Diffusive Hamilton-Jacobi equations with super-quadratic growth

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Singular problems associated to quasilinear equations,
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in honor of my friends Marie-Françoise and Laurent

Diffusive Hamilton-Jacobi equations:

$$u_t - \Delta u + H(x, Du) = 0 \qquad \text{in} \ \ (0, T) \times \Omega$$

where Ω is a smooth bounded set in $\mathbb{R}^N$.

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- Stationary solutions: regularity, existence & uniqueness.

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- Evolution of smooth solutions: when and how do we lose them ?

  $\rightsquigarrow$ gradient blow-up, loss (and recovery !) of boundary data, ...

What is *natural* in the natural growth ?

Remind: under natural growth conditions ($H \leq c(1 + |Du|^2)$)

- Smooth data $\Rightarrow$ bounded solutions are smooth

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- Global existence for the evolution problem (either convergence or infinite time blow-up of $\|u\|_\infty$)

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The solution of

$$\begin{cases} \lambda u - \Delta u + |Du|^p = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is the value function of a stochastic control problem

$$u(x) = \inf_{a \in \mathcal{A}} \mathbb{E} \left\{ \int_0^{\tau_x} \left[c_p |a_t|^{\frac{p}{p-1}} + f(X_t) \right] \mathrm{e}^{-\lambda t} \, dt \right\} , \qquad (1)$$

where X_t is a controlled process:

$$\begin{cases} dX_t = a_t \, dt + \sqrt{2} \, dB_t \, , \\ X_0 = x \ \in \Omega \, , \end{cases}$$

B_t a Brownian motion in $\mathbb{R}^N$

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Important: the optimal drift would be given in feedback form by

$$a_t = a(X_t) = -p|Du(X_t)|^{p-2} Du(X_t)$$

$\rightsquigarrow$ when $p > 2$ singular drifts are less expensive...

The stationary problem

$$-\Delta u + \lambda\, u + |Du|^p = f(x) \quad \text{with } p > 2$$

[Capuzzo Dolcetta-Leoni-P. '10]:
viscosity solutions framework (fully nonlinear)

$$\to \text{extends to } F(x, D^2 u) + \lambda\, u + |Du|^p \leq f,$$

(see also [Barles '10], [Barles-Koike-Ley-Topp '14] [Barles-Topp '15] for further extensions to state constraint, nonlocal diffusions etc...)

[Dall'Aglio-P. '14]:
distributional solutions framework (divergence form)

$$\to \text{extends to } -\mathrm{div}(a(x, Du)) + \lambda\, u + |Du|^p \leq f,$$

(similar with m-Laplacian and $p > m$)

A list of un-natural properties due to super-quadratic Hamiltonian:

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- Sub solutions are Hölder continuous

  f bounded $\Rightarrow$ USC bounded viscosity subsolutions are $\frac{p-2}{p-1}$-Hölder

  Proof by doubling variables & comparison ([Capuzzo Dolcetta-Leoni-P])

$$u(x) \le u(y) + k\left(\frac{|x-y|}{d(x)^{1-\alpha}} + L\,|x-y|^{\alpha}\right) \qquad \alpha = \frac{p-2}{p-1}$$

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$f \in L^m, m > \frac{N}{p} \Rightarrow$ distributional subsolutions are α-Hölder with $\alpha = \min(1 - \frac{N}{m\,p}, 1 - \frac{1}{p-1})$.

Proof by a Morrey-type estimate ([Dall'Aglio-P]):

$$\int_{B_r} |\nabla u|^p\, dx \leq K\, r^{N-\gamma}\,, \qquad \text{where } \gamma = \max(\tfrac{N}{m}, p')$$

[...]

$$-\Delta u + \lambda\, u + |Du|^p = f(x) \quad \text{with } p > 2$$

- Interior Hölder regularity extends up to the boundary (independently of boundary data !)

  Consequence: Hölder regularity is necessary for boundary data !

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- Global universal bounds for u^+:

$$\|u^+\|_{L^\infty(\Omega)} \le M$$

  where $M = M(\Omega, \frac{1}{\lambda}, \|f\|_{L^m(\Omega)})$, $m > N/p$.

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  $\rightsquigarrow$ loss of boundary data

  Size and regularity conditions are needed to have compatibility of boundary data.

A sample result on the solvability of the Dirichlet problem

$$\begin{cases} \lambda u - \Delta u + |Du|^p = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $p > 2$ and, now, f is continuous, $\lambda > 0$.

