On (p, N) problems with critical exponential nonlinearities

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June 1–3, 2020 — Workshop on

Singular problems associated to quasilinear equations

by **Quoc–Hung Nguyen** and **Phuoc–Tai Nguyen**

in honor of ***Marie–Françoise Bidaut–Véron*** and ***Laurent Véron***

June 2, 2020 – 10:30-11:20

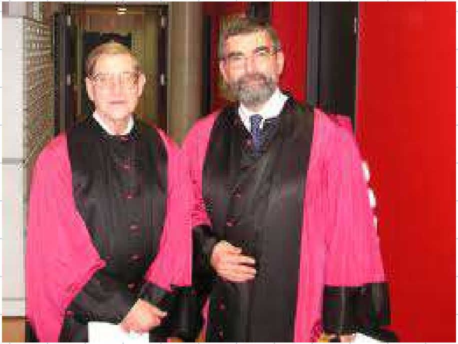
Berkeley 1985

Workshop on Nonlinear Diffusion Equations and their Equilibrium States, held on August 1989 at the *Gregynog Center* of the University College of Wales organized by J. Serrin, L.A. Peletier and W.-M. Ni.

Reaction Diffusion Systems, held on October 1995 at the University of Trieste organized by G. Caristi, E. Mitidieri and K.P. Rybakowski.

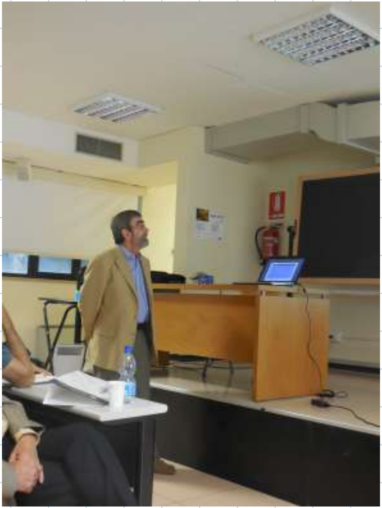
USA–Chile Workshop on Nonlinear Analysis held on January 2000 at the Universidad Federico Santa Maria at Vina del Mar organized by P. Felmer, M. Del Pino, R. Manasevich, P. Rabinowitz and E. Tuma.

Vina del Mar, January 2000



Nonlinear Partial Differential Equations and Applications held on June 2005 at the University of Tours organized by G. Barles and L. Véron.

Liouville Theorems and Detours held on May 2008 at Palazzone of Cortona organized by E. Lanconelli, E. Mitidieri, S. Pokhozhaev and A. Tertikas.



Recent Trends in Nonlinear Partial Differential Equations and Applications – on the occasion of the 60th birthday of Enzo Mitidieri held on May 2014 at the University of Trieste organized by L. D’Ambrosio, D. Del Santo, F. Gazzola, J. Lopez–Gomez, P. Omari and P. Pucci.

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Environment

The importance of studying problems involving (p, q) operators, or operators with non–standard growth conditions, begins with the following pioneering papers

- P. Marcellini, *On the definition and the lower semicontinuity of certain quasiconvex integrals*, Ann. Inst. H. Poincaré Anal. Non Linéaire 3 (1986).
- P. Marcellini, *Regularity and existence of solutions of elliptic equations with (p, q)–growth conditions*, J. Differential Equations 90 (1991).
- P. Marcellini, *Regularity for elliptic equations with general growth conditions*, J. Differential Equations 105 (1993).
- V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986).

Environment

We recall that a (p,q) elliptic operator of *Marcellini* type is an operator whose energy functional is given as

$$I(u) = \int_\Omega \mathcal{A}(x, \nabla u(x))dx, \qquad u : \Omega \to \mathbb{R},$$

with energy density $\mathcal{A} : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$|t|^p \leq \mathcal{A}(x,t) \leq |t|^q + 1, \qquad 1 \leq p \leq q$$

for any $(x,t) \in \Omega \times \mathbb{R}$. This definition covers the canonical examples as

- $\Delta_p u + \Delta_q u = \operatorname{div}(|\nabla u|^{p-2}u + |\nabla u|^{q-2}u)$ the (p,q) Laplace operator;
- $\operatorname{div}(|\nabla u|^{p-2}u + a(x)|\nabla u|^{q-2}u)$ the double phase operator;
- $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}u)$ the $p(x)$ Laplace operator.

Last References

- A. Fiscella, P. P., *Degenerate Kirchhoff (p, q)–fractional systems with critical nonlinearities*, submitted for publication, pages 21.
- P. P., L. Temperini, *Existence for (p, q) critical systems in the Heisenberg group*, Adv. Nonlinear Anal. 9 (2020), 895–922.
- A. Fiscella, P. P., *(p, q) systems with critical terms in $\mathbb{R}^N$*, Special Issue Nonlinear PDEs and Geometric Function Theory, in honor of Carlo Sbordone on his 70th birthday, Nonlinear Anal. 177 Part B (2018), 454–479.
- Y. Fu, H. Li, P. P., *Existence of Nonnegative Solutions for a Class of Systems Involving Fractional (p, q)–Laplacian Operators*, Chin. Ann. Math. Ser. B, special volume dedicated to Professor Philippe G. Ciarlet on the occasion of his 80th birthday, 39 (2018), 357–372.

Main equation

In [FP1] we study the equation in $\mathbb{R}^N$

$$(\mathcal{E}) \quad -\Delta_p u - \Delta_N u + |u|^{p-2}u + |u|^{N-2}u = \lambda h(x) u_+^{q-1} + \gamma f(x,u),$$

where

- $-\Delta_p u - \Delta_N u = \operatorname{div}(|\nabla u|^{p-2}u + |\nabla u|^{N-2}u)$;
- $1 < p < N < \infty$;
- $1 < q < N$;
- $u_+ = \max\{u, 0\}$;
- $h \in L^\theta(\mathbb{R}^N)$ is positive, with $\theta = N/(N-q)$;
- λ and γ are positive parameters.

