SOME RECENT LIOUVILLE TYPE RESULTS AND THEIR APPLICATIONS

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Workshop "Singular problems associated to quasilinear equations"
In honor of Marie-Françoise Bidaut-Véron and Laurent Véron

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OUTLINE

1. **The Lane-Emden equation**
2. **The nonlinear heat equation**
3. **The diffusive Hamilton-Jacobi equation**
4. **A mixed elliptic equation**

I – THE LANE-EMDEN EQUATION

$$-\Delta u = u^p, \quad x \in \mathbb{R}^n \qquad (p > 1) \tag{1}$$

• Classical Gidas-Spruck Liouville theorem

Theorem. [Gidas-Spruck CPAM 81] *Equation (1) does not admit any positive classical solution in* $\mathbb{R}^n$ *if (and only if)* $p < p_S = (n+2)/(n-2)_+$.

See also simplified proof in [Bidaut-Véron–Véron, Invent. Math. 91]

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- Half-space $\mathbb{R}^n_+ = \{(x_1, \dots, x_n);\ x_n > 0\}$

$$\begin{cases} -\Delta u = u^p, & x \in \mathbb{R}^n_+, \\ u = 0, & x \in \partial\mathbb{R}^n_+ \end{cases} \qquad (p > 1) \tag{2}$$

Theorem. [Gidas-Spruck CPDE 81] *Problem (2) does not admit any positive classical solution if* $p \leq p_S$.

Applications: a priori estimates for $p < p_S$ by rescaling method, and existence for Dirichlet boundary value problems via degree theory

HALF-SPACE: BEYOND SOBOLEV EXPONENT

Exponent p_S is optimal for nonexistence in whole space. What about half-space ?

For **bounded** solutions:

- $p < p_S(n-1) = (n+1)/(n-3)_+$ [Dancer, Bull. Austral. Math. Soc. 92]
- $p < p_{JL}(n-1) := (n^2 - 10n + 8\sqrt{n} + 13)/(n-3)(n-11)_+$ [Farina, JMPA 07]
- $p > 1$ [Chen-Li-Zou, JFA 14]

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Theorem 1. [Dupaigne-Sirakov-Souplet 2020] Let $p > 1$.

(i) Problem (2) has no positive classical solution bounded on finite strips

(ii) Problem (2) has no positive classical solution with $u_{x_n} \geq 0$

Finite strip: $\Sigma_R := \left\{x \in \mathbb{R}^n_+;\ 0 < x_n < R\right\} \quad (R > 0)$

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Remarks

- u bounded on finite strips $\implies u_{x_n} \geq 0 \implies u$ stable
- Theorem 1 remains true for any f *convex* C^2, with $f(0) = 0$ and $f > 0$ on $(0, \infty)$
- Open question: if there still exists a positive classical solution, it would have to **blow up for x_n bounded** (and $|x'_n| \to \infty$). Is this possible ?

SKETCH OF PROOF OF THEOREM 1

Step 1. Basic strategy

Show that u is convex in the normal direction (idea from Chen-Li-Zou 14).
(leads to contradiction with basic local L^1 estimates)

Moving planes: u bounded on finite strips $\implies u_{x_n} > 0$

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Key auxiliary function:

$$\xi := \frac{u_{x_n x_n}}{(1+x_n)u_{x_n}}$$

Elliptic operator:

$$\mathcal{L} := z^{-2}\nabla\cdot(z^2\nabla) \quad \text{with weight } z := (1+x_n)u_{x_n} > 0$$

Equation for ξ (using convexity of nonlinearity):

$$\mathcal{L}\xi \geq 2\xi^2$$

Also $\xi = 0$ on $\partial\mathbb{R}^n_+$ (due to $u_{x_n x_n} = \Delta u = -f(0) = 0$)

Does this imply $\xi \geq 0$?

