Asymptotic profiles of groundstates for a class of Choquard equations

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Choquard equation

Choquard equation introduced by Lieb in 1977:

$$-\Delta u + u = \left(|x|^{-1} * |u|^2\right) u \quad \text{in } \mathbb{R}^3$$

The corresponding energy is

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 - \frac{1}{4}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}\,dx\,dy.$$

- existence of a positive ground–state
- radial monotone decreasing + exponential decay
- unique ground–state!

Schrödinger–Newton, attractive/focusing Hartree...

NLS with nonlocal interactions

Attractive nonlocal interactions (Choquard type)

$$\underbrace{-\Delta u + V(x)u}_{\text{Schrödinger}} = \underbrace{\left(C_\alpha |x|^{-(N-\alpha)} * |u|^2\right) u}_{\text{nonlocal intercation}} \pm \dots \quad \text{in } \mathbb{R}^N$$

Repulsive nonlocal interactions (Schrödinger–Poisson type)

$$\underbrace{-\Delta u + V(x)u}_{\text{Schrödinger}} = -\underbrace{\left(C_\alpha |x|^{-(N-\alpha)} * |u|^2\right) u}_{\text{nonlocal intercation}} + \underbrace{|u|^{q-2}u \pm \dots}_{\text{short-range}} \quad \text{in } \mathbb{R}^N$$

- $|u(x)|^2$ – particles density in a multi-particle system
- $C_\alpha |x|^{-(N-\alpha)} \to \delta_0$ as $\alpha \to 0$ – short-range local interaction

Equivalent system representation

$$\begin{cases} -\Delta u + V(x)u = \pm \phi u \pm \dots & \text{in } \mathbb{R}^N, \\ (-\Delta)^{\alpha/2}\phi = |u|^2 & \text{in } \mathbb{R}^N. \end{cases}$$

Generalised Choquard equation

Jean Van Schaftingen and VM, 2013–15...

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in} \ \ \mathbb{R}^N$$

- $I_\alpha(x) := C_\alpha |x|^{-(N-\alpha)} \sim (-\Delta)^{-\alpha/2}$ – Riesz potential
- $p > 1$, $\alpha \in (0, N)$ and $N \geq 2$

Local prototype ($\alpha = 0$): $-\Delta u + V(x)u = |u|^{2p-2}u$ in $\mathbb{R}^N$

- Liouville type theorems and decay of solutions
- existence and properties of ground states
- general nonlinearities $F(u)$ instead of $|u|^p$
- semiclassical concentration at minima of $V(x)$
- least–action nodal solutions (with Marco Ghimenti)

Theorem (VM and Van Schaftingen '13)

If $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ *then Choquard equation*

$$-\Delta v + v = (I_\alpha * |v|^p)|v|^{p-2}v \quad \text{in } \mathbb{R}^N$$

admits a radial decreasing positive ground state $v \in H^1(\mathbb{R}^N)$. *Further,* $v \in C^2(\mathbb{R}^N)$ *and as* $|x| \to \infty$,

- *if* $p > 2$: $v(x) \simeq C|x|^{-\frac{N-1}{2}} e^{-|x|}$
- *if* $p = 2$: $v(x) \simeq C|x|^{-\frac{N-1}{2}} e^{-\int_\nu^{|x|} \sqrt{1-\frac{\nu^{N-\alpha}}{s^{N-\alpha}}}\,ds}$,
- *if* $p < 2$: $v(x) \simeq C|x|^{\frac{N-\alpha}{2-p}}$.

- existence interval $(\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ is optimal
- uniqueness is mostly open beyond Lieb's setting $p = 2$, $\alpha = 2$

Setting of the problem (Zeng Liu and VM, 2020)

We study positive solutions of Choquard type equation

$$(P_\varepsilon) \qquad -\Delta u + \varepsilon u + |u|^{q-2}u - (I_\alpha * u^p)|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^N$$

- $I_\alpha(x) := C_\alpha |x|^{-(N-\alpha)} \sim (-\Delta)^{-\alpha/2}$ – Riesz potential
- $\alpha \in (0, N)$ and $N \geq 3$
- $p > 1$, $q > 2$
- $\varepsilon \geq 0$ – small (or large) parameter