[FP1] A. Fiscella, P. P., (p,N) *equations with critical exponential nonlinearities in* $\mathbb{R}^N$, J. Math. Anal. Appl. *Special Issue New Developments in Nonuniformly Elliptic and Nonstandard Growth Problems*, doi.org/10.1016/j.jmaa.2019.123379

Main equation

The function f is of exponential type and satisfies

(f_1) *f is a Carathéodory function, with $f(\cdot, u) = 0$ for all $u \leq 0$, and such that there exists $\alpha_0 > 0$ with the property that for all $\varepsilon > 0$ there exists $\kappa_\varepsilon > 0$ such that*

$$f(x,u) \leq \varepsilon\, u^{N-1} + \kappa_\varepsilon \left(e^{\alpha_0 u^{N'}} - S_{N-2}(\alpha_0, u) \right)$$

for a.e. $x \in \mathbb{R}^N$ and all $u \in \mathbb{R}_0^+$, where $\mathbb{R}_0^+ = [0, \infty)$,

$$N' = \frac{N}{N-1} \quad \text{and} \quad S_{N-2}(\alpha_0, u) = \sum_{j=0}^{N-2} \frac{\alpha_0^j u^{jN'}}{j!};$$

(f_2) *there exists a number $\nu > N$ such that $0 < \nu F(x,u) \leq uf(x,u)$ for a.e $x \in \mathbb{R}^N$ and any $u \in \mathbb{R}^+$, $\mathbb{R}^+ = (0, \infty)$, where $F(x,u) = \int_0^u f(x,t)dt$ for a.e. $x \in \mathbb{R}^N$ and all $u \in \mathbb{R}$.*

Preliminaries

Lemma (Trudinger 1967, Moser 1971)

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. For any $u \in W_0^{1,N}(\Omega)$ with $\|u\|_{W_0^{1,N}} \le 1$, there exists $C = C(N,\Omega) > 0$ such that

$$\int_\Omega e^{\alpha |u|^{\frac{N}{N-1}}}\, dx \le C,$$

for any $\alpha \le \alpha_N$, where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the $(N-1)$–dimensional measure of the unit sphere S^{N-1} of $\mathbb{R}^N$.

Lemma (do Ó 1997)

For any $u \in W^{1,N}(\mathbb{R}^N)$ with $\|\nabla u\|_N \le 1$ and $\|u\|_N \le M$, if $\alpha < \alpha_N$ there exists $C = C(N,M,\alpha) > 0$ such that

$$\int_{\mathbb{R}^N} \left(e^{\alpha |u|^{\frac{N}{N-1}}} - S_{N-2}(\alpha, |u|) \right) dx \le C, \quad \textit{where}$$

$$S_{N-2}(\alpha,|u|) = \sum_{j=0}^{N-2} \frac{\alpha^j |u|^{jN'}}{j!}, \quad N' = \frac{N-1}{N}.$$

Main equation

When $N = 2$, a classical example of function verifying (f_1)–(f_2) is given by

$$f(u) = u_+\left(e^{u_+^2} - 1\right), \quad u \in \mathbb{R}.$$

For this model the main involved numbers are $\alpha_0 > 1$ and $\nu = 4 > 2 = N$.

Similarly, in the general case $N > 2$ the example becomes

$$f(u) = u_+^{N-1}\left(e^{u_+^{N'}} - S_{N-2}(1, u_+)\right), \quad \text{with}$$

$$S_{N-2}(1, u_+) = \sum_{j=0}^{N-2} \frac{u_+^{jN'}}{j!},$$

so that $\alpha_0 > 1$ and $\nu = 2N$.

Clearly, any function $g(x, u) = a(x)f(u)$, where a is a positive measurable function, with $a \in L^\infty(\mathbb{R}^N)$ and $\operatorname{ess\,inf}_{x \in \mathbb{R}^N} a(x) > 0$, and $f(u)$ defined as above, verifies (f_1)–(f_2).

Main equation

The natural space where finding solutions of

$$(\mathcal{E}) \;\; -\Delta_p u - \Delta_N u + |u|^{p-2}u + |u|^{N-2}u = \lambda h(x) u_+^{q-1} + \gamma f(x,u)$$

is the intersection space

$$W = W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N),$$

endowed with the norm

$$\|u\| = \|u\|_{W^{1,p}} + \|u\|_{W^{1,N}},$$

where $\|u\|_{W^{1,\mathfrak{p}}} = \left(\|u\|_{\mathfrak{p}}^{\mathfrak{p}} + \|\nabla u\|_{\mathfrak{p}}^{\mathfrak{p}}\right)^{1/\mathfrak{p}}$ for all $u \in W^{1,\mathfrak{p}}(\mathbb{R}^N)$ and $\|\cdot\|_{\mathfrak{p}}$ denotes the canonical $L^{\mathfrak{p}}(\mathbb{R}^N)$ norm for any $\mathfrak{p} > 1$.

First solution

Theorem 1.1 of [FP1]

Let $1 < p < N < \infty$ *and* $1 < q < N$. *Let* h *be a positive function in* $L^{\theta}(\mathbb{R}^N)$, *with* $\theta = N/(N-q)$. *Suppose that* f *verifies* (f_1)–(f_2). *Then, there exists* $\widetilde{\lambda} > 0$ *such that equation*

$$(\mathcal{E}) \quad -\Delta_p u - \Delta_N u + |u|^{p-2}u + |u|^{N-2}u = \lambda h(x) u_+^{q-1} + \gamma f(x,u)$$

admits at least one nontrivial nonnegative solution $u_{\lambda,\gamma}$ *in* W *for all* $\lambda \in (0, \widetilde{\lambda})$ *and all* $\gamma > 0$. *Moreover,* $\lim\limits_{\lambda \to 0^+} \|u_{\lambda,\gamma}\| = 0$.

The proof of Theorem 1.1 is based on the application of the Ekeland variational principle.

[FP1] A. Fiscella, P. P., (p,N) *equations with critical exponential nonlinearities in* $\mathbb{R}^N$, J. Math. Anal. Appl. *Special Issue New Developments in Nonuniformly Elliptic and Nonstandard Growth Problems*, doi.org/10.1016/j.jmaa.2019.123379

Comments

Theorem 1.1 extends the existence results of

- C.O. Alves, L.R. de Freitas, S.H.M. Soares, *Indefinite quasilinear elliptic equations in exterior domains with exponential critical growth*, Differential Integral Equations 24 (2011).
- D.G. de Figueiredo, O.H. Miyagaki, B. Ruf, *Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range*, Calc. Var. 3 (1995).
- J.M. do Ó, *N–Laplacian equations in $\mathbb{R}^N$ with critical growth*, Abstr. Appl. Anal. 2 (1997).
- J.M. do Ó, E. Medeiros, U. Severo, *On a quasilinear nonhomogeneous elliptic equation with critical growth in $\mathbb{R}^N$*, *J. Differential Equations* 246 (2009).
- G.M. Figueiredo, F.B.M. Nunes, *Existence of positive solutions for a class of quasilinear elliptic problems with exponential growth via the Nehari manifold method*, Rev. Mat. Complut. 32 (2019).

