Singularity formation in Nonlinear Evolution Equations

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Workshop: Singular problems associated to quasilinear equations
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Outline

1 - Introduction

Singularity formation in Nonlinear PDEs - Motivations

- Applied point of view:
  - Understanding the physical limitation of mathematical models.

    Can the equations always do their job?

    What additional conditions of physical effects to have a proper model.
  - Singularities are physically *relevant* in natural sciences: concentration of laser beam in media (blowup in NLS), concentration of energy to smaller scales in fluid mechanics, concentration of density of bacteria population, etc.
- Mathematical point of view:
  - The long-time dynamic of solutions to PDEs is of significant interest. However, solutions may develop singularities in finite time.

    How to extend solutions beyond their singularities?
  - The study of singularity formation requests *new tools* to handle many delicate problems such as stability of a family of special solutions, classification of all possible asymptotic behaviors , etc.
  - The numerical study of singularities is challenging.
  - ...

Model examples

■ **Reaction-Diffusion equations:** Non-variational semilinear parabolic systems

$$\begin{cases} \partial_t u = \Delta u + v|v|^{p-1}, \\ \partial_t v = \mu\Delta v + u|u|^{q-1}, \end{cases} \quad \mu > 0, \quad p, q > 1. \tag{RD}$$

Application: thermal reaction, chemical reaction, population dynamics, ...

■ **Geometric evolution equations:** Harmonic heat flows and wave maps (σ-model): $\Phi(t) : \mathbb{R}^d \to \mathbb{S}^d$,

$$\partial_t \Phi = \Delta\Phi + |\nabla\Phi|^2\Phi, \tag{HF}$$

$$\partial_t^2 \Phi = \Delta\Phi + \left(|\nabla\Phi|^2 - |\partial_t\Phi|^2\right)\Phi. \tag{WM}$$

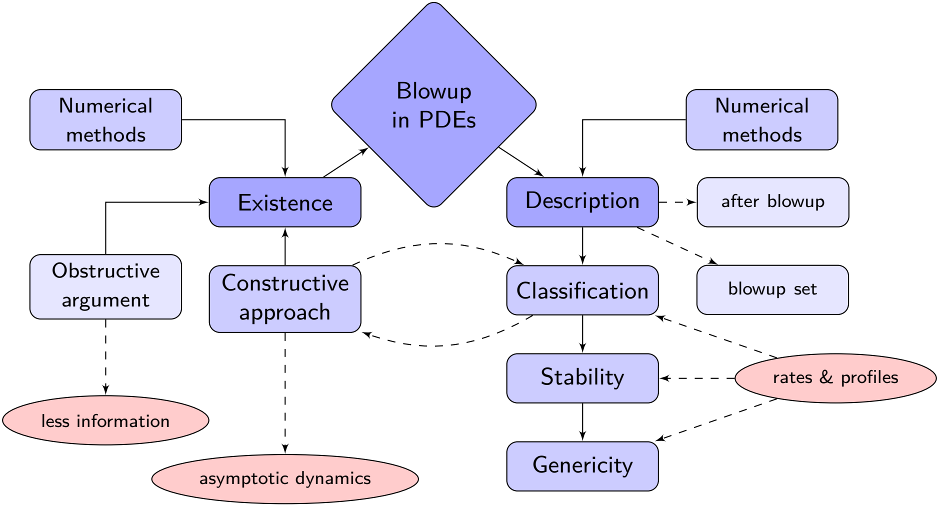
Application: geometry, topology, simplified model for Einstein's equation of general relativity,...

■ **Aggregation-Diffusion equations:** the 2D Keller-Segel system

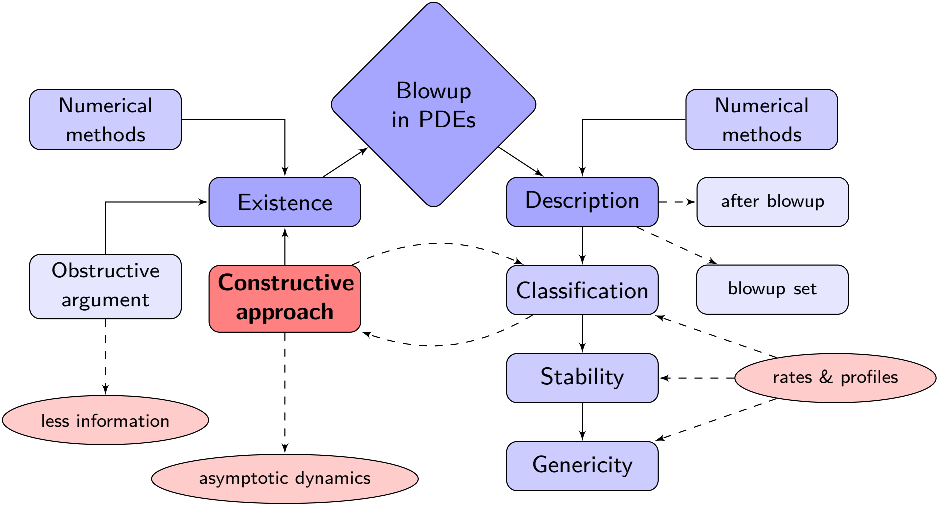
$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u\nabla\Phi_u), \\ 0 = \Delta\Phi_u + u, \end{cases} \quad \text{in } \mathbb{R}^2. \tag{KS}$$

Application: biology (chemotaxis), interacting many-particle system, ...

