\,|H|\right] v\,dx$$

for each non-negative C^1 function h with compact support in Ω, where

$$v = (1 + |Du|^2)^{1/2}$$

$$g^{ij} = \delta_{ij} - \nu_i \nu_j\,, \qquad \nu_i = u_{x_i}/v, \qquad i, j = 1, \cdots, n$$

$$H = \frac{1}{n} v^{-1} \sum_{i,j=1}^{n} g^{ij} u_{x_i x_j}\,,$$

and where c is a constant depending only on n. (See the discussion in §2 below.) The quantity H appearing in this inequality is in fact the mean curvature of the hypersurface $x_{n+1} = u(x)$ and in the special case when $H \equiv 0$ (*i.e.* when u

Verifying uniform ellipticity

Minimizers of

$$v \mapsto \int_\Omega [F(Dv) - fv]\, dx \quad \text{for} \quad F(z) := \frac{|z|^p}{p}$$

are solutions to

$$-\operatorname{div} \partial F(Du) = f\ .$$

In this case we have

$$(p-1)|z|^{p-2} Id \leq \partial^2 F(z) \leq c|z|^{p-2} Id$$

therefore

$$\frac{\text{highest eigenvalue of } \partial^2 F(z)}{\text{lowest eigenvalue of } \partial^2 F(z)} \approx \frac{p}{\min\{p-1, 1\}}$$

Non-uniformly elliptic problems

I consider functionals

$$v \mapsto \int_\Omega [F(Dv) - fv]\, dx \, ,$$

so that the Euler-Lagrange reads as

$$-\operatorname{div} \partial F(Du) = f$$

and non-uniform ellipticity reads as

$$\lim_{|z|\to\infty} \mathcal{R}(z) = \lim_{|z|\to\infty} \frac{\text{highest eigenvalue of } \partial^2 F(z)}{\text{lowest eigenvalue of } \partial^2 F(z)} = \infty \, .$$

Connection: functionals with non-standard growth of polynomial type (Marcellini)

$$W^{1,1} \ni v \mapsto \int_\Omega F(Dv)\,dx \qquad \Omega \subset \mathbb{R}^n$$

with

$$|z|^p \lesssim F(z) \lesssim |z|^q + 1 \qquad \text{and } q > p > 1$$

A first example: almost polynomial

This means

$$W^{1,1} \ni v \mapsto \int_\Omega |Dv|^p \log(1 + |Dv|)\, dx \qquad p \geq 1$$

in particular, we have the almost linear growth condition

$$W^{1,1} \ni v \mapsto \int_\Omega |Dv| \log(1 + |Dv|)\, dx$$

Oscillating and polynomial

This time it is

$$W^{1,3} \ni v \mapsto \int_\Omega F(|Dv|)\, dx$$

with

$$F(t) := \begin{cases} et^3 & \text{if } t \leq e \\ t^{4+\sin(\log\log t)} & \text{if } t > e \end{cases}$$

Anisotropic growth conditions

In this case the model is

$$W^{1,1} \ni v \mapsto \int_\Omega |Dv|^p + \sum_{i=1}^{n} |D_i v|^{p_i} \, dx$$

with

$$1 \leq p \leq p_1 \leq \ldots \leq p_n$$

Fast growth conditions

This means we are considering functionals of the type

$$v \mapsto \int_{\Omega} \exp(\exp(\ldots \exp(|Dv|^p)\ldots))\,dx\ , \quad p \geq 1\ ,$$

Duc & Eells (1991), Lieberman (1992), Marcellini (1996)

A basic condition

$$W^{1,1} \ni v \mapsto \int_\Omega F(Dv)\,dx \qquad \Omega \subset \mathbb{R}^n$$

with

$$|z|^p \lesssim F(z) \lesssim |z|^q + 1 \qquad \text{and } q > p > 1$$

then

$$\frac{q}{p} < 1 + o(n)$$

is a sufficient (Marcellini) and necessary (Giaquinta and Marcellini) condition for regularity