Main equation

In order to get also a mountain pass solution for $(\mathcal{E})$, we need to replace (f_1) with the stronger assumption

$(f_1)'$ *$\partial_u f$ is a Carathéodory function, with $\partial_u f(\cdot, u) = 0$ for all $u \leq 0$, and such that there exists $\alpha_0 > 0$ with the property that for all $\varepsilon > 0$ there exists $\kappa_\varepsilon > 0$ such that*

$$\partial_u f(x, u)u \leq \varepsilon\, u^{N-1} + \kappa_\varepsilon \left(e^{\alpha_0 u^{N'}} - S_{N-2}(\alpha_0, u)\right)$$

for a.e. $x \in \mathbb{R}^N$ and all $u \in \mathbb{R}_0^+$,

and to assume furthermore that condition

(f_3) *there exist $\wp > N$ and $C > 0$ such that*

$$F(x, u) \geq \frac{C}{2\wp} u^{\wp} \quad \textit{for a.e. } x \in \mathbb{R}^N \textit{ and any } u \in \mathbb{R}_0^+$$

holds true.

Main equation

Of course for the prototype

$$f(u) = u_+\left(e^{u_+^2} - 1\right), \quad u \in \mathbb{R},$$

when $N = 2$, the associated partial derivative $\partial_u f$ verifies $(f_1)'$, with $\alpha_0 > 1$, while its primitive

$$F(u) = \left(e^{u_+^2} - 1 - u_+^2\right)/2$$

satisfies (f_3), with $\wp = \nu = 4$ and $C = 2$.

In the case $N > 2$, for the example

$$f(u) = u_+^{N-1}\left(e^{u_+^{N'}} - S_{N-2}(1, u_+)\right), \quad \text{with}$$

$$S_{N-2}(1, u_+) = \sum_{j=0}^{N-2} \frac{u_+^{jN'}}{j!},$$

again $\partial_u f$ verifies $(f_1)'$, with $\alpha_0 > 1$. While, the corresponding primitive F satisfies (f_3), with $\wp = 2N$ e $C = 2/(N-1)!$.

Second solution

Theorem 1.2 of [FP1]

Let $1 < p < N < \infty$ *and* $1 < q < N$. *Let* h *be a positive function in* $L^\theta(\mathbb{R}^N)$, *with* $\theta = N/(N-q)$. *Suppose that* f *verifies* $(f_1)'$, (f_2)–(f_3).

Then, there exists $\gamma^* > 0$ *such that for all* $\gamma > \gamma^*$ *there exists* $\widehat{\lambda} = \widehat{\lambda}(\gamma) > 0$ *with the property that equation*

$$(\mathcal{E}) \quad -\Delta_p u - \Delta_N u + |u|^{p-2}u + |u|^{N-2}u = \lambda h(x) u_+^{q-1} + \gamma f(x,u)$$

admits a nontrivial nonnegative solution $v_{\lambda,\gamma}$ *in* W *for all* $\lambda \in (0, \widehat{\lambda}]$. *Furthermore, if* $\lambda < \min\{\widetilde{\lambda}, \widehat{\lambda}\}$, *then* $v_{\lambda,\gamma}$ *is a second solution of* $(\mathcal{E})$ *independent of* $u_{\lambda,\gamma}$ *constructed in Theorem* 1.1.

[FP1] A. Fiscella, P. P., (p,N) *equations with critical exponential nonlinearities in* $\mathbb{R}^N$, J. Math. Anal. Appl. *Special Issue New Developments in Nonuniformly Elliptic and Nonstandard Growth Problems*, doi.org/10.1016/j.jmaa.2019.123379

Comments

The proof of Theorem 1.2 is based on the application of a tricky step analysis of the critical mountain pass level, somehow inspired by

- A. Fiscella, P. Pucci, *Degenerate Kirchhoff (p,q)–fractional problems with critical nonlinearities*, submitted for publication.

Beside the works quoted before, Theorem 1.2 generalizes the multiplicity result proved in the paper

- L.R. de Freitas, *Multiplicity of solutions for a class of quasilinear equations with exponential critical growth*, Nonlinear Anal. 95 (2014).

which deals with the equation in $\mathbb{R}^N$

Comments

$$(*) \qquad -\Delta_N u + |u|^{N-2}u = \lambda h(x)|u|^{q-2}u + \gamma f(u),$$

where f does not depend on $x \in \mathbb{R}^N$, such as in $(\mathcal{E})$. This fact is due to the variational approach in

- L.R. de Freitas, *Multiplicity of solutions for a class of quasilinear equations with exponential critical growth*, Nonlinear Anal. 95 (2014).

which is strongly based on the study of a critical level for $(*)$ when $\lambda = 0$. In order to get this critical level, the use of the homogeneity of the N–Laplace operator is a crucial requirement.

In [FP1], the presence of the (p, N) operator in $(\mathcal{E})$ does not allow us to adopt the same approach. A crucial point in our argument is, among others, the use of a completely new Brézis and Lieb type lemma for exponential nonlinearities.

Lemma 4.3 of [FP1]

Let $(u_k)_k$ be a sequence in W and let u be in W such that $u_k \rightharpoonup u$ in W, $\|u_k\|_{W^{1,N}} \to \upsilon_N$, $u_k \to u$ a.e. in $\mathbb{R}^N$, $\nabla u_k \to \nabla u$ a.e. in $\mathbb{R}^N$ and

$$\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,N}}^{N'} < \frac{\alpha_N}{2^z\alpha_0}$$

hold true, with $z \geq N' + 1$ and $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, where ω_{N-1} is the $(N-1)$–dimensional measure of the unit sphere S^{N-1} of $\mathbb{R}^N$. Then,

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \left| f(x,u_k)u_k - f(x,u-u_k)(u-u_k) - f(x,u)u \right| dx = 0.$$

Moreover, $\displaystyle\lim_{k\to\infty} \int_{\mathbb{R}^N} \left| F(x,u_k) - F(x,u-u_k) - F(x,u) \right| dx = 0.$

Proof

[FP1] A. Fiscella, P. P., (p,N) *equations with critical exponential nonlinearities in* $\mathbb{R}^N$, J. Math. Anal. Appl. *Special Issue New Developments in Nonuniformly Elliptic and Nonstandard Growth Problems*, doi.org/10.1016/j.jmaa.2019.123379

Comments

For the case $N = 1$, on a fractional framework $H^{1/2}(\mathbb{R})$, we can refer to

- J.M. do Ó, O.H. Miyagaki, M. Squassina, *Ground states of nonlocal scalar field equations with Trudinger–Moser critical nonlinearity*, Topol. Methods Nonlinear Anal. 48 (2016).