Theorem (Capuzzo Dolcetta-Leoni-P. '10)

There exists a constant $M_0 > 0$ such that if φ satisfies

$$|\varphi(x) - \varphi(y)| \leq M|x-y|^\alpha \qquad \forall x, y \in \partial\Omega\,, \qquad \alpha = \frac{p-2}{p-1}\,.$$

with $M < M_0$ and if $\lambda \inf \varphi \leq \inf f$, then (2) has a unique viscosity solution $u \in C^{0,(p-2)/(p-1)}(\overline{\Omega})$ such that $u(x) = \varphi(x)$ for every $x \in \partial\Omega$.

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Alternatives:

- for Lipschitz solutions one can use [P-L. Lions '80]: there exists a $W^{1,\infty}$ solution if and only if there exists a $W^{1,\infty}$ *sub solution* ψ.
- For a general theory... $\rightsquigarrow$ relaxed formulation of boundary conditions, viscosity solutions theory.

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$$\begin{cases} -\Delta u + |\nabla u|^p = 0 & \text{in } \Omega \\ u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega}) \end{cases}$$

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$\rightsquigarrow$ Typical first order effects

- the L^∞-bound does not bring enough information...
- loss of boundary data
  $\rightarrow$ need of relaxed formulation of boundary conditions, viscosity solutions theory.

  (uniqueness results for viscosity solutions: [Barles-Rouy-Souganidis '99], [Barles-Da Lio '04], [Barles '10]...)

The evolution problem

We consider now the evolution problem

$$\begin{cases} u_t - \Delta u = |Du|^p \qquad \text{in} \quad (0,T)\times\Omega \\ u = 0 \qquad \text{on} \quad (0,T)\times\partial\Omega \\ u(0) = u_0 \end{cases}$$

- $p > 2$
- Ω smooth
- $u_0 \in C^0(\overline{\Omega})$; $u_0 = 0$ on $\partial\Omega$.

We discuss here the (more interesting) reaction case:

$$u_0 \geq 0 \quad (\Rightarrow u(t) \geq 0 \;\; \forall t)$$

Many other contributions in different directions (absorption problems, local theory in Lebesgue spaces, initial traces, etc...): [Alaa, Pierre, Barles, Da Lio, Ben Artzi, Souplet, Weissler, Bidaut Véron, Dao, Benachour, Laurençot, Dabuleanu,....]

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- $u_0 \in C_0^1(\overline{\Omega}) \Rightarrow$ (classical parabolic theory) there exists a maximal **classical** C^1 **solution** in $[0, T^*)$, where $T^* = T^*(u_0) \in (0, \infty]$.

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- But: $\|u(t)\|_{C^1}$ may blow-up

$$\rightsquigarrow \quad \underline{\text{gradient blow-up:}} \ \lim_{t\uparrow T^*} \|Du(t)\|_\infty = +\infty.$$

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Previous contributions to the question of blow-up of classical solutions: see e.g. [Alikakos-Bates-Grant],[Conner-Grant], [Guo-Hu], [Fila-Lieberman], [Arrieta-Bernal Rodriguez-Souplet], [Li-Souplet], [Quittner-Souplet], [Souplet-Zhang], [Souplet-Vazquez],...

Gradient blow-up

Well known facts (see e.g. [Souplet '02], [Souplet-Zhang '06]):

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(ii) u_t is bounded for all times (L^∞-contraction):

$$\|u(t+h) - u(t)\|_\infty \leq \|u(t_0+h) - u(t_0)\|_\infty \qquad \forall t > t_0$$

and is enough to take some t_0 smaller than the blow-up time so

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(iii) At any time t, $u(t)$ solves

$$-\Delta u = |Du|^p + f \qquad \text{with } f\in L^\infty$$

Stationary gradient bounds [Lions '85] imply

$$|Du(t,x)| \le C\, d(x)^{-\frac{1}{p-1}} \qquad d(x) = \text{dist}(x,\partial\Omega)$$

Continuation after blow-up

- There exists a unique global *relaxed* viscosity solution

$$u \in C([0,\infty) \times \overline{\Omega}) , \quad u_t - \Delta u = |Du|^p$$

[Barles-Da Lio '04] : existence and uniqueness of continuous viscosity solutions with relaxed boundary conditions:

for $x \in \partial\Omega$, if $u(t,x) > 0$ then $u_t - \Delta u \leq |Du|^p$ in viscosity sense