A model result by Marcellini

We consider functionals of the type

$$\mathcal{F}(v) := \int_\Omega F(Dv)\, dx \qquad v\colon \Omega \to \mathbb{R}$$

assuming that $z \mapsto F(z)$ is C^2 and

$$\begin{cases} \nu|z|^p \leq F(z) \leq L(1+|z|^q) \\ \nu(|z|^2+1)^{\frac{p-2}{2}}|\lambda|^2 \leq \langle \partial^2 F(z)\lambda, \lambda\rangle \leq L(|z|^2+1)^{\frac{q-2}{2}}|\lambda|^2 \end{cases}$$

A model result by Marcellini

Theorem (Marcellini JDE 1991)

Under the above assumptions, if

$$\frac{q}{p} < 1 + \frac{2}{n}$$

then any local $W^{1,p}$-minimizer is locally Lipschitz continuous. Moreover, we have

$$\|Du\|_{L^\infty(B_{R/2})} \lesssim \left(\fint_{B_R} F(Du)\, dx \right)^{\frac{2}{(n+2)p-nq}} + 1$$

for every ball $B_R \Subset \Omega$

Additional interesting results

- Bella & Shäffner, Analysis & PDE, to appear

$$\frac{q}{p} < 1 + \frac{2}{n-1} \Longrightarrow Du \in L^{\infty}_{\mathrm{loc}}$$

- Shäffner, Arxiv 2020, to appear

$$\frac{q}{p} < 1 + \frac{2}{n-1} \Longrightarrow Du \in L^{q}_{\mathrm{loc}} \text{ (vectorial case)}$$

- Hirsch & Shäffner, Comm. Cont. Math., to appear.

$$\frac{1}{p} - \frac{1}{q} < \frac{1}{n-1} \Longrightarrow u \in L^{\infty}_{\mathrm{loc}}$$

- De Filippis & Kristensen & Koch, to appear

$$\frac{q}{p} < 1 + \frac{2}{n-2} \Longrightarrow Du \in L^{\infty}_{\mathrm{loc}}$$

by duality methods, under special assumptions

Non-standard growth conditions

Bounded minimisers give better bounds

$$q < p + 1$$

the first example of this result I know is from a paper of Uraltseva & Urdaletova (1984).
Later results by Choe (Nonlinear Anal. 1992) – Kristensen & co. (Ann. IHP 2011) – De Filippis & Min. (JGA 2020).

Theorem (Bousquet & Brasco Rev. Mat. Iber. to appear)

If u is a local minimizer of the functional

$$v \mapsto \int_\Omega \sum_{k=1}^{n} |D_k v|^{p_k}\, dx \;,$$

where

$$2 \leq p_1 \leq \ldots \leq p_n$$

Then

$$u \in L^\infty_{\rm loc} \Longrightarrow Du \in L^\infty_{\rm loc} \;.$$

Special structures

Theorem (Bousquet & Brasco Rev. Mat. Iber. to appear)

If u is a local minimizer of the functional

$$v \mapsto \int_\Omega \sum_{k=1}^n |D_k v|^{p_k}\, dx \ ,$$

where

$$2 \leq p_1 \leq \ldots \leq p_n$$

Then

$$u \in L^\infty_{\rm loc} \Longrightarrow Du \in L^\infty_{\rm loc} \ .$$

No upper bound on p_n/p_1 is needed.

The variational setting

- We consider functionals

$$v \mapsto \int_\Omega [F(Dv) - fv] \, dx \,,$$

- Double control on the eigenvalues

$$g_1(|z|) Id \lesssim \partial^2 F(z) \lesssim g_2(|z|) Id$$

- Balance condition

$$\mathcal{R}(z) \lesssim \frac{g_2(|z|)}{g_1(|z|)} \lesssim H\left(\int_0^{|z|} g_1(s) s \, ds \right)$$

for a suitable increasing function $H(\cdot)$ which is of power type

The non-uniformly elliptic case

Theorem (Beck & Min. CPAM 2020)

If u is a local minimizer and $f \in L(n,1)$, then $Du \in L^\infty_{\rm loc}(\Omega)$. Moreover the estimate

$$\int_0^{\|Du\|_{L^\infty(B_{R/2})}} g_1(s)s\,ds \lesssim \left(\fint_{B_R} F(Du)\,dx\right)^{\gamma_2} + \|f\|^{\gamma_1}_{L(n,1)(B_R)} + 1$$

holds for every ball. The result still holds in the vectorial case provided $F(Du) \equiv \tilde{F}(|Du|)$.