Main system

Question: what happens on a vectorial setting?

Main system

In [CFPT] and [FP2] we study the system in $\mathbb{R}^N$

$$(\mathcal{S})\begin{cases} -\Delta_p u - \Delta_N u + |u|^{p-2}u + |u|^{N-2}u = \lambda h(x) u_+^{q-1} + \gamma F_u(x,u,v), \\ -\Delta_p v - \Delta_N v + |v|^{p-2}v + |v|^{N-2}v = \mu h(x) v_+^{q-1} + \gamma F_v(x,u,v), \end{cases}$$

where

- $1 < p < N < \infty$;
- $1 < q < N$;
- $u_+ = \max\{u, 0\}$;
- $h \in L^\theta(\mathbb{R}^N)$ is positive, with $\theta = N/(N-q)$;
- λ, μ and γ are positive parameters.

[CFPT] S. Chen, A. Fiscella, P. P., X. Tang, *Coupled elliptic systems in $\mathbb{R}^N$ with (p,N) Laplacian and critical exponential nonlinearities*, submitted for publication.
[FP2] A. Fiscella, P. P., *Entire solutions for (p,N) systems with coupled critical exponential nonlinearities*, submitted for publication.

Main system

While F_u, F_v are partial derivatives of a Carathéodory function F, of exponential type, satisfying

(H_1) *$F(x,\cdot,\cdot) \in C^1(\mathbb{R}^2)$ for a.e. $x \in \mathbb{R}^N$, $F_u(x,u,v) = 0$ for all $u \le 0$ and $v \in \mathbb{R}$, $F_v(x,u,v) = 0$ for all $u \in \mathbb{R}$ and $v \le 0$, $F_u(x,u,0) = 0$ for all $u \in \mathbb{R}$, and $F_v(x,0,v) = 0$ for all $v \in \mathbb{R}$. Furthermore, there is $\alpha_0 > 0$ with the property that for all $\varepsilon > 0$ there exists $\kappa_\varepsilon > 0$ such that*

$$|F_z(x,z)| \le \varepsilon\, |z|^{N-1} + \kappa_\varepsilon \left(e^{\alpha_0 |z|^{N'}} - S_{N-2}(\alpha_0, |z|) \right)$$

for a.e. $x \in \mathbb{R}^N$ and all $z = (u,v)$, with $u,\ v \in \mathbb{R}^+_0$, where $|z| = \sqrt{u^2+v^2}$, $F_z = (F_u, F_v)$,

$$N' = \frac{N}{N-1} \quad \text{and} \quad S_{N-2}(\alpha,t) = \sum_{j=0}^{N-2} \frac{\alpha^j t^{jN'}}{j!},\ \alpha > 0,\ t \in \mathbb{R};$$

(H_2) *there exists $\nu > N$ such that $0 < \nu F(x,z) \le F_z(x,z)\cdot z$ for a.e. $x \in \mathbb{R}^N$ and any $z = (u,v)$, with $u,\ v \in \mathbb{R}^+$.*

Main system

When $N = 2$, a classical example of function verifying (H_1)–(H_2) is given by

$$F(u,v) = e^{u_+v_+} - u_+v_+ - 1,$$

for $(u,v) \in \mathbb{R}^2$, with partial derivatives

$$F_u(u,v) = v_+\left(e^{u_+v_+} - 1\right), \qquad F_v(u,v) = u_+\left(e^{u_+v_+} - 1\right).$$

For this model the main involved numbers are $\alpha_0 > 1/2$ and $\nu = 4 > 2 = N$. Moreover, $F_u(t,t) = F_v(t,t)$ for all $t \in \mathbb{R}$.

Another interesting example in the case $N = 2$ is given by

$$F(u,v) = \begin{cases} u^2(e^{v^2} - v^2 - 1) & \text{if } (u,v) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ 0 & \text{otherwise.} \end{cases}$$

Again F satisfies (H_1)–(H_2), with $\alpha_0 > 1$ and $\nu = 6 > 2 = N$. But in this case $0 < F_u(t,t) < F_v(t,t)$ for all $t \in \mathbb{R}^+$.

Main system

The natural space where finding solutions of

$$(\mathcal{S})\begin{cases} -\Delta_p u - \Delta_N u + |u|^{p-2}u + |u|^{N-2}u = \lambda h(x)u_+^{q-1} + \gamma F_u(x,u,v), \\ -\Delta_p v - \Delta_N v + |v|^{p-2}v + |v|^{N-2}v = \mu h(x)v_+^{q-1} + \gamma F_v(x,u,v), \end{cases}$$

is the intersection space

$$\mathbf{W} = \left(W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)\right) \times \left(W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)\right),$$

endowed with the norm

$$\|(u,v)\| = \|u\|_{W^{1,p}} + \|v\|_{W^{1,p}} + \|u\|_{W^{1,N}} + \|v\|_{W^{1,N}},$$

where $\|u\|_{W^{1,\mathfrak{p}}} = \left(\|u\|_{\mathfrak{p}}^{\mathfrak{p}} + \|\nabla u\|_{\mathfrak{p}}^{\mathfrak{p}}\right)^{1/\mathfrak{p}}$ for all $u \in W^{1,\mathfrak{p}}(\mathbb{R}^N)$ and any $\mathfrak{p} > 1$.

We say that a pair (u,v) is nonnegative in $\mathbb{R}^N$, if both components are nonnegative in $\mathbb{R}^N$.