Typical model: self-gravitating Bose-Einstein condensate [Chavanis]

$$-\Delta u + \varepsilon u + |u|^2 u - (I_2 * u^2)u = 0 \quad \text{in } \mathbb{R}^3$$

Defocusing Gross-Pitaevskii equation with nonlocal attarction

Local equivalent (as $\alpha = 0$)

$$(L_\varepsilon) \qquad -\Delta u + \varepsilon u + |u|^{q-2}u - |u|^{2p-2}u = 0 \quad \text{in} \ \mathbb{R}^N$$

Theorem (Berestycki–Lions '83; Serrin–Tang '00)

Assume that

- $2 < 2p < \frac{2N}{N-2}$, $2 < q < 2p$ *and* $\varepsilon > 0$, *or*
- $2 < 2p < q$ *and* $\varepsilon \in (0, \varepsilon_*)$.

Then (L_ε) *admits unique positive radial monotone decreasing solution* $u_\varepsilon \in H^1 \cap L^q(\mathbb{R}^N)$. *Further,* $u_\varepsilon \in C^2(\mathbb{R}^N)$ *and has exponential decay.*

Muratov, VM (2014) – asymptotic profiles of u_ε as $\varepsilon \to 0$

Albalawi, Mercuri, VM (2019) – extensions to p-Laplacian

Theorem

Assume that:

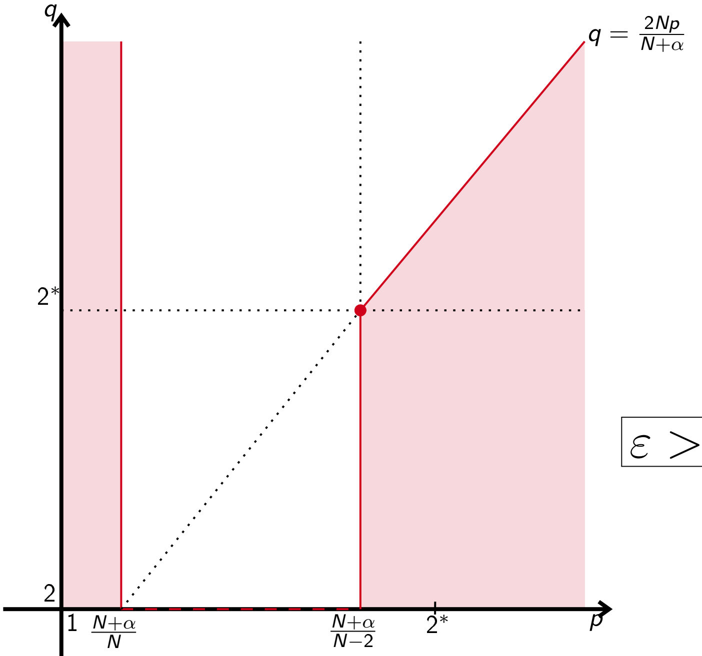
- $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ *and* $q > 2$, *or*
- $p \geq \frac{N+\alpha}{N-2}$ *and* $q > \frac{2Np}{N+\alpha}$

Then for any $\varepsilon > 0$

$$(P_\varepsilon) \qquad -\Delta u + \varepsilon u + |u|^{q-2}u - (I_\alpha * u^p)|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^N$$

admits a radial decreasing positive ground state $u_\varepsilon \in H^1 \cap L^q(\mathbb{R}^N)$. *Further,* $u_\varepsilon \in C^2(\mathbb{R}^N)$ *and*

- *if* $p > 2$: $u_\varepsilon(x) \simeq C_\varepsilon |x|^{-\frac{N-1}{2}} e^{-\sqrt{\varepsilon}|x|}$
- *if* $p = 2$: $u_\varepsilon(x) \simeq C_\varepsilon |x|^{-\frac{N-1}{2}} e^{-\int_\nu^{|x|} \sqrt{\varepsilon - \frac{\nu^{N-\alpha}}{s^{N-\alpha}}}\, ds}$,
- *if* $p < 2$: $u_\varepsilon(x) \simeq C_\varepsilon |x|^{-\frac{N-\alpha}{2-p}}$.