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- The unique global (viscosity) solution u is the increasing limit of smooth solutions of truncated problems:

$$\begin{cases} \partial_t u_n - \Delta u_n = F_n(\nabla u_n), \\ u_n(0) = u_0, \quad u_{n|\partial\Omega} = 0, \end{cases} \quad \Rightarrow \quad u_n \uparrow u$$

where F_n has natural growth and $F_n(\xi) \uparrow |\xi|^p$

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$$\int_\Omega u(t)\varphi_1\,dx \geq \int_\Omega u_0\varphi_1\,dx + \int_0^t\int_\Omega |\nabla u|^p\varphi_1\,dxds - \lambda_1\int_0^t\int_\Omega u\varphi_1\,dxds$$

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As long as $u = 0$ on the boundary, Poincaré inequality implies

$$\begin{aligned}\left(\int_\Omega u\varphi_1\,dx\right)^p &\leq \left(\int_\Omega u^k\,dx\right)^{p/k} \leq C\left(\int_\Omega |\nabla u|^k\,dx\right)^{p/k}\\ &\leq \left(\int_\Omega |\nabla u|^p\varphi_1\,dx\right)\underbrace{\left(\int_\Omega \varphi_1^{-k/(p-k)}\,dx\right)^{p/k-1}}_{\leq c}\end{aligned}$$

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by choosing $1 < k < \frac{p}{2}$. Therefore

$$\int_\Omega u(t)\varphi_1\, dx \geq \int_\Omega u_0\varphi_1\, dx + c_0 \int_0^t \left[\left(\int_\Omega u\varphi_1\, dx\right)^p - c_1^p\right] ds$$

and this would lead to a blow-up if $\int_\Omega u_0\varphi_1\, dx$ is too large.

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$\rightsquigarrow$ u must lose the boundary condition ($\Rightarrow$ gradient must blow-up)

- Long time behavior

  [Benachour-Dabuleanu Hapca-Laurencot '07], [P.-Zuazua '12]

  For all initial data u_0, there exists C such that

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  For all initial data u_0, there exists C such that

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  What's more: there exists time $T_0 \gtrsim K \|u_0\|_\infty$ such that

  $$u(t) \in W_0^{1,\infty}(\Omega) \quad \text{and } \underline{\text{is a classical } C^1\text{-solution}} \text{ for } t \ge T_0 \text{ and}$$

  $$\|Du(t)\|_\infty \le C \frac{e^{-\lambda_1 t}}{\sqrt{t}} \qquad \forall t \ge T_0$$

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  For all initial data u_0, there exists C such that

$$\|u(t)\|_\infty \leq Ce^{-\lambda_1 t}$$

  What's more: there exists time $T_0 \gtrsim K \|u_0\|_\infty$ such that

$u(t) \in W_0^{1,\infty}(\Omega)$ and is a classical C^1-solution for $t \geq T_0$ and

$$\|Du(t)\|_\infty \leq C \frac{e^{-\lambda_1 t}}{\sqrt{t}} \qquad \forall t \geq T_0$$

So not only there is life after blow-up, but there is also a happy ending

$\rightsquigarrow$ whatever u_0 is, solutions eventually become classical again and behave like the heat equation

So far, three main features appear in the qualitative behavior of the evolution problem:

(i) gradient blow-up

(blow-up rate, blow-up profile,...?)

(ii) loss of boundary condition

(when and how does it occur ?)

(iii) recovery of boundary condition and regularization

(a totally new issue ! How this smoothing property can be described?...)

Jointly with with P. Souplet, we investigated the relations between these 3 issues.

1. Blow-up rate (and regularization rate)

In [P-Souplet '19] we give some estimates for the rate of blow-up and regularization:

$T^* :=$ blow-up time $\qquad\qquad$ $T^r :=$ regularization time

The regularization time $T^r = T^r(u_0)$ means the (ultimate) time after which the solution remains classical for ever:

$$T^r(u_0) := \inf\left\{\tau > T^*(u_0);\ u(t,\cdot) \in C^1_0(\overline{\Omega}) \text{ for all } t > \tau\right\} \in [T^*, \infty).$$

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In both cases, we show that the minimal rate has order $1/(p-2)$:

if blow-up time $T^* < \infty$, then $\quad \|\nabla u(t)\|_\infty \geq \dfrac{C}{(T^* - t)^{1/(p-2)}}, \quad t \uparrow T^*$

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and similarly at the regularization time:

$$\|\nabla u(t)\|_\infty \geq \frac{C_2}{(t - T^r)^{1/(p-2)}}, \quad t \downarrow T^r$$

Remarks:

- The blow-up rate $(T^* - t)^{-1/(p-2)}$ is known to be optimal for time-increasing solutions (1d or radial, see [Conner-Grant], [Guo-Hu],[Attouchi-Souplet]).