First solution

Theorem 1.1 of [*CFPT*]

Let $1 < p < N < \infty$ *and* $1 < q < N$. *Let* h *be a positive function in* $L^{\theta}(\mathbb{R}^N)$, *with* $\theta = N/(N-q)$. *Suppose that* F *verifies* (H_1)–(H_2). *Then, there exists* $\widetilde{\lambda} > 0$ *such that system*

$$(\mathcal{S}) \begin{cases} -\Delta_p u - \Delta_N u + |u|^{p-2}u + |u|^{N-2}u = \lambda h(x) u_+^{q-1} + \gamma F_u(x,u,v), \\ -\Delta_p v - \Delta_N v + |v|^{p-2}v + |v|^{N-2}v = \mu h(x) v_+^{q-1} + \gamma F_v(x,u,v), \end{cases}$$

admits at least one nonnegative solution $(u_{\lambda,\mu}, v_{\lambda,\mu})$ *in* $\mathbf{W}$, *with both nontrivial components, for all* $(\lambda,\mu) \in (0,\widetilde{\lambda}) \times (0,\widetilde{\lambda})$ *and all* $\gamma > 0$. *Moreover,* $\displaystyle\lim_{(\lambda,\mu)\to(0^+,0^+)} \|(u_{\lambda,\mu}, v_{\lambda,\mu})\| = 0$.

[CFPT] S. Chen, A. Fiscella, P. P., X. Tang, *Coupled elliptic systems in* $\mathbb{R}^N$ *with* (p,N) *Laplacian and critical exponential nonlinearities*, submitted for publication.

Furthermore, if $F_u(x,t,t) \equiv F_v(x,t,t)$ *for a.e.* $x \in \mathbb{R}^N$ *and all* $t \in \mathbb{R}$*, then the solution* $(u_{\lambda,\mu}, v_{\lambda,\mu})$ *has the property that* $u_{\lambda,\mu} \not\equiv v_{\lambda,\mu}$ *in* $\mathbb{R}^N$*, whenever* $\lambda \neq \mu$*. Finally, if* $\lambda = \mu$*, but* $F_u(x,t,t) \neq F_v(x,t,t)$ *for a.e.* $x \in \mathbb{R}^N$ *and all* $t \in \mathbb{R}^+$*, then* $u_{\lambda,\lambda} \not\equiv v_{\lambda,\lambda}$ *in* $\mathbb{R}^N$*.*

1st case $\quad F(u,v) = e^{u_+ v_+} - u_+ v_+ - 1 \quad \text{for } (u,v) \in \mathbb{R}^2,$

2nd case $\quad F(u,v) = \begin{cases} u^2(e^{v^2} - v^2 - 1) & \text{if } (u,v) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ 0 & \text{otherwise.} \end{cases}$

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Main system

In order to get a mountain pass solution for $(\mathcal{S})$, we need to replace (H_1) and (H_2) with the stronger assumptions

$(H_1)'$ *$F(x,\cdot,\cdot) \in C^2(\mathbb{R}^2)$ for a.e. $x \in \mathbb{R}^N$, $F_u(x,u,v) = 0$ for all $u \leq 0$ and $v \in \mathbb{R}$, $F_v(x,u,v) = 0$ for all $u \in \mathbb{R}$ and $v \leq 0$, $F_u(x,u,0) = 0$ for all $u \in \mathbb{R}$, $F_v(x,0,v) = 0$ for all $v \in \mathbb{R}$ and $F_u(x,u,v) > 0$, $F_v(x,u,v) > 0$ for all $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$. Furthermore, there is $\alpha_0 > 0$ with the property that for all $\varepsilon > 0$ there exists $\kappa_\varepsilon > 0$ such that*

$$\left|F_{uu}(x,u,v)\right| u \leq \varepsilon\,|z|^{N-1} + \kappa_\varepsilon \left(e^{\alpha_0 |z|^{N'}} - S_{N-2}(\alpha_0, |z|)\right),$$

$$\left|F_{vv}(x,u,v)\right| v \leq \varepsilon\,|z|^{N-1} + \kappa_\varepsilon \left(e^{\alpha_0 |z|^{N'}} - S_{N-2}(\alpha_0, |z|)\right),$$

$$\left|F_{uv}(x,u,v)\right| u \leq \varepsilon\,|z|^{N-1} + \kappa_\varepsilon \left(e^{\alpha_0 |z|^{N'}} - S_{N-2}(\alpha_0, |z|)\right),$$

$$\left|F_{vu}(x,u,v)\right| v \leq \varepsilon\,|z|^{N-1} + \kappa_\varepsilon \left(e^{\alpha_0 |z|^{N'}} - S_{N-2}(\alpha_0, |z|)\right),$$

Main system

for a.e. $x \in \mathbb{R}^N$ *and all* $z = (u, v)$, *with* $u,\ v \in \mathbb{R}_0^+$, *where* $\mathbb{R}_0^+ = [0, \infty)$, $|z| = \sqrt{u^2 + v^2}$,

$$N' = \frac{N}{N-1}, \qquad S_{N-2}(\alpha, t) = \sum_{j=0}^{N-2} \frac{\alpha^j t^{jN'}}{j!},\ \alpha > 0,\ t \in \mathbb{R},$$

and $F_{uu} = \partial^2 F/\partial u^2$, $F_{vv} = \partial^2 F/\partial v^2$;

$(H_2)'$ *there exists* $\nu > N$ *such that* $0 < \nu F(x, z) \leq F_z(x, z) \cdot z$ *for a.e.* $x \in \mathbb{R}^N$ *and any* $z = (u, v)$, *with* $u,\ v \in \mathbb{R}^+$ *and* $F_z = (F_u, F_v)$ *and there is* $R \geq 1$ *such that*

$$c \in L^\infty(\mathbb{R}^N) \text{ and } c \not\equiv 0, \text{ where } \quad c(x) = \inf_{u_+ v_+ \geq R} F(x, u, v);$$

and need to assume the further condition

(H_3) *there exist* $\wp \geq N$ *and* $C > 0$ *such that*

$$F(x, u, v) \geq \frac{C}{2\wp}(uv)^\wp \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and any } u,\ v \in \mathbb{R}_0^+.$$

Main system

Clearly, the example

$$F(u,v) = e^{u_+v_+} - u_+v_+ - 1 \geq (u_+v_+)^2/2$$

verifies $(H_1)'$, $(H_2)'$ and (H_3). For this model the main involved numbers are $\alpha_0 > 1/2$, $\nu = 4 > 2 = N$, $R = 1$, $c(x) \equiv e - 2$, $\wp = 2$ and $C = 1$. Moreover, $F_u(t,t) = F_v(t,t)$ for all $t \in \mathbb{R}$. While, taking into account the new regularity required in $(H_1)'$, we can consider another interesting example

$$F(u,v) = \begin{cases} u^3(e^{v^2} - v^2 - 1) + e^{uv} - uv - 1 & \text{if } (u,v) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ 0 & \text{otherwise.} \end{cases}$$

Again F satisfies $(H_1)'$, $(H_2)'$ and (H_3), with $\alpha_0 > 1$, $\nu = 4 > 2 = N$, $c > e^R - R - 1$ for all $R \geq 1$, $\wp = 2$ and $C = 1$. But in this case $0 < F_u(t,t) < F_v(t,t)$ for all $t \in \mathbb{R}^+$.