$\boxed{\varepsilon > 0}$

Essential tools

The total energy has the form

$$E_\varepsilon(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{\varepsilon}{2}\int |u|^2 + \frac{1}{q}\int |u|^q - \frac{1}{2p}\int (I_\alpha * |u|^p)|u|^p$$

Critical points of E_ε satisfy Pohožaev identity

$$\underbrace{\frac{N-2}{2}\int |\nabla u|^2 + \frac{\varepsilon N}{2}\int |u|^2 + \frac{N}{q}\int |u|^q - \frac{N+\alpha}{2p}\int (I_\alpha * |u|^p)|u|^p}_{\mathcal{P}_\varepsilon(u)} = 0$$

Nonlocal term controlled via Hardy-Littlewood-Sobolev inequality

$$\boxed{\int (I_\alpha * |u|^p)|u|^p \le C_{N,\alpha}\Big(\int |u|^{\frac{2Np}{N+\alpha}}\Big)^{\frac{N+\alpha}{N}}}$$

To prove the existence of a ground state we minimize E_ε on the Pohožaev manifold (PM):

$$S_\varepsilon = \inf\left\{E_\varepsilon(u) : u \in H^1 \cap L^q_{rad}(\mathbb{R}^N),\ \mathcal{P}_\varepsilon(u) = 0\right\}.$$

The energy E_ε is bounded below on PM, and PM is a *natural constraint* for E_ε, i.e. if u_ε is a minimizer for S_ε then $\nabla E_\varepsilon(u_\varepsilon) = 0$.

- existence for any $\varepsilon > 0$
- existence (p, q)-region is sharp
- uniqueness of u_ε is open
- $u_\varepsilon \in L^\infty(\mathbb{R}^N)$ is hard and relies on the contraction inequality

Lemma (Exercise 4.15 in the Augusto Ponce's book)

Let $q \geq 2$, $s \geq 1$, $0 \leq f \in L^s(\mathbb{R}^N)$ and $0 \leq u \in L^1_{loc}(\mathbb{R}^N)$ satisfies

$$-\Delta u + u^{q-1} \leq f \quad \text{in } \mathscr{D}'(\mathbb{R}^N).$$

Then $u^{q-1} \in L^s(\mathbb{R}^N)$ and $\|u\|_{(q-1)s} \leq \|f\|_s$.

To understand asymptotic of u_ε as $\varepsilon \to 0$ consider the formal "zero mass" limit equation

$$(P_0) \qquad -\Delta u + |u|^{q-2}u - (I_\alpha * |u|^p)|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^N$$

Theorem

Assume that:

- $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ *and* $2 < q < \frac{2Np}{N+\alpha}$, *or*
- $p > \frac{N+\alpha}{N-2}$ *and* $q > \frac{2Np}{N+\alpha}$.

Then (P_0) *admits a radial decreasing positive ground state* $u_0 \in D^1 \cap L^q(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$. *Moreover,*

- *if* $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ *then* $u \in L^1(\mathbb{R}^N)$
- *if* $p > \frac{N+\alpha}{N-2}$ *then* $u \gtrsim |x|^{-(N-2)}$

Consider the family of equations for $\varepsilon \geq 0$:

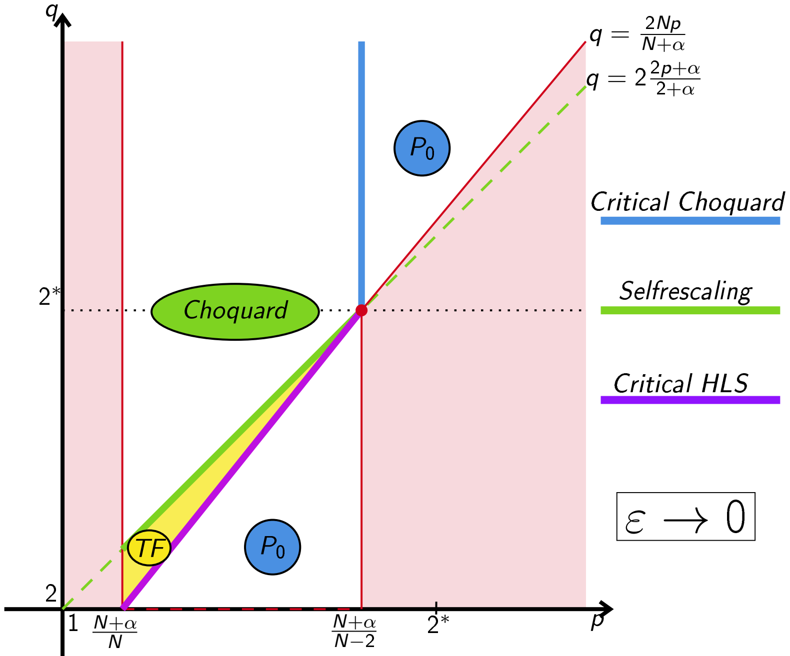
$$(P_\varepsilon) \qquad -\Delta u + \varepsilon u + |u|^{q-2}u - (I_\alpha * u^p)|u|^{p-2}u = 0 \quad \text{in} \ \mathbb{R}^N$$