Step 2. Key Lemma based on Moser iteration

Lemma 1. *Let $q > 1$ and consider the diffusion operator*

$$\mathcal{L} = A^{-1}\nabla \cdot (A\nabla)$$

where the weight $A \in L^\infty_{loc}(\overline{\mathbb{R}^n_+})$, $A > 0$ a.e., satisfies

$$\int_{B_R^+} A\,dx = \exp\big(o(R^2)\big), \quad R \to \infty. \tag{H}$$

Let $\xi \in H^1_{loc} \cap C(\overline{\mathbb{R}^n_+})$, with $\xi \geq 0$ on $\partial\mathbb{R}^n_+$, be a weak solution of

$$-\mathcal{L}\xi \geq (\xi_-)^q \quad \text{in } \mathbb{R}^n_+.$$

Then $\xi \geq 0$ a.e. in $\mathbb{R}^n_+$.

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• Gaussian assumption (H) is optimal ! Counter-example:

$$A(x) = \exp\big[(x_n)^k\big], \quad \xi = -x_n, \quad \text{with } k > 2 \text{ and } q = k - 1$$

• Idea of proof of Lemma 1: Moser type iteration, testing with powers of $(\xi_-)^m$ times suitably scaled cut-off $\phi(x/R)$ where $m = \varepsilon R^2$.

Step 3. Conclusion via stability estimates.

Theorem 1 follows if we show $\xi \geq 0$, i.e. $u_{x_n x_n} \geq 0$.

To apply Lemma 1 we need sub-Gaussian integral bounds on the weight A.

Here $A = ((1 + x_n)u_{x_n})^2$.

Recall: $u_{x_n} \geq 0 \implies u$ stable

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Estimates for stable solutions (e.g. Farina 07):

Lemma 2. *Let $p > 1$ and let $u \in C^2(\Omega)$ be a nonnegative stable solution of $-\Delta u = u^p$ in B_1. Then we have*

$$\int_{B_{1/2}} |\nabla u|^2 \, dx \leq C(n, p).$$

Lemma 2 + similar boundary estimates for half-balls

$\implies \int_{B_R^+} A \, dx \leq C(1 + R)^{n+2}$ □

• **Remark:** general case f convex: analogue of Lemma 2 is consequence of recent estimates of [Cabré-Figalli-RosOthon-Serra, Acta Math. 19]

II – THE SEMILINEAR HEAT EQUATION

Theorem 2. [Quittner 2020] *Let* $p > 1$. *Then the equation*

$$u_t - \Delta u = u^p, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^n$$

has no positive classical solution if (and only if) $p < p_S$.

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Previous results

- $p \leq (n+2)/n$ (consequence of [Fujita 66], true for global solutions on $[0, \infty) \times \mathbb{R}^n$)
- $p < n(n+2)/(n-1)^2$ [Bidaut-Véron, special vol. in honor of JL Lions 98]
- Radial case for $p < p_S$ [Polacik-Quittner NA06, Polacik-Quittner-Souplet IUMJ07]
- $n = 2$ [Quittner Math Ann. 16]

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Liouville for half-space case $\mathbb{R} \times \mathbb{R}^n_+$ (with $u = 0$ on $\mathbb{R} \times \partial\mathbb{R}^n_+$)

- bounded solutions for $p < p_S$ [Polacik-Quittner-Souplet IUMJ07]
- $p < p_S$ (possibly unbounded) [Quittner 2020]

Rem: true in a larger range for bounded solutions; optimality unknown

Related: Liouville type theorem for *ancient solutions* [Merle-Zaag, CPAM 98]

SKETCH OF PROOF OF THEOREM 2

- Pass to similarity variables to get modified equation (cf. [Giga-Kohn CPAM85])

$$w := w_{a,k}(y,s) = e^{-\beta s} u\big(a + y e^{-s/2}, k - e^{-s}\big), \qquad s = -\log(k-t), \qquad \beta = 1/(p-1).$$

$$(E') \qquad w_s = \Delta w - \frac{y}{2} \cdot \nabla w + w^p - \beta w \quad \text{in } \mathbb{R}^n \times \mathbb{R} \quad (\text{for each integer } k)$$