Second solution

Theorem 1.1 of [FP2]

Let $1 < q < p < N < \infty$. *Let* h *be a positive function in* $L^\theta(\mathbb{R}^N)$, *with* $\theta = N/(N-q)$. *Suppose that* F *verifies* $(H_1)'$, $(H_2)'$ *and* (H_3).
Then, there is $\gamma^* > 0$ *such that for all* $\gamma > \gamma^*$ *there exists* $\widehat{\lambda} = \widehat{\lambda}(\gamma) > 0$ *with the property that system*

$$(\mathcal{S})\begin{cases} -\Delta_p u - \Delta_N u + |u|^{p-2}u + |u|^{N-2}u = \lambda h(x) u_+^{q-1} + \gamma F_u(x,u,v), \\ -\Delta_p v - \Delta_N v + |v|^{p-2}v + |v|^{N-2}v = \mu h(x) v_+^{q-1} + \gamma F_v(x,u,v), \end{cases}$$

admits at least one nonnegative solution (u,v) *in* $\mathbf{W}$, *with both nontrivial components, for all* $(\lambda,\mu) \in (0,\widehat{\lambda}] \times (0,\widehat{\lambda}]$.

[FP2] A. Fiscella, P. P., *Entire solutions for* (p,N) *systems with coupled critical exponential nonlinearities*, submitted for publication.

Furthermore, if $F_u(x,t,t) \equiv F_v(x,t,t)$ *for a.e.* $x \in \mathbb{R}^N$ *and all* $t \in \mathbb{R}$*, then the solution* (u,v) *has the property that* $u \not\equiv v$ *in* $\mathbb{R}^N$*, whenever* $\lambda \neq \mu$*. While, if* $\lambda = \mu$*, but* $F_u(x,t,t) \neq F_v(x,t,t)$ *for a.e.* $x \in \mathbb{R}^N$ *and all* $t \in \mathbb{R}^+$*, then* $u \not\equiv v$ *in* $\mathbb{R}^N$*.*
Finally, if $\lambda, \mu < \min\{\widetilde{\lambda}, \widehat{\lambda}\}$*, then* (u,v) *is a second solution of* $(\mathcal{S})$ *independent of the one obtained in* [*CFPT*].

[CFPT] S. Chen, A. Fiscella, P. P., X. Tang, *Coupled elliptic systems in* $\mathbb{R}^N$ *with* (p,N) *Laplacian and critical exponential nonlinearities*, submitted for publication.

Comments

Let me point out that:

• none of the previous contributions on coupled exponential systems presents explicitly conditions under which the two components of the constructed solution are different. Actually, it seems that this question is not addressed at all;

• passing into the vectorial case, the assumption $1 < q < p$ is used only to show that both components are nontrivial. Indeed, if for contradiction $u \neq 0$, but $v = 0$ a.e. in $\mathbb{R}^N$, the weak formulation of $(\mathcal{S})$ and $(H_1)'$ give

$$\|u\|_{W^{1,p}}^p + \|u\|_{W^{1,N}}^N = \lambda \int_{\mathbb{R}^N} h(x) u^q dx.$$

While, since $(u, 0) \in \mathbf{W}$ would be solution at positive critical moutain pass level, again $(H_1)'$ and the fact that $N > p$ force

$$
\begin{aligned}
\frac{1}{p}\{\|u\|^p_{W^{1,p}} + \|u\|^N_{W^{1,N}}\} &\geq \frac{1}{p}\|u\|^p_{W^{1,p}} + \frac{1}{N}\|u\|^N_{W^{1,N}} \\
&> \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x)u^q dx + \gamma \int_{\mathbb{R}^N} F(x,u,0)dx \\
&\geq \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x)u^q dx.
\end{aligned}
$$

Combining the two estimates, we get

$$
\|u\|^p_{W^{1,p}} + \|u\|^N_{W^{1,N}} \leq \frac{q}{p}\{\|u\|^p_{W^{1,p}} + \|u\|^N_{W^{1,N}}\},
$$

which gives a contradiction, since $1 < q < p$.

(p, Q) equations in the Heisenberg group

Question: what happens on (p, Q) equations with critical exponential growth at infinity and a singular behavior at the origin in the Heisenberg group?

Main (p, Q) equation

In [PT] we consider the equation in $\mathbb{H}^n$

$$(\mathcal{E})\ -\Delta_{H,p}u - \Delta_{H,Q}u + |u|^{p-2}u + |u|^{Q-2}u = \frac{f(\xi,u)}{r(\xi)^\beta} + h(\xi)$$

- $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$;
- $1 < p < Q$, $0 \le \beta < Q$;
- h is a nontrivial nonnegative functional of $HW^{-1,Q'}(\mathbb{H}^n)$, where $HW^{-1,Q'}(\mathbb{H}^n)$ is the dual space of $HW^{1,Q}(\mathbb{H}^n)$;
- $r(\xi) = r(z,t) = (|z|^4 + t^2)^{1/4}$ is the *Korányi norm* in $\mathbb{H}^n$, with $\xi = (z,t) \in \mathbb{H}^n$, $z = (x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, $t \in \mathbb{R}$, $|z|$ the Euclidean norm in $\mathbb{R}^{2n}$;
- $\Delta_{H,\wp}\varphi = \mathrm{div}_H(|D_H\varphi|_H^{\wp-2} D_H\varphi)$, with $\wp \in \{p, Q\}$, is the well known $\wp$ Kohn–Spencer Laplacian.

[PT] P. P., L. Temperini, *Existence for singular critical exponential* (p,Q) *equations in the Heisenberg group*, submitted for publication

Here and in the sequel $D_H\varphi$ is the horizontal gradient of a regular function φ, that is,

$$D_H\varphi = (X_1\varphi, \cdots, X_n\varphi, Y_1\varphi, \cdots, Y_n\varphi),$$

where $\{X_j, Y_j\}_{j=1}^n$ is the standard basis of the horizontal left invariant vector fields on $\mathbb{H}^n$, that is

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t},$$

for $j = 1, \ldots, n$.