New features

- Provides a nonlinear potential theoretic approach to the regularity of non-uniformly elliptic problems, yielding new and optimal estimates already in the case $f \equiv 0$.

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- In the case $f \equiv 0$ it recovers the classical theory of Marcellini for both functionals with polynomial and non-polynomial growth conditions and the theory by Lieberman for anisotropic integrals
- It recovers the usual theory from Orlicz spaces setting

Example 1: Estimates in the polynomial case

- In the case $F(Du) \approx |Du|^p$ we recover the classical estimate

$$\|Du\|_{L^\infty(B_{R/2})} \lesssim \left(\fint_{B_R} |Du|^p \, dx \right)^{\frac{1}{p}} + \|f\|_{L(n,1)(B_R)}^{\frac{1}{p-1}}$$

- In the case $|Du|^p \lesssim F(Du) \lesssim |Du|^q + 1$ and $f \equiv 0$ we recover the classical estimate

$$\|Du\|_{L^\infty(B_{R/2})} \lesssim \left(\fint_{B_R} F(Du) \, dx \right)^{\frac{2}{(n+2)p-nq}} + 1$$

which is the basic result of Marcellini

Example 2: Fast growth conditions

Theorem (Beck & Min. CPAM 2020)

If u is a local minimizer of the functional

$$v \mapsto \int_\Omega [\exp(\exp(\ldots \exp(|Dv|^p)\ldots)) - fv]\, dx \,, \quad p \geq 1 \,,$$

with

$$f \in L(n,1) \,.$$

Then

$$Du \in L^\infty_{\mathrm{loc}}(\Omega) \,.$$

Holds in the vectorial case too.

Example 3: Exponentials

Theorem (Beck & Min. CPAM 2020)

If u is a local minimizer of the functional

$$v \mapsto \int_\Omega \exp(|Dv|^p)\, dx\ , \qquad p \geq 1\ .$$

Then

$$\|Du\|^p_{L^\infty(B/2)} \lesssim \log\left(\fint_B \exp(|Du|^p)\, dx\right) + 1\ .$$

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If u is a local minimizer of the functional

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Then

$$\|Du\|^p_{L^\infty(B/2)} \lesssim \log\left(\fint_B \exp(|Du|^p)\,dx\right) + 1\ .$$

Previous estimates looked like

$$\|Du\|_{L^\infty(B/2)} \lesssim \left(\fint_B \exp(|Du|^p)\,dx\right)^\gamma + 1$$

Example 4: Natural growth estimates and no $\triangle_2$

Theorem (Beck & Min. CPAM 2020)

If the functional has the form

$$v \mapsto \int_\Omega A(|Dv|)\, dx \ ,$$

where $A(t)$ does not satisfy the $\triangle_2$-condition, then

$$\|Du\|_{L^\infty(B_{R/2})} \lesssim A^{-1}\left(\fint_{B_R} A(|Du|)\, dx\right) + 1$$

Example 4: Natural growth estimates and no $\triangle_2$

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If the functional has the form

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where $A(t)$ does not satisfy the $\triangle_2$-condition, then

$$\|Du\|_{L^\infty(B_{R/2})} \lesssim A^{-1}\left(\fint_{B_R} A(|Du|)\,dx\right) + 1$$

The result was known only assuming the $\triangle_2$-condition

$$A(2t) \lesssim A(t)$$

No available technique was catching natural estimates under fast growth conditions.