  However, faster blow-up rates can occur !! (see later....)

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  However, faster blow-up rates can occur !! (see later....)

- This is not the self-similar rate of the equation. The self-similar scaling would lead to a rate of order $1/(p-1)$.

  $$u_\lambda(x,y,t) := \lambda^m u(\lambda x, \lambda y, \lambda^2 t) \quad \text{with } m = (2-p)/(p-1),$$

  $$\rightsquigarrow \qquad \nabla u_\lambda = \lambda^{1/(p-1)} \nabla u(\lambda x, \lambda y, \lambda^2 t).$$

  The self-similar rate is actually the rate of the spatial blow-up of normal derivative (see also [Filippucci-Pucci-Souplet])

  $\rightsquigarrow$ for single point blow-up, in $2-d$ we proved ([P-Souplet '17]) that the blow-up profile is strongly anisotropic:
  $1/(p-2)$ for the time rate
  $1/(p-1)$ of the normal spatial rate
  $2/(p-2)$ for the tangential spatial rate

2. Loss of boundary conditions

In [P.-Souplet '17], we show that loss of boundary condition may or may not occur, strongly depending on the initial data.

- there exist initial data for which the loss of boundary conditions occurs **everywhere on $\partial\Omega$**

  (see also [Quaas-Rodriguez] in the setting of viscosity solutions)

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  (see also [Quaas-Rodriguez] in the setting of viscosity solutions)

- one can find solutions for which the loss of boundary conditions occurs essentially only on a prescribed open subset of $\partial\Omega$

  $\forall\omega$ open set of R^n, and $\forall\varepsilon > 0$, there exists u_0:

  $$\omega \cap \partial\Omega \;\subset\; \Sigma_u \;\subset\; [\omega + B_\varepsilon(0)] \cap \partial\Omega$$

  where Σ_u is the set where the boundary condition is lost.

  In other words, one can prepare the initial datum u_0 so that loss of boundary condition occurs at his/her favorite subset of $\partial\Omega$

- gradient blowup may occur without loss of boundary conditions

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  [P-Souplet] in 1-d, and [Filippucci-Pucci-Souplet] in more generality

  Actually, for fixed $u_0 \geq 0$, this happens at the critical value

$$\bar{\lambda} := \inf\{\lambda > 0 \,:\, T^*(\lambda u_0) < \infty\}$$

  If $\lambda u_0 \rightsquigarrow u_\lambda$, then

  - u_λ blows-up if and only if $\lambda \geq \bar{\lambda}$
  - u_λ loses the boundary value if and only if $\lambda > \bar{\lambda}$

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This distinguishes between *minimal blow-up solutions* and *non minimal* ones:

u is a minimal blow-up solution if, for any $v_0 \leq u_0$,
the corresponding solution v is smooth for all times

Be careful: minimal blow-up solutions may have faster blow-up rates!

3. The one-dimensional case: a precise picture of loss and recovery of boundary conditions.

Take initial data u_0 which satisfy

$$u_0 \text{ is symmetric w.r.t. } x = \tfrac{1}{2}, \;\; u_0' \geq 0 \text{ on } [0, \tfrac{1}{2}],$$
$$u_0(0) = u_0''(0) + {u_0'}^p(0) = 0$$

$$\text{there exists } a \in (0,1) \text{ such that } u_0'' + {u_0'}^p \begin{cases} > 0 & \text{on } [0, a) \\ < 0 & \text{on } (a, \tfrac{1}{2}]. \end{cases} \tag{3}$$

Set as before

$$T^* = \text{blow-up time}$$
$$T^r = \text{regularization time}$$

[P-Souplet '19]: a detailed picture of minimal and non minimal solutions in 1-d

1. u is a minimal blow-up solution if and only if it does not lose the boundary condition. Moreover:

   (i) There is instantaneous and permanent regularization at T^*, u becomes immediately smooth once and for ever after T^*.

   (ii) both the blow-up and the regularization rate are faster than the minimal rate:

$$(T^* - t)^{1/(p-2)} \|u_x(t)\|_\infty \to \infty, \quad t \uparrow T^*$$
$$(t - T^*)^{1/(p-2)} \|u_x(t)\|_\infty \to \infty, \quad t \downarrow T^*$$

   Rmk: for a class of non monotone in time solutions, [Attouchi-Souplet] recently proved a blow-up rate of $2/(p-2)$!...