Theorem (Formal limit)

If (P_ε) and (P_0) both have a ground–state then $u_\varepsilon \to u_0$ as $\varepsilon \to 0$ in $D^1 \cap L^q(\mathbb{R}^N)$ and $C^2(\mathbb{R}^N)$.

Prove $S_\varepsilon \to S_0$ by testing S_0 with u_ε and S_ε with (a cutoff of) u_0

- not a small perturbation result: (P_ε) is well posed in $H^1(\mathbb{R}^N)$ and (P_0) in $D^1(\mathbb{R}^N)$
- $u_0 \notin L^2(\mathbb{R}^N)$ at least when $N = 3, 4$



q
$q = \frac{2Np}{N+\alpha}$
$q = 2\frac{2p+\alpha}{2+\alpha}$
P_0
Critical Choquard
Selfrescaling
Critical HLS
2^*
Choquard
TF
P_0
$\varepsilon \to 0$
2
1
$\frac{N+\alpha}{N}$
$\frac{N+\alpha}{N-2}$
2^*
p

For $s, t \in \mathbb{R}$ the rescaling

$$v(x) := \varepsilon^s u(\varepsilon^t x),$$

transforms (P_ε) into the family of equivalent equations

$$(*) \quad -\varepsilon^{-s-2t}\Delta v + \varepsilon^{1-s} v + \varepsilon^{-(q-1)s}|v|^{q-2}v \\ -\varepsilon^{-(2p-1)s+\alpha t}(I_\alpha * |v|^p)|v|^{p-2}v = 0 \quad \text{in } \mathbb{R}^N.$$

If $q = 2\frac{2p+\alpha}{2+\alpha}$ we can take $s = -\frac{2+\alpha}{4(p-1)}$ and $t = -1/2$ to balance of all *four* terms in $(*)$ to get

$$(P_1) \qquad -\Delta v + v - (I_\alpha * |v|^p)|v|^{p-2}v + |v|^{q-2}v = 0 \quad \text{in } \mathbb{R}^N.$$

This means any solution of (P_ε) is a rescaling of a solution of (P_1).

Theorem (Self-rescaling regimes)

If $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $q = 2\frac{2p+\alpha}{2+\alpha}$ then $u_\varepsilon(x) = \varepsilon^{\frac{2+\alpha}{4(p-1)}} u_1(\sqrt{\varepsilon}x)$

I. First rescaling. The choice $\varepsilon^{-s-2t} = \varepsilon^{1-s} = \varepsilon^{-(2p-1)s+\alpha t}$ leads to $s = -\frac{2+\alpha}{4(p-1)}$, $t = -\frac{1}{2}$ and the equation

$$-\Delta v + v + \varepsilon^{\frac{q(2+\alpha)-2(2p+\alpha)}{4(p-1)}} |v|^{q-2}v - (I_\alpha * |v|^p)|v|^{p-2}v = 0 \quad \text{in } \mathbb{R}^N.$$

As $\varepsilon \to 0$ and $q > 2\frac{2p+\alpha}{2+\alpha}$ we have $\varepsilon^{\frac{q(2+\alpha)-2(2p+\alpha)}{4(p-1)}} \to 0$ and the *Choquard equation* as the formal limit:

$$(\mathscr{C}) \qquad -\Delta v + v - (I_\alpha * |v|^p)|v|^{p-2}v = 0 \quad \text{in } \mathbb{R}^N.$$

Theorem (Choquard limit)

Let $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $q > 2\frac{2p+\alpha}{2+\alpha}$. Then as $\varepsilon \to 0$,

$$v_\varepsilon(x) := \varepsilon^{-\frac{2+\alpha}{4(p-1)}} u_\varepsilon\left(\frac{x}{\sqrt{\varepsilon}}\right) \to v(x) \quad \text{in } H^1 \cap L^q(\mathbb{R}^N) \text{ and } C^2(\mathbb{R}^N),$$

where v is a ground state of $(\mathscr{C})$.