Non-autonomous functionals

Finally, we consider

$$v \mapsto \int_{\Omega} F(x, Dv)\, dx$$

New phenomena appear in this situation, and the presence of x *is not any longer a perturbation*.

Two functionals of Zhikov

Zhikov introduced, between the 80s and the 90s, the following functionals:

$$v \mapsto \int_\Omega |Dv|^{p(x)}\, dx \qquad p(x) \geq 1$$

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx \qquad a(x) \geq 0$$

motivations: modelling of strongly anisotropic materials, Elasticity, Homogenization, Lavrentiev phenomenon etc.

A counterexample

Theorem (Fonseca-Malý-Min. ARMA 2004)

For every choice of $n \geq 2$, $\Omega \subset \mathbb{R}^n$ *and of* $\varepsilon, \sigma > 0$, $\alpha > 0$, *there exists a non-negative function* $a(\cdot) \in C^{[\alpha]+\{\alpha\}}$, *a boundary datum* $u_0 \in W^{1,\infty}(B)$ *and exponents* p, q *satisfying*

$$n - \varepsilon < p < n < n + \alpha < q < n + \alpha + \varepsilon$$

such that the solution to the Dirichlet problem

$$\begin{cases} u \mapsto \min\limits_{w} \displaystyle\int_B (|Dv|^p + a(x)|Dv|^q)\, dx \\ w \in u_0 + W_0^{1,p}(B) \end{cases}$$

has a singular set of essential discontinuity points of Hausdorff dimension larger than $n - p - \sigma$

A counterexample

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For every choice of $n \geq 2$, $\Omega \subset \mathbb{R}^n$ and of $\varepsilon, \sigma > 0$, $\alpha > 0$, there exists a non-negative function $a(\cdot) \in C^{[\alpha]+\{\alpha\}}$, a boundary datum $u_0 \in W^{1,\infty}(B)$ and exponents p, q satisfying

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such that the solution to the Dirichlet problem

$$\begin{cases} u \mapsto \min\limits_{w} \displaystyle\int_B (|Dv|^p + a(x)|Dv|^q)\, dx \\ \qquad\qquad w \in u_0 + W^{1,p}_0(B) \end{cases}$$

has a singular set of essential discontinuity points of Hausdorff dimension larger than $n - p - \sigma$

See also a recent, very interesting paper by Balci & Diening & Surnachev (Arxiv 2019)

Maximal regularity

Theorem (Baroni & Colombo & Min., Calc. Var. 2018)

Let $u \in W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$, be a local minimiser of the functional

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx \qquad 0 \le a(\cdot) \in C^{0,\alpha}(\Omega)$$

and assume that one of the following assumptions holds:

- $$q/p \le 1 + \alpha/n$$
- $$u \in L^\infty \qquad \textit{and} \qquad q \le p + \alpha$$
- $$u \in C^{0,\gamma} \qquad \textit{and} \qquad q < p + \frac{\alpha}{1-\gamma}$$

then

Du is Hölder continuous

More

- The first regularity results are in papers by Colombo & Min. (ARMA 2015 + ARMA 2015 + JFA 2016), also with different proofs.
- The result holds for more general functionals of the type

$$v \mapsto \int_\Omega F(x, v, Dv)\, dx$$

modelled on the double phase functional.

- A recent paper of Balci & Diening & Surnachev (Arxiv 2019) features examples, still related to a fractal construction, showing that also the third condition

$$u \in C^{0,\gamma} \qquad \text{and} \qquad q < p + \frac{\alpha}{1-\gamma}$$

is sharp.