2. u is a non minimal blow-up solution if and only if it loses the boundary condition. Moreover:

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(i) The blow-up and regularization rates are both minimal:

$$c_1(T^*-t)^{-1/(p-2)} \leq \|u_x(t)\|_\infty \leq c_2(T^*-t)^{-1/(p-2)}, \qquad \text{as } t \to T^*_-,$$

$$c_3(t-T^r)^{-1/(p-2)} \leq \|u_x(t)\|_\infty \leq c_4(t-T^r)^{-1/(p-2)}, \qquad \text{as } t \to T^r_+,$$

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(ii) u linearly detaches from the boundary condition and linearly reconnects:

$$u(t,0) \sim \ell_1(t - T^*), \qquad \text{as } t \to T^*_+,$$
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(iii) There is immediate and permanent regularization after the recovery of boundary conditions

$u(t,0) > 0$ for all $t \in (T^*, T^r)$,

then u becomes a classical solution again and for ever

We also describe the life of the solution in the time interval when the boundary condition is relaxed. Actually, we have

(iii) In the interval $[T^*, T^r]$, the solution behaves like a shifted copy of the singular stationary profile:

$$u(t,x) \simeq u(t,0) + c_p\, x^{-1/(p-1)} \qquad \text{as } x \to 0^+,$$

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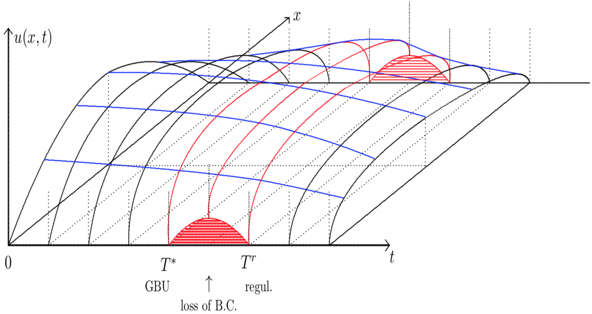
Moreover

$u(t,0)$ is C^1 in $[T^*, T^r]$, admits a unique max. at some $T_m \in (T^*, T^r)$, it is increasing on $[T^*, T_m]$ and decreasing on $[T_m, T^r]$



Key point in the above results is the zero number argument applied to u_t.

- u_t satisfies a linear equation $z_t - z_{xx} = b z_x$, $b = p|u_x|^{p-2}u_x$, so up to $t = T^*$ one can apply the zero number property ([Matano], [Angenent])

$\rightarrow$ the number of sign changes of u_t, say $N(t)$, decreases with time.

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$\rightsquigarrow$ there are (roughly speaking) only two cases:

(i) either the zero of u_t collapses at the boundary as $t \uparrow T^*$
Then $u_t \le 0$ at $t = T^*$ (and later), and the boundary condition can not be lost (minimal solution)

(ii) or u_t remains positive near $x = 0$ at $t = T^*$, then $u_t(t, 0)$ jumps at $t = T^*$ and u leaves the boundary condition and becomes positive on $\partial\Omega$ (non minimal solution)

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Important: Recent results by [Mizoguchi-Souplet] show that the behavior can be much more weird for other initial data: gradient blow-up and loss/recovery of boundary condition may happen several times !!

Conclusions, comments,...

- Perturbing the Laplace (or the heat) equation with super quadratic coercive first order terms lead to a surprisingly rich hybrid model of a second order equation which shares properties of first order equations.

- The stationary equation already shows the possible competition between oscillations (due to the Laplacian) and coercivity (given by the Hamiltonian).
  It gives much more rigidity compared to a second order equation: the coercive term may prevent too singular oscillations and gives prescriptions on size and regularity of solutions up the boundary

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- The evolution case shows a completely new issue for second order equations: the loss and recovery of boundary conditions.

  Once more, the competition between the possible formation of singularity (gradient blow-up) and the stabilizing effects (due to the maximum principle) leaves room for a rich variety of phenomena: anisotropic blow-up rates, profiles, etc...

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Thanks for the attention !

Happy Birthday to Marie-Françoise and Laurent !

Wishing you and us all that for your next celebration we will have recovered our boundary conditions...:

- coffee breaks
- dinners to share
- random walks in stranger countries
- personal discussions
- claps after the talks...:)
- ....