Framework of studying singularities in PDEs



Framework of studying singularities in PDEs



2 - Constructive approach

Underlying problem

Existence and Stability of blowup solutions.

- Obstructive argument (Virial law): Existence, no blowup dynamics.
- **Constructive approach**: Existence + blowup dynamics.
  - Kenig (Chicago), Rodnianski (Princeton), Merle (Cergy Pontoise & IHES), Raphaël (Cambridge), Martel (École Polytechnique), Collot (CNRS & Cergy Pontoise), ...
  - del Pino (Bath), Musso (Bath), Wei (UBC), Davila (Bath), ...
  - Krieger (EPFL), Schlag (Yale), Tataru (Berkeley), ...
  - Herrero(UCM), Velázquez (Bonn), Zaag (CNRS & Paris Nord), ...

Architecture of the constructive approach

- Constructing a good approximate solution;
- Reduction of the linearized problem to a finite dimensional one:
  - Modulation technique: Kenig, Merle, Raphaël, Martel, ... $\rightsquigarrow$ existence/stability;
  - Inner-outer gluing method: del Pino, Wei, Musso, Davila, ... $\rightsquigarrow$ existence/stability;
  - Iterative technique: Krieger, Schlag, Tataru, ... $\rightsquigarrow$ existence;
  - Spectral analysis $\rightsquigarrow$ existence/stability + **classification**;
  - ...
- Solving the finite dimensional problem (if necessary).

3 - Results: The non-variational semilinear parabolic system

$$\begin{cases} \partial_t u = \ \Delta u + v|v|^{p-1}, \\ \partial_t v = \mu \Delta v + u|u|^{q-1}, \end{cases} \quad \mu > 0, \quad p, q > 1. \tag{RD}$$

Mathematical analysis:

- No variational structure $\rightsquigarrow$ Energy-type methods break down;
- $\mu \neq 1$ $\rightsquigarrow$ breaks any symmetry of the problem;
- The linearized operator is not self-adjoint even for the case $\mu = 1$.

Literature: Andreucci-Herrero-Velázquez '97, Souplet '09, Zaag '98 & '01, Mahmoudi-Souplet-Tayachi '15, ...

Type I (ODE-type) blowup solutions for (**RD**) via spectral analysis

- Type I blowup: "∂_t dominates Δ" $\rightsquigarrow$ the blowup rate, **unknown blowup profiles**.

$$\begin{cases} \bar{u}' = \bar{v}^p, \\ \bar{v}' = \bar{u}^q \end{cases} \rightsquigarrow \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} \Gamma(T-t)^{-\alpha} \\ \gamma(T-t)^{-\beta} \end{pmatrix}, \quad \alpha = \frac{p+1}{pq-1}, \; \beta = \frac{q+1}{pq-1}.$$

Theorem 1 (Ghoul-Ng.-Zaag '18]).

- $\exists (u_0, v_0) \in L^\infty \times L^\infty$ such that the solution (u, v) to System (**RD**) blows up in finite time T and admits the asymptotic dynamic

$$(T-t)^\alpha u(x,t) - \Phi_0(\xi) \to 0, \quad (T-t)^\beta v(x,t) - \Psi_0(\xi) \to 0,$$

as $t \to T$ in L^∞, where
  - (blowup variable) $\xi = \frac{x}{\sqrt{(T-t)|\ln(T-t)|}}$;
  - (profiles) $\Phi_0(\xi) = \Gamma(1 + b|\xi|^2)^{-\alpha}, \quad \Psi_0(\xi) = \gamma(1 + b|\xi|^2)^{-\beta}$ with $b > 0$.
- The constructed solution is **stable** under perturbation of initial data.

Remark: - Other profiles are possible, but they are suspected to be unstable.
- The existence of Type II blowup solutions remains unknown.

Constructive proof for (**RD**): approximate blowup solution

- **Self-similar variables:** $y = \frac{x}{\sqrt{T-t}}$, $s = -\ln(T-t)$, $\begin{cases} \phi(y,s) = (T-t)^{\alpha} u(x,t) \\ \psi(y,s) = (T-t)^{\beta} v(x,t) \end{cases}$,

$$\begin{cases} \partial_s \phi + \frac{1}{2} y.\nabla\phi + \alpha\phi = \Delta\phi + |\psi|^{p-1}\psi, \\ \partial_s \psi + \frac{1}{2} y.\nabla\psi + \beta\psi = \mu\Delta\psi + |\phi|^{q-1}\phi, \end{cases}$$

- Nonzero constant solutions: (Γ, γ)

$$\alpha\Gamma = \gamma^p, \quad \beta\gamma = \Gamma^q.$$

- **Linearizing:** $(\phi, \psi) = (\Gamma, \gamma) + (\bar\phi, \bar\psi)$,

$$\partial_s \begin{pmatrix} \bar\phi \\ \bar\psi \end{pmatrix} = (\mathcal{H} + \mathcal{M}) \begin{pmatrix} \bar\phi \\ \bar\psi \end{pmatrix} + \text{"nonlinear quadratic term"};$$

where $\mathcal{H} + \mathcal{M}$ has two positive eigenvalues 1 and $\frac{1}{2}$, a zero eigenvalue and an infinite many discrete negative spectrum.