- Good energy structure associated with (E') for Gaussian weight $\rho(y) = e^{-y^2/4}$

$$E_{a,k}(s) = \frac{1}{2} \int_{\mathbb{R}^n} \big(|\nabla w_{a,k}|^2 + w_{a,k}^2\big) \rho \, dy - \frac{1}{p+1} \int_{\mathbb{R}^n} w_{a,k}^{p+1} \rho \, dy.$$

- Hard energy estimates of the form $E_{a_i,k}(s) \le k^{\gamma_j}$ for $k \gg 1$, with suitable centers a_i, powers $\gamma_j > 0$ and time intervals.

  Obtained by bootstrap procedure + covering and measure arguments.

- Appropriate rescaled of $w_k \to w$ positive solution of $-\Delta w = w^p$ in $\mathbb{R}^n$ as $k \to \infty$: contradiction with Gidas-Spruck elliptic Liouville.

APPLICATIONS OF PARABOLIC LIOUVILLE THEOREM

Estimates for nonnegative solutions of $u_t - \Delta u = u^p$ with $1 < p < p_S$.
Csq of Thm 2 + Rescaling + Doubling Lemma [Poláčik-Quittner-S. DMJ & IUMJ07]

• Blow-up rate estimates (final *and* initial), in any smooth domain (incl. nonconvex !) and with *universal constants*

$$u \text{ solution in } (0,T)\times\mathbb{R}^n \Longrightarrow \boxed{u \le C(n,p)\big[t^{-\beta} + (T-t)^{-\beta}\big]} \qquad \beta := \frac{1}{p-1}$$

$$u \text{ solution in } (0,T)\times\Omega \text{ with zero B.C.} \quad \Longrightarrow \boxed{u \le C(p,\Omega)\big[1 + t^{-\beta} + (T-t)^{-\beta}\big]}$$

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• Decay estimates for all global solutions in $\mathbb{R}^n$

$$u \text{ solution in } (0,\infty)\times\mathbb{R}^n \Longrightarrow \boxed{u \le C(n,p)\, t^{-\beta}}$$

• Universal bounds away from $t = 0$ for global solutions in any smooth domain

$$u \ge 0 \text{ solution of (E) in } (0,\infty)\times\Omega \text{ with zero B.C.} \Longrightarrow \boxed{u \le C(p,\Omega)\big[1 + t^{-\beta}\big]}$$

• Local universal estimate in space and time

$$u \text{ solution in } (0,T)\times\Omega \Longrightarrow \boxed{u \le C(n,p)\big[t^{-\beta} + (T-t)^{-\beta} + (\mathrm{dist}(x,\partial\Omega))^{-2\beta}\big]}$$

III – DIFFUSIVE HAMILTON-JACOBI EQUATION

(DHJ)
$$\boxed{\begin{cases} u_t - \Delta u = |\nabla u|^p, & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}}$$

$\Omega \subset \mathbb{R}^n$ smooth bounded domain, $p > 2$ (superquadratic).

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Some key features: [cf. A. Porretta's lecture]

• Finite time gradient blow-up (GBU) occurs for large initial data:

$$\lim_{t \to T} \|\nabla u(t)\|_\infty = \infty$$

• Continuation as unique global viscosity condition (with possible loss of classical boundary conditions)

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Related elliptic problem:

(1)
$$\boxed{\begin{cases} -\Delta v = |\nabla v|^p, & x \in \mathbb{R}^n_+, \\ v = 0, & x \in \partial\mathbb{R}^n_+ \end{cases}}$$

• Whole space case: [PL Lions, JAM 85]

if $p > 1$ and v classical solution of $-\Delta v = |\nabla v|^p$ in $\mathbb{R}^n$, then v is constant

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• Elliptic half-space case is important for study of GBU (see later)

Theorem 3. [Filippucci-Pucci-Souplet CPDE 2019]
Let $p > 2$ and let $v \in C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ be a solution of (1). Then v depends only on the variable x_n.