Heuristic explanation - dependence on α of the bound

The Euler equation of the functional is

$$\operatorname{div} A(x, Du) = \operatorname{div}\left(|Du|^{p-2}Du + (q/p)a(x)|Du|^{q-2}Du\right) = 0$$

on a ball B_R where

$$B_R \cap \{a(x) = 0\} \neq \emptyset \, .$$

Then

$$\frac{\sup_{B_R} \text{ highest eigenvalue of } \partial_z A(x, Du)}{\inf_{B_R} \text{ lowest eigenvalue of } \partial_z A(x, Du)} \approx 1 + R^{\alpha}|Du|^{q-p}$$

Heuristic explanation - the bound $q \leq p + \alpha$

Consider the usual p-capacity for $p < n$

$$\mathrm{cap}_p(B_r) = \inf \left\{ \int_{\mathbb{R}^n} |Df|^p \, dx \, : \, f \in W^{1,p}, f \geq 1 \text{ on } B_r \right\}$$

we have

$$\mathrm{cap}_p(B_r) \approx r^{n-p}$$

then consider the weighted capacity

$$\mathrm{cap}_{q,\alpha}(B_r) = \inf \left\{ \int_{\mathbb{R}^n} |x|^\alpha |Df|^q \, dx \, : \, f \in W^{1,p}, f \geq 1 \text{ on } B_r \right\}$$

we then have

$$\mathrm{cap}_{q,\alpha}(B_r) \approx r^{n-q+\alpha}$$

Heuristic explanation - the bound $q \leq p + \alpha$

We then ask for

$$\mathrm{cap}_{q,\alpha}(B_r) \lesssim \mathrm{cap}_p(B_r)$$

that is

$$r^{n-q+\alpha} \leq r^{n-p}$$

for r small enough, so that

$$q \leq p + \alpha$$

Theorem (De Filippis & Oh JDE 2019)

Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the energy

$$w \mapsto \int_\Omega \left[|Dw|^p + \sum_{j=1}^{k} a_j(x)|Dw|^{p_j} \right] dx,$$

where

$$a_j(\cdot) \in C^{0,\alpha_j}(\Omega), \quad 1 < \frac{p_j}{p} \leq 1 + \frac{\alpha_j}{n}, \quad 1 < p < p_1 \leq \cdots \leq p_k.$$

Then $Du \in C^{0,\beta}_{loc}(\Omega)$ for some universal $\beta \in (0,1)$.

Fully nonlinear equations – The viscosity setting

Theorem (De Filippis, Proc. Royal Soc. Edin. 2020)

Let u be a continuous viscosity solution to problem

$$[|Du|^p + a(x)|Du|^q]\, F(D^2u) = f(x),$$

with $f \in L^\infty$, $0 \le a(\cdot)$ continuous and $0 < p \le q$. Then $u \in C^{1,\gamma}_{loc}$ for some $\gamma \in (0,1)$.

Viscosity solutions can also be considered in a non-local version of double phase operators. See another paper from De Filippis & Palatucci JDE 2019

The space $W^{1,H(\cdot)}(\Omega, \mathbb{S}^N)$

For $z \in \mathbb{R}^{N\times n}$ we define the integrand

$$H(x,z) = |z|^p + a(x)|z|^q,$$

where

$$\begin{cases} 0 \le a(\cdot) \le L, \ \ a \in C^{0,\alpha}(\Omega), \ \ \alpha \in (0,1] \\ q - p < \alpha, \ \ 1 < p \le q < N. \end{cases}$$

The space $W^{1,H(\cdot)}(\Omega, \mathbb{S}^N)$ is defined as

$$W^{1,H(\cdot)}(\Omega, \mathbb{S}^N) \colon = \left\{ w \colon \Omega \to \mathbb{S}^N \text{ such that } H(\cdot, Dw) \in L^1(\Omega) \right\}.$$

The subset of smooth functions might not be dense in $W^{1,H(\cdot)}(\Omega, \mathbb{S}^N)$.