Constructive proof for (RD): approximate blowup solution

- **Null mode is dominant**: $\begin{pmatrix}\bar\phi\\ \bar\psi\end{pmatrix}(y,s) = \theta_2(s)\begin{pmatrix}f_2\\ g_2\end{pmatrix}, \quad \begin{pmatrix}f_2\\ g_2\end{pmatrix} = \begin{pmatrix}a_2\\ b_2\end{pmatrix}y^2 + \begin{pmatrix}a_0\\ b_0\end{pmatrix},$

$$\theta_2' = \bar c\theta_2^2 + \mathcal{O}(|\theta_2|^3), \quad \bar c > 0, \quad \rightsquigarrow \theta_2 \sim -\frac{1}{\bar c s}.$$

- **Inner approximation:** for $|y| \leq C$,

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix}(y,s) \sim \begin{pmatrix}\Gamma\\ \gamma\end{pmatrix} - \frac{|y|^2}{s}\begin{pmatrix}a_2'\\ b_2'\end{pmatrix} \quad \rightsquigarrow \quad \boxed{\xi = \frac{|y|}{\sqrt{s}} = \frac{x}{\sqrt{(T-t)|\ln(T-t)|}}}$$

Constructive proof for (**RD**): approximate blowup solution

- **Shape of profiles:**

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix}(y,s) = \begin{pmatrix}\Phi_0\\ \Psi_0\end{pmatrix}(\xi) + \frac{1}{s}\begin{pmatrix}\Phi_1\\ \Psi_1\end{pmatrix}(\xi) + \dots, \quad \xi = \frac{|y|}{\sqrt{s}},$$

where

$$-\frac{\xi}{2}\Phi_0' - \alpha\Phi_0 + \Psi_0^p = 0, \quad -\frac{\xi}{2}\Psi_0' - \beta\Psi_0 + \Phi_0^q = 0.$$

- Solving ODEs: $\Phi_0(0) = \Gamma$, $\Psi_0(0) = \gamma$,

$$\boxed{\Phi_0(\xi) = \Gamma(1 + b|\xi|^2)^{-\alpha}, \quad \Psi_0(\xi) = \gamma(1 + b|\xi|^2)^{-\beta}, \quad b \in \mathbb{R}.}$$

- **Matching asymptotic expansions:** $\rightsquigarrow$ value of $b = b(p, q, \mu) > 0$. $\square$

Constructive proof for (**RD**): Control of the remainder

- **Linearized problem:** $(\phi, \psi) = (\Phi_0, \Psi_0) + (\Lambda, \Upsilon)$,

$$\partial_s \begin{pmatrix} \Lambda \\ \Upsilon \end{pmatrix} = \Big(\mathcal{H} + \mathcal{M} + \mathcal{V}\Big) \begin{pmatrix} \Lambda \\ \Upsilon \end{pmatrix} + \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \text{"quadratic term"}. \qquad (\star)$$

- Constructing for $(\star)$ a global in time solution (Λ, Υ) such that

$$\|\Lambda(s)\|_{L^\infty} + \|\Upsilon(s)\|_{L^\infty} \longrightarrow 0 \quad \text{as} \quad s \to +\infty.$$

Constructive proof for (**RD**): Control of the remainder

- **Spectral properties of the linear part:** for $K \gg 1$,
  - For $|y| \geq K\sqrt{s}$: $\mathcal{H} + \mathcal{M} + \mathcal{V}$ has a negative spectrum. $\rightsquigarrow$ *Control of* $\binom{\Lambda}{\Upsilon}$ *is simple.*
  - For $|y| \leq K\sqrt{s}$: the potential $\mathcal{V}$ is regarded as a perturbation of $\mathcal{H} + \mathcal{M}$. We decompose

$$\begin{pmatrix} \Lambda \\ \Upsilon \end{pmatrix} = \sum_{n=0}^{2} \theta_n \begin{pmatrix} f_n \\ g_n \end{pmatrix} + \begin{pmatrix} \Lambda_- \\ \Upsilon_- \end{pmatrix},$$

where $\binom{\Lambda_-}{\Upsilon_-} = \Pi_- \binom{\Lambda}{\Upsilon}$ with Π_- being the projection on the subspace associated to the negative eigenvalues of $\mathcal{H} + \mathcal{M}$. $\rightsquigarrow$ $\binom{\Lambda_-}{\Upsilon_-}$ *is controllable