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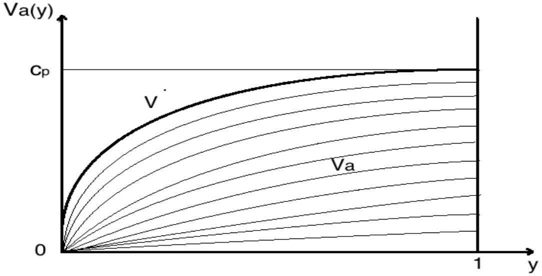
Remarks

• Thm 3 $\implies$ v solves the ODE $-v'' = |v'|^p,\ s > 0$ with $v(0) = 0$

$v \equiv 0 \quad$ or $\quad v(s) = c_p\big[(s+a)^{1-\beta} - a^{1-\beta}\big]$

for some $a \geq 0$, with $\beta = 1/(p-1)$

including singular sol. $V = c_p s^{1-\beta}$



• Thm 3 also true for $1 < p \leq 2$ [Porretta-Véron, Adv. Nonl. Stud. 06]

SKETCH OF PROOF OF THEOREM 3

• Write $x = (\tilde{x}, y) \in \mathbb{R}^{n-1} \times [0, \infty)$ and fix any $h \in \mathbb{R}^{n-1} \setminus \{0\}$. Let

$$z(\tilde{x}, y) = v(\tilde{x} + h, y) - v(\tilde{x}, y), \qquad (\tilde{x}, y) \in \mathbb{R}^{n-1} \times [0, \infty)$$

Goal: show $z \equiv 0$ by contradiction, assuming $\sup_{\mathbb{R}^n_+} z > 0$.

• Use local Bernstein estimate [PL Lions 85]:

$$|\nabla v(\tilde{x}, y)| \leq C(n, p) y^{-\beta}, \quad \text{for all } (\tilde{x}, y) \in \mathbb{R}^{n-1} \times (0, \infty)$$

$\Longrightarrow$ supremum of z localized in a *finite strip*

• Translations parallel to the boundary + compactness procedure

$\Longrightarrow$ supremum of z localized at a *finite point*

• The new function z_∞ satisfies a linear equation with (locally bounded) drift, along with $z = 0$ on $\partial \mathbb{R}^n_+$

$\Longrightarrow$ contradiction with Strong Maximum Principle

APPLICATIONS OF THEOREM 3

[Filippucci-Pucci-Souplet CPDE 19]

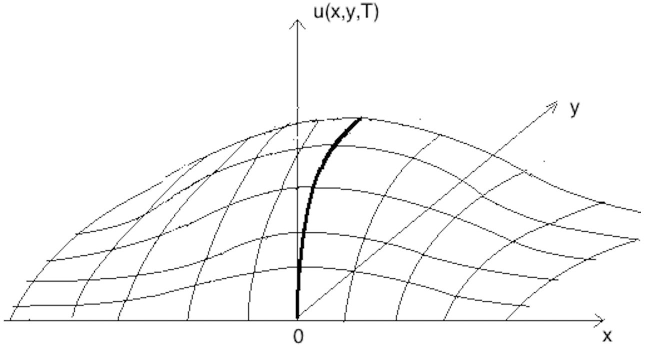
• **Sharp GBU profile in normal direction:** for any GBU point $a \in \partial\Omega$,

$$\boxed{\lim_{s\to 0} s^\beta \nabla u(a + s\nu_a, T) = d_p \nu_a} \implies |\nabla u(x,T)| \sim d_p \delta^{-\beta}, \text{ as } x \to a,\ x - a \perp \partial\Omega$$