We also define

$$H_B^-(z) := |z|^p + \left(\inf_B a(x)\right)|z|^q$$

$C^{1,\beta}$-partial regularity

Theorem (De Filippis & Min. JGA 2020)

Let $u \in W^{1,1}_{loc}(\Omega, \mathbb{S}^N)$ be a constrained minimizer of the double phase functional. Then

- *There exists $\delta > 0$ such that $H(x, Du) \in L^{1+\delta}_{\mathrm{loc}}(\Omega)$.*
- *There exists $\beta > 0$ and an open subset $\Omega_u \subset \Omega$, with full measure, such that $Du \in C^{0,\beta}_{\mathrm{loc}}(\Omega_u)$.*
- *There exists $\varepsilon > 0$ such that $x_0 \in \Omega_u$ iff*

$$\fint_{B_{2r}(x_0)} H(x, Du)\, dx \leq H^-_{B_{2r}(x_0)}\left(\frac{\varepsilon}{2r}\right)$$

  holds for some $B_{2r}(x_0) \Subset \Omega$
- *$\delta, \beta, \varepsilon$ are universal, i.e. they are indipendent of the minimizer*

This extends and recovers classical works of Schoen & Uhlenbeck

We consider a function $\Phi\colon \Omega \times [0,\infty) \to [0,\infty)$, non decreasing in the second variable (+ some technical, easy-to-verify conditions), and define

$$h_\Phi(B) = \int_B \Phi\left(x, 1/\mathrm{radius}(B)\right)\, dx$$

and

$$\mathcal{H}_{\Phi,\kappa}(E) = \inf_{\mathcal{C}_E^\kappa} \sum_j h_\Phi(B_j)\ ,$$

$$\mathcal{C}_E^\kappa = \{\ \{B_j\}_{j\in\mathbb{N}} \text{ covers } E \ \text{ with radius}(B_j) \leq \kappa\ \}$$

Finally, we define

$$\mathcal{H}_\Phi(E) := \lim_{\kappa\to 0} \mathcal{H}_{\Phi,\kappa}(E) = \sup_{\kappa>0} \mathcal{H}_{\Phi,\kappa}(E)\ .$$

Some examples

These definitions unify several instances of similar objects, and introduce new ones

- $\Phi(x,t) \equiv t^p$, $p \leq n$, then $\mathcal{H}_\Phi \approx \mathcal{H}^{n-p}$;
- $\Phi(x,t) \equiv t^{p(x)}$, $p(\cdot) \leq n$, and this falls into the realm of variable exponent Hausdorff measures;
- $\Phi(x,t) \equiv \omega(x)t^p$, weighted Hausdorff measures, studied in particular when $\omega(\cdot)$ is a Muckenhoupt weight;
- $\Phi(x,t) = [H(x,t)]^{1+\sigma} \equiv [t^p + a(x)t^q]^{1+\sigma}$ for some $\sigma \geq 0$, $q(1+\sigma) \leq n$.

Classical references are

- E. Nieminen, *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, (1991).
- B. O. Turesson, *Lecture Notes in Math.*, (2000).

How to measure the singular set

Theorem (De Filippis & Min. JGA 2020)

Let $u \in W^{1,1}_{loc}(\Omega, \mathbb{S}^N)$ be a local minimizer and let $\Omega_u \subset \Omega$ be its regular set. If $q(1+\delta) \leq n$, then it holds that

$$\mathcal{H}_{H^{1+\delta}}(\Sigma_u) = 0 \, .$$

- These measure naturally connect to the standard intrinsic capacities
- Chlebicka & De Filippis show that these measures can be used to characterize the removability sets for solutions to non-uniformly elliptic problems

Removable sets

- A function $u \in W^{1,H}(\Omega)$, $\Omega \subset \mathbb{R}^n$ being open, is H-harmonic in $\Omega \setminus E$, where E is a closed subset, iff

$$-\mathrm{div}\,(|Du|^{p-2}Du + a(x)|Du|^{q-2}Du) = 0 \qquad \text{in } \Omega \setminus E\,.$$

- The set is removable for u if the above condition automatically implies that u is H-harmonic in Ω.
- Here we assume that

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega), \quad \frac{q}{p} \leq 1 + \frac{\alpha}{n}, \quad 1 < p < q \leq n\,.$$

Removable sets

Theorem (Chlebicka & De Filippis AMPA 2020)

Let $E \subset \Omega$ be a closed subset and u be H-harmonic in $\Omega \setminus E$, and such that, for all $x_1 \in E$, $x_2 \in \Omega$,

$$|u(x_1) - u(x_2)| \lesssim |x_1 - x_2|^{\beta_0} \qquad 0 < \beta_0 \leq 1 \,.$$

If

$$\mathcal{H}_{H^\sigma}(E) = 0 \quad \textit{for} \quad \sigma := 1 - \frac{\beta_0}{q}(p-1)$$

then u is H-harmonic in Ω and therefore E is removable.

This extends classical results by Carleson, Serrin, Verons, Kilpeläinen.

General functionals

$$
\begin{cases}
\nu|z|^p \leq F(x,z) \leq L(1+|z|^q) \\[1em]
\nu\left(\lambda^2+|z_1|^2+|z_2|^2\right)^{\frac{p-2}{2}}|z_1-z_2|^2 \\
\leq (\partial_z F(x,z_1)-\partial_z F(x,z_2))\cdot(z_1-z_2) \\[1em]
|\partial_z F(x,z)-\partial_z F(y,z)| \leq L|x-y|^\alpha(1+|z|^{q-1})\,,
\end{cases}
$$

General functionals

Theorem (De Filippis & Min. JGA 2020)

Let $u \in W^{1,p}(\Omega)$ *be a bounded local minimiser of the functional*

$$v \mapsto \int_\Omega F(x, Dv)\, dx$$

under the above assumptions. Furthermore, assume that

$$2 \leq p < q < p + \alpha$$

and no Lavrentiev phenomenon occurs. Then $Du \in L^{\tilde{p}}_{\mathrm{loc}}(\Omega)$ *provided*

$$q < \tilde{p} < p + \alpha\ .$$

Proof uses several things, amongst them, interpolation inequalities in fractional Sobolev spaces and delicate approximation and penalization methods.

A cheap trick, De Filippis & Min. JGA 2020

We now consider the assumptions

$$\begin{cases} (|z|^2+1)^{\frac{p}{2}} \leq F(x,v,z) \leq L(|z|^2+1)^{\frac{q}{2}} \\ \\ (|z|^2+1)^{\frac{p-2}{2}}|\xi|^2 \leq \left\langle \partial^2 F(x,v,z)\xi,\xi \right\rangle \\ \\ |\partial_{zz}F(x,z)| + \dfrac{|\partial_{xz}F(x,z)|}{(1+|z|^2)^{1/2}} \leq L(|z|^2+1)^{\frac{q-2}{2}} \end{cases}$$

with

$$\frac{q}{p} < 1 + \frac{1}{n}$$

A cheap trick, De Filippis & Min. JGA 2020

We now consider the assumptions

$$\begin{cases} (|z|^2+1)^{\frac{p}{2}} \leq F(x,v,z) \leq L(|z|^2+1)^{\frac{q}{2}} \\ \\ (|z|^2+1)^{\frac{p-2}{2}}|\xi|^2 \leq \left\langle \partial^2 F(x,v,z)\xi,\xi \right\rangle \\ \\ |\partial_{zz}F(x,z)| + \dfrac{|\partial_{xz}F(x,z)|}{(1+|z|^2)^{1/2}} \leq L(|z|^2+1)^{\frac{q-2}{2}} \end{cases}$$

with

$$\frac{q}{p} < 1 + \frac{1}{n}$$

we have the apriori estimate

$$\|Du\|_{L^\infty(B_{R/2})} \lesssim R^{-\frac{n}{p-n(q-p)}} \left(\|Du\|_{L^p(B_R)}^{\frac{p}{p-n(q-p)}} + 1 \right)$$

Step 1: Tracking the constants in the case $p = q$

The following estimate holds when $p = q$

$$\|Du\|_{L^\infty(B_{\tau_1})} \leq \frac{cL^{n/p}}{(\tau_2 - \tau_1)^{n/p}} \left(\|Du\|_{L^p(B_{\tau_2})} + 1\right) ,$$

where

$$B_{\tau_1} \Subset B_{\tau_2}$$

are arbitrary balls. This follows tracking the constants in the classical proof.