II. Second rescaling. The choice $\varepsilon^{1-s} = \varepsilon^{-(q-1)s} = \varepsilon^{-(2p-1)s+\alpha t}$ leads to $s = -\frac{1}{q-2}$, $t = -\frac{2p-q}{\alpha(q-2)}$ and rescaled equation

$$\varepsilon^{\frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}}(-\Delta)v + v + |v|^{q-2}v - (I_\alpha * |v|^p)|v|^{p-2}v = 0 \quad \text{in } \mathbb{R}^N.$$

As $\varepsilon \to 0$ and $2 < q < 2\frac{2p+\alpha}{2+\alpha}$ we have $\varepsilon^{\frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}} \to 0$ and the *Thomas–Fermi equation* as a formal limit

$$(TF) \qquad w + |w|^{q-2}w - (I_\alpha * |w|^p)|w|^{p-2}w = 0 \quad \text{in } \mathbb{R}^N.$$

The corresponding energy is

$$E^{TF}(w) = \frac{1}{2}\int |w|^2 + \frac{1}{q}\int |w|^q - \frac{1}{2p}\int (I_\alpha * |w|^p)|w|^p$$

is well–defined on $L^2 \cap L^q(\mathbb{R}^N)$, provided that $q > \frac{2Np}{N+\alpha} > 2$

HLS + Hölder lead to the interpolation inequality

$$(\star) \qquad \int (I_\alpha * |u|^p)|u|^p \le C_{N,\alpha} \|u\|^{2p}_{\frac{2Np}{N+\alpha}} \le C_{N,\alpha} \|u\|_2^{2p\theta} \|u\|_q^{2p(1-\theta)}$$

Theorem

Let $p > \frac{N+\alpha}{N}$ and $q > \frac{2Np}{N+\alpha}$. Then $(\star)$ admits an optimizer which corresponds to a nonnegative radial nonincreasing ground state $w \in L^1 \cap L^q(\mathbb{R}^N)$ of (TF).

Further, $w \in L^\infty(\mathbb{R}^N)$, locally Hölder inside the support and:

(a) *if $p < 2$ then $\mathrm{Supp}(w) = \mathbb{R}^N$ and $w(x) \simeq C|x|^{-\frac{N-\alpha}{2-p}}$*

(b) *if $p \ge 2$ then $\mathrm{Supp}(w) = \bar{B}_R$ and*

$$w = \lambda \chi_{B_R} + \phi$$

where $\phi : B_R \to \mathbb{R}$, $\phi(0) > 0$ and $\lim_{|x| \to R} \phi(|x|) = 0$

if $p > 2$ then $\lambda > 0$

if $p = 2$ then $\lambda = 0$

$$\varepsilon^{\frac{2(2p+\alpha)-q(2+\alpha)}{\alpha(q-2)}}(-\Delta)v + v - (I_\alpha * |v|^p)|v|^{p-2}v + |v|^{q-2}v = 0 \quad \text{in } \mathbb{R}^N$$

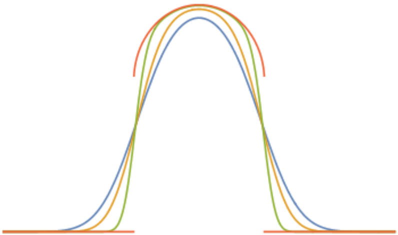


Figure: $v_\varepsilon \to w = \lambda\chi_{B_R} + \phi$ in the case $p > 2$

Theorem (Thomas–Fermi limit)