As in In [FP1]

(f_1) *f is a Carathéodory function, with $f(\cdot,u)=0$ for all $u\le 0$, and such that there exists $\alpha_0>0$ with the property that for all $\varepsilon>0$ there exists $\kappa_\varepsilon>0$ such that*

$$f(\xi,u)\le \varepsilon\, u^{Q-1}+\kappa_\varepsilon\left(e^{\alpha_0 u^{Q'}}-S_{Q-2}(\alpha_0,u)\right)$$

for a.e. $\xi\in\mathbb{H}^n$ and all $u\in\mathbb{R}^+_0$, where $\mathbb{R}^+_0=[0,\infty)$,

$$Q'=\frac{Q}{Q-1}\quad\text{and}\quad S_{Q-2}(\alpha_0,u)=\sum_{j=0}^{Q-2}\frac{\alpha_0^j u^{jQ'}}{j!};$$

(f_2) *there exists a number $\nu>Q$ such that $0<\nu F(\xi,u)\le uf(\xi,u)$ for a.e $\xi\in\mathbb{H}^n$ and any $u\in\mathbb{R}^+$, $\mathbb{R}^+=(0,\infty)$, where $F(\xi,u)=\int_0^u f(\xi,v)dv$ for a.e $\xi\in\mathbb{H}^n$ and all $u\in\mathbb{R}$.*

[FP1] A. Fiscella, P. P., *(p,N) equations with critical exponential nonlinearities in $\mathbb{R}^N$*, J. Math. Anal. Appl. *Special Issue New Developments in Nonuniformly Elliptic and Nonstandard Growth Problems*, doi.org/10.1016/j.jmaa.2019.123379

The natural solution space of

$$(\mathcal{E}) \quad -\Delta_{H,p}u - \Delta_{H,Q}u + |u|^{p-2}u + |u|^{Q-2}u = \frac{f(\xi,u)}{r(\xi)^{\beta}} + h(\xi)$$

is the separable reflexive Banach space

$$W = HW^{1,p}(\mathbb{H}^n) \cap HW^{1,Q}(\mathbb{H}^n),$$

endowed with the norm $\|u\| = \|u\|_{HW^{1,p}} + \|u\|_{HW^{1,Q}}$, where $\|u\|_{HW^{1,\wp}} = \left(\|u\|_{\wp}^{\wp} + \|D_H u\|_{\wp}^{\wp}\right)^{1/\wp}$, $\wp \in \{p, Q\}$, for all $u \in HW^{1,\wp}(\mathbb{H}^n)$.

Existence and multiplicity of nontrivial nonnegative solutions for equations in $\mathbb{H}^n$, involving elliptic operators with standard Q–growth as well as critical Trudinger–Moser nonlinearities, have been proved in a series of papers. The key tool applied in the above papers is the Trudinger–Moser inequality in the whole space $\mathbb{H}^n$, combined with the mountain pass theorem, minimization arguments, the Ekeland variational principle and the concentration compactness principle *à la* Lions.

Nevertheless, in literature, there are very few contributions devoted to the study of exponential nonlinear problems driven by elliptic operators with nonstandard growth in the Heinseberg group, and the existence of nontrivial solutions to the (p, Q) equation

$$(\mathcal{E})\ -\Delta_{H,p}u - \Delta_{H,Q}u + |u|^{p-2}u + |u|^{Q-2}u = \frac{f(\xi, u)}{r(\xi)^{\beta}} + h(\xi)$$

on $\mathbb{H}^n$ has not been established yet so far.
An equation similar to $(\mathcal{E})$ in the Euclidean context first appears in

- Y. Yang, K. Perera, (N, q)–*Laplacian problems with critical Trudinger–Moser nonlinearities*, Bull. London Math. Soc. **48** (2016), 260–270.

but set on a bounded domain Ω, and the minimax argument used there strongly relies on the requirement that Ω is bounded.
More recently, existence of one solution for critical exponential problems, set on bounded domains Ω of $\mathbb{R}^N$ and driven by a general (p, N) operator, is given in

- G.M. Figueiredo, F.B.M. Nunes, *Existence of positive solutions for a class of quasilinear elliptic problems with exponential growth via the Nehari manifold method*, Rev. Mat. Complut. **32** (2019), 1–18.

via the Nehari manifold approach. Finally, existence and multiplicity of nontrivial nonnegative solutions for a (p,N) equation in the whole space $\mathbb{R}^N$ have been established in

- [FP1] A. Fiscella, P. P., *(p,N) equations with critical exponential nonlinearities in $\mathbb{R}^N$*, J. Math. Anal. Appl. *Special Issue New Developments in Nonuniformly Elliptic and Nonstandard Growth Problems*, doi.org/10.1016/j.jmaa.2019.123379

The approach in this last article is based on a combination of the Ekeland variational principle and the mountain pass theorem, as well as a crucial new Brézis–Lieb lemma in the exponential context.

Motivated by the works quoted above, we study for the first time in literature a singular critical exponential (p, Q) equation set in the Heisenberg group $\mathbb{H}^n$, and we prove the existence of a nontrivial nonnegative solution for $(\mathcal{E})$. It is also worth to emphasize that, contrary to the other references cited before, in this paper we do not consider the presence of a potential V which ensures that the embedding of the solution space W into the space $L^Q(\mathbb{H}^n)$ is compact. This further lack of compactness makes the proof arguments more delicate. Moreover, the conditions (f_1) and (f_2) required on the function f in this paper are milder with respect to the usual assumptions made in

- N. Lam, G. Lu, H. Tang, *Sharp subcritical Moser–Trudinger inequalities on Heisenberg groups and subelliptic PDEs*, Nonlinear Anal. **95** (2014), 77–92.
- N. Lam, G. Lu, *Sharp Moser–Trudinger inequality on the Heisenberg group at the critical case and applications*, Adv. Math. **231** (2012), 3259–3287.
- N. Lam, G. Lu, H. Tang, *On nonuniformly subelliptic equations of Q–sub–Laplacian type with critical growth in the Heisenberg group*, Adv. Nonlinear Stud. **12** (2012), 659–681.
- J. Li, G. Lu, M. Zhu, *Concentration–compactness principle for Trudinger–Moser inequalities on Heisenberg groups and existence of ground state solutions*, Calc. Var. Partial Differential Equations **57** (2018), Art. 84, 26 pp.