ν_a = inner unit normal vector

$\delta(x) = \mathrm{dist}(x, \partial\Omega)$

$\beta = \frac{1}{p-1}$, $d_p = \beta^\beta$



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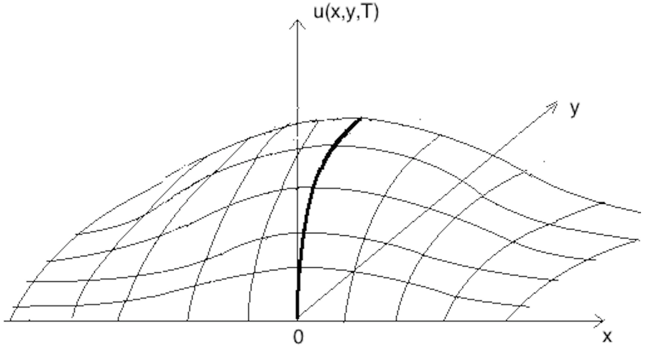

• **More singular tangential behavior:** $\lim\limits_{x\to a,\ x\in\partial\Omega} |x-a|^\beta u_\nu(x,T) = \infty$

Sharp tangential exponent known only in special cases ($2 < p \le 3$, flat symm. case)

$u_\nu(x,0,T) \sim |x|^{-2/(p-2)}$ [Porretta-Souplet, IMRN 17]

APPLICATIONS OF THEOREM 3 (cont'd)

[Filippucci-Pucci-Souplet CPDE 19]

• **Asymptotic ODE type singular behavior in space-time:**

$\boxed{-u_{\nu\nu} \sim |u_\nu|^p}$ in the region of $(0,T) \times \Omega$ where $|\nabla u| \gg 1$.

Asymptotic scheme: $u_t - \boxed{u_{\nu\nu}} - u_{\tau\tau} = \left(\boxed{u_\nu^2} + u_\tau^2\right)^{p/2}$

Rem: Analogue of [Merle-Zaag CPAM 98]

$u_t \sim u^p$ in $\{u \gg 1\}$ for semilinear heat equation $u_t - \Delta u = u^p \quad (p < p_S)$

Proved by means of Liouville type theorem for ancient solutions

Significant difference: normal spatial direction instead of time direction

• **GBU viscosity solutions without loss of boundary conditions**

(existence known from [Porretta-Souplet AIHP 17])

Liouville $\Rightarrow$ such solutions are exceptional: completely unstable from above and below

Thresholds between global classical and GBU solutions

IV – A MIXED ELLIPTIC EQUATION

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[Filippucci-Pucci-Souplet, Adv. Nonl. Stud. 20]

(special volume in honor of Marie-Françoise and Laurent)

Theorem 4.
Let $q > 2$, $p > 0$. Then any bounded solution $u \geq 0$ of (1) is constant.

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• Case $0 < q \leq 2$: studied in detail [Bidaut-Véron, Garcia-Huidobro, Véron DMJ 19] (also [Burgos-Pérez, García-Melían, Quaas DCDS 16])

Various regions of nonexistence / existence

• Theorem 4 fails for *supersolutions*: they exist if $(n-2)q + (n-1)p > n$ and $n \geq 3$

• Open question for $q > 2$: can one relax assumption u bounded ?

SKETCH OF PROOF OF THEOREM 4

(A) Basic tool: monotone decreasing (resp. increasing) property of spherical averages of superharmonic (resp. subharmonic) functions

(B) $v := u - \inf u \geq 0$ superharmonic

How to find a good subharmonic quantity ?

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How to find a good subharmonic quantity ?

(C) Show: $w := (u - \inf u)^m \geq 0$ subharmonic for $m \gg 1$

Lemma. *If u positive bounded solution of (1), then $u^{q+1}|\nabla u|^{p-2}$ bounded.*

Proof by a local Bernstein argument

Lemma + simple computation $\implies$ (C)

(D) Combination of opposite monotonicity properties of spherical averages obtained from (A), (B), (C) forces $u \equiv \text{const}$.

BON ANNIVERSAIRE

MARIE-FRANÇOISE ET LAURENT !!