Let $p > \frac{N+\alpha}{N}$ and $\frac{2Np}{N+\alpha} < q < 2\frac{2p+\alpha}{2+\alpha}$ and $\alpha > 1$. As $\varepsilon \to 0$,

$$v_\varepsilon(x) := \varepsilon^{-\frac{1}{q-2}} u_\varepsilon\left(\varepsilon^{-\frac{2p-q}{\alpha(q-2)}} x\right) \to w(x) \text{ in } L^2 \cap L^q(\mathbb{R}^N) \text{ and } C_{loc}(B_R)$$

III. Third rescaling. The choice $\varepsilon^{-s-2t} = \varepsilon^{1-s} = \varepsilon^{-(q-1)s}$ leads to $s = -\frac{1}{q-2}$, $t = -\frac{1}{2}$ and rescaled equation

$$-\Delta v + v + |v|^{q-2}v - \varepsilon^{\frac{2(2p+\alpha)-q(\alpha+2)}{2(q-2)}}(I_\alpha * |v|^p)|v|^{p-2}v = 0 \quad \text{in } \mathbb{R}^N.$$

As $\varepsilon \to 0$ and $2 < q < 2\frac{2p+\alpha}{2+\alpha}$ we have $\varepsilon^{\frac{2(2p+\alpha)-q(\alpha+2)}{2(q-2)}} \to 0$ as $\varepsilon \to 0$ and the local limit equation

$$-\Delta v + v + |v|^{q-2}v = 0 \quad \text{in } \mathbb{R}^N.$$

This has no nonzero finite energy solutions!

Theorem (Critical Choquard limit ($\alpha = 0$: Muratov, VM - 2014))

Let $p = \frac{N+\alpha}{N-2}$ *and* $q > \frac{2Np}{N+\alpha} = \frac{2N}{N-2}$. *There exists a rescaling* $\lambda_\varepsilon : (0,\infty) \to (0,\infty)$ *such that as* $\varepsilon \to 0$, *the rescaled family*

$$v_\varepsilon(x) := \lambda_\varepsilon^{\frac{N-2}{2}} u_\varepsilon(\lambda_\varepsilon x)$$

converges to ground state $\widetilde{U}(x)$ $\left(= c(1+|x|^2)^{-\frac{N-2}{2}} \text{ if } N = 3,4\ldots \right)$ *of the critical Choquard equation*

$$-\Delta U = (I_\alpha * |U|^{\frac{N+\alpha}{N-2}})|U|^{\frac{N+\alpha}{N-2}-2}U, \quad U \in D^1(\mathbb{R}^N)$$

in $D^1 \cap L^q(\mathbb{R}^N)$ *and* $C^2(\mathbb{R}^N)$. *Moreover,*

$$\lambda_\varepsilon \simeq \begin{cases} \varepsilon^{-\frac{1}{q-4}} & \text{if } N = 3, \\ \left(\varepsilon \ln \frac{1}{\varepsilon}\right)^{-\frac{1}{q-2}} & \text{if } N = 4, \\ \varepsilon^{-\frac{2}{(q-2)(N-2)}} & \text{if } N \geq 5. \end{cases}$$

Theorem (Critical HLS limit)

Let $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ *and* $q = \frac{2Np}{N+\alpha}$*. There exists a rescaling* $\lambda_\varepsilon : (0,\infty) \to (0,\infty)$ *such that as* $\varepsilon \to 0$*, the rescaled family*

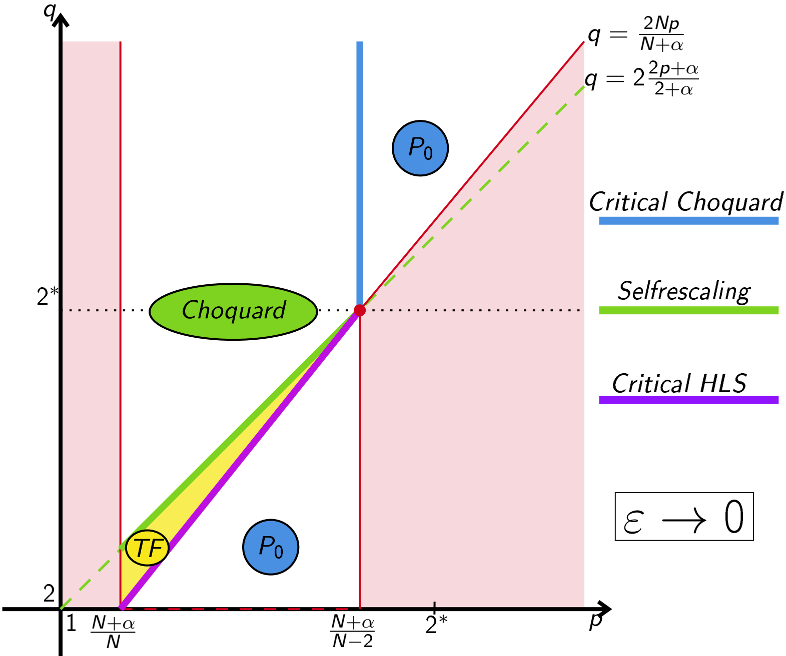
$$v_\varepsilon(x) := \lambda_\varepsilon^{\frac{N+\alpha}{2p}} u_\varepsilon(\lambda_\varepsilon x)$$