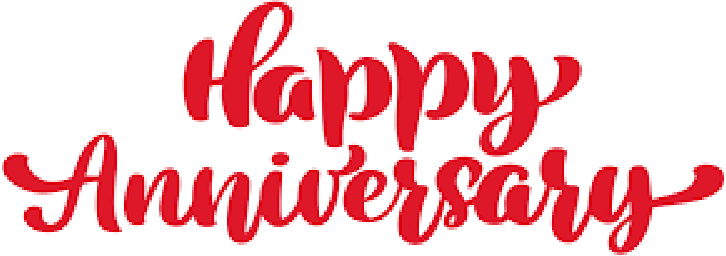
Last but not least the presence of the singular coefficient extends and complements the results introduced in [FP1]

Theorem 1.1 of [PT]

Let $1 < p < Q$ and $0 \leq \beta < Q$. Suppose that f verifies (f_1)–(f_2) and that h is a nontrivial nonnegative functional of $HW^{-1,Q'}(\mathbb{H}^n)$. Then, there exists a constant $\sigma > 0$ such that

$$(\mathcal{E}) \; -\Delta_{H,p}u - \Delta_{H,Q}u + |u|^{p-2}u + |u|^{Q-2}u = \frac{f(\xi,u)}{r(\xi)^{\beta}} + h(\xi)$$

on $\mathbb{H}^n$ admits at least a nontrivial nonnegative solution u_h in W, provided that $\|h\|_{HW^{-1,Q'}} < \sigma$. Moreover, $\lim_{h\to 0}\|u_h\| = 0$.



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Putting $v_k = u_k - u$, we get

$$|f(x, v_k + u)(v_k + u) - f(x, v_k)v_k| \leq 2^{N-1}(|v_k|^{N-1}|u| + |u|^N) + 2\tilde{\kappa}|u| \left(e^{\alpha\,(|v_k|+|u|)^{N'}} - S_{N-2}(\alpha, |v_k| + |u|)\right).$$

From this fact, setting
$\mathfrak{f}_k(x) = |f(x, v_k + u)(v_k + u) - f(x, v_k)v_k - f(x, u)u|$, we easily obtain

$$\mathfrak{f}_k(x) \leq 2^{N-1}|v_k|^{N-1}|u| + (2^{N-1} + 1)|u|^N + 2\tilde{\kappa}|u|Q_k + 2\tilde{\kappa}|u|Q,$$

where $Q_k = e^{\alpha\,(|v_k|+|u|)^{N'}} - S_{N-2}(\alpha, |v_k| + |u|)$ and
$Q = e^{\alpha\,|u|^{N'}} - S_{N-2}(\alpha, |u|)$. Of course $Q_k \to Q$ a.e. in $\mathbb{R}^N$, since $v_k \to 0$ a.e. in $\mathbb{R}^N$.
Now, by the structural assumptions on $(u_k)_k$ and the Brézis and Lieb lemma, we have

$$\begin{aligned} \|v_k\|^N_{W^{1,N}} &= \|u_k\|^N_{W^{1,N}} - \|u\|^N_{W^{1,N}} + o(1) \\ &\leq \|u_k\|^N_{W^{1,N}} + o(1) \end{aligned}$$

as $k \to \infty$. Thus,

$$\limsup_{k\to\infty} \|v_k\|_{W^{1,N}}^N = \lim_{k\to\infty} \|v_k\|_{W^{1,N}}^N \le \ell_N^N$$

$$\le \sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,N}}^N < \left(\frac{\alpha_N}{2^z\alpha_0}\right)^{N-1}.$$

Hence there exists J such that

$$\sup_{k\ge J} \|v_k\|_{W^{1,N}}^N < \left(\frac{\alpha_N}{2^z\alpha_0}\right)^{N-1}.$$

Of course, the assumption implies at once that

$$\|u\|_{W^{1,N}}^N \le \ell_N^N \le \sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,N}}^N < \left(\frac{\alpha_N}{2^z\alpha_0}\right)^{N-1}.$$

Hence, we have

$$\sup_{k\ge J} \||v_k| + |u|\|_{W^{1,N}}^N < 2^N \left(\frac{\alpha_N}{2^z\alpha_0}\right)^{N-1} \le \left(\frac{\alpha_N}{2\alpha_0}\right)^{N-1},$$

if the exponent $z \ge N' + 1$.

Thus we can apply a key lemma to the sequence $(|v_k| + |u|)_{k \geq J}$, with fixed $m \in (\alpha_N/2\alpha_0, \alpha_N/\alpha_0)$, $\alpha > \alpha_0$ and t, where $1 < t \leq N'$ is so close to 1 that $t\alpha m < \alpha_N$. Then, the Hölder inequality we have for all $k \geq J$

$$\begin{aligned} \int_{\mathbb{R}^N} \mathfrak{f}_k(x)dx &\leq \left(2^{N-1}\|v_k\|_N^{N-1} + 2^{N-1} + 1\right) \cdot \\ &\qquad \times \int_{\mathbb{R}^N} |u|^N dx + 2\tilde{\kappa}\left(\|Q_k\|_t^t + \|Q\|_t^t\right) \int_{\mathbb{R}^N} |u|^{t'} dx \\ &\leq C_Q \int_{\mathbb{R}^N} \left(|u|^N + |u|^{t'}\right) dx, \end{aligned}$$

where $C_Q = 2^{N-1}\left(\sup_{k\in\mathbb{N}} \|v_k\|_N^{N-1} + 1\right) + 1 + 2\tilde{\kappa}\left(\sup_{k\geq J} \|Q_k\|_t^t + \|Q\|_t^t\right) < \infty$, since $(u_k)_k$ is bounded in W and $(Q_k)_{k\geq J}$ is bounded in $L^t(\mathbb{R}^N)$ by the choices of the parameters taken above. Since $u \in W$ and $t' \geq N$, then $|u|^N + |u|^{t'} \in L^1(\mathbb{R}^N)$. This shows that $(\mathfrak{f}_k)_{k\geq J}$ is bounded in $L^1(\mathbb{R}^N)$. Thus the sequence $(\mathfrak{f}_k)_{k\geq J}$ of $L^1(\mathbb{R}^N)$ verifies the two properties of Vitali. Dietro