Step 2: Reduction to the uniform case

On B_{τ_2}

$$|\partial_{zz} F(x,z)| + \frac{|\partial_{xz} F(x,z)|}{(1+|z|^2)^{1/2}} \leq cL\left(\|Du\|^{q-p}_{L^\infty(B_{\tau_2})} + 1\right)(|z|^2+1)^{\frac{p-2}{2}}$$

therefore we have standard growth conditions with L replaced by

$$cL\left(\|Du\|^{q-p}_{L^\infty(B_{\tau_2})} + 1\right)$$

and c is an absolute constant.

Step 2: Reduction to the uniform case

On B_{τ_2}

$$|\partial_{zz}F(x,z)| + \frac{|\partial_{xz}F(x,z)|}{(1+|z|^2)^{1/2}} \le cL\left(\|Du\|_{L^\infty(B_{\tau_2})}^{q-p}+1\right)(|z|^2+1)^{\frac{p-2}{2}}$$

therefore we have standard growth conditions with L replaced by

$$cL\left(\|Du\|_{L^\infty(B_{\tau_2})}^{q-p}+1\right)$$

and c is an absolute constant. The a priori estimate in the case $p=q$ from the previous slide gives

$$\|Du\|_{L^\infty(B_{\tau_1})} \le \frac{c\left[L\|Du\|_{L^\infty(B_{\tau_2})}^{q-p}+1\right]^{n/p}}{(\tau_2-\tau_1)^{n/p}}\|Du\|_{L^p(B_{\tau_2})}\,.$$

Step 2: Reduction to the uniform case

The assumed bound

$$\frac{q}{p} < 1 + \frac{1}{n}$$

implies

$$(q-p)\frac{n}{p} < 1$$

so that, Young's inequality gives

$$\|Du\|_{L^\infty(B_{\tau_1})} \leq \frac{1}{2}\|Du\|_{L^\infty(B_{\tau_2})} + \frac{c\|Du\|_{L^p(B_{\tau_2})}^{\frac{p}{p-n(q-p)}} + c}{(\tau_2-\tau_1)^{\frac{n}{p-n(q-p)}}}$$

Step 3: Iteration lemma

Lemma (Giaquinta & Giusti, Acta Math. 1982)

Let $\mathcal{Z}\colon [\varrho, R) \to [0,\infty)$ *be a function which is bounded on every interval* $[\varrho, R_*]$ *with* $R_* < R$. *Let* $\varepsilon \in (0,1)$, $a, \gamma \geq 0$ *be numbers. If*

$$\mathcal{Z}(\tau_1) \leq \varepsilon \mathcal{Z}(\tau_2) + \frac{a}{(\tau_2 - \tau_1)^\gamma} \ ,$$

for all $\varrho \leq \tau_1 < \tau_2 < R$, *then*

$$\mathcal{Z}(\varrho) \leq \frac{ca}{(R-\varrho)^\gamma} \ ,$$

holds with $c \equiv c(\varepsilon, \gamma)$.

Step 3: Iteration lemma

$$\|Du\|_{L^\infty(B_{R/2})} \lesssim R^{-\frac{n}{p-n(q-p)}} \left(\|Du\|_{L^p(B_R)}^{\frac{p}{p-n(q-p)}} + 1 \right)$$

and the a priori estimate is ready.

Step 3: Iteration lemma

$$\|Du\|_{L^\infty(B_{R/2})} \lesssim R^{-\frac{n}{p-n(q-p)}} \left(\|Du\|_{L^p(B_R)}^{\frac{p}{p-n(q-p)}} + 1 \right)$$

and the a priori estimate is ready.

Notice that in the case $p = q$ the above estimate gives

$$\|Du\|_{L^\infty(B_{R/2})} \lesssim \left(\fint_{B_R} (|Du| + 1)^p \, dx \right)^{1/p} ,$$

which is the usual $L^\infty - L^p$ estimate typical of harmonic functions