converges to the ground state $\widetilde{V}(x) = c(1+|x|^2)^{-\frac{N+\alpha}{2p}}$ *of the critical HLS equation*

$$|V|^{2-q}V = (I_\alpha * |V|^p)|V|^{p-2}V, \quad V \in L^2 \cap L^q(\mathbb{R}^N)$$

in $L^q(\mathbb{R}^N)$*. Moreover, if* $N \geq 4$ *then* $\lambda_\varepsilon \simeq \varepsilon^{-\frac{1}{2}}$*, while if* $N = 3$ *then*

$$\begin{cases} \lambda_\varepsilon \simeq \varepsilon^{-\frac{1}{2}}, & p \in \left(\frac{3+\alpha}{3}, \frac{2(3+\alpha)}{3}\right), \\ \varepsilon^{-\frac{1}{2}}(\ln \frac{1}{\varepsilon})^{-\frac{1}{2}} \lesssim \lambda_\varepsilon \lesssim \varepsilon^{-\frac{1}{2}}(\ln\frac{1}{\varepsilon})^{\frac{1}{6}}, & p = \frac{2(3+\alpha)}{3}, \\ \varepsilon^{\frac{p-(3+\alpha)}{p}} \lesssim \lambda_\varepsilon \lesssim \varepsilon^{\frac{(3+\alpha)(3+\alpha-2p)}{p(3p-(3+\alpha))}}, & p \in \left(\frac{2(3+\alpha)}{3}, 3+\alpha\right). \end{cases}$$



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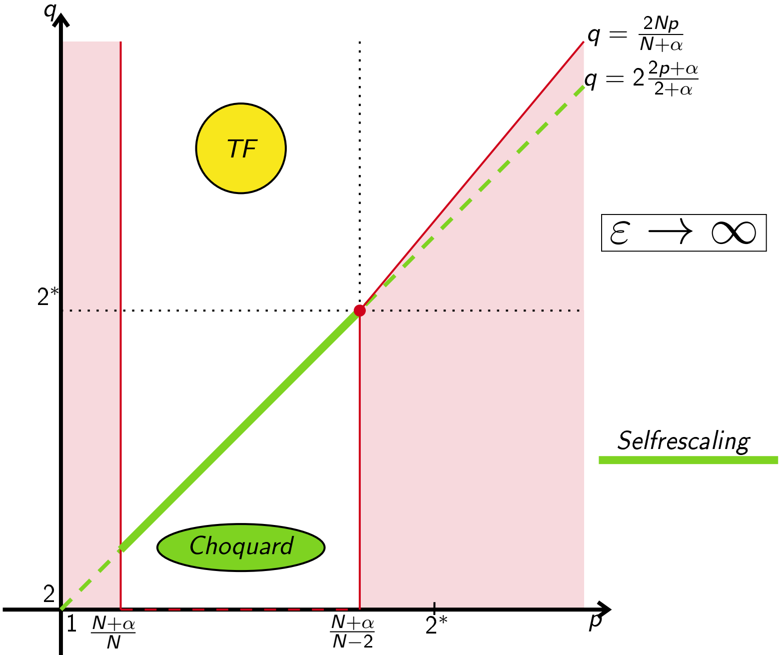

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Многая лета!

to Laurent and Marie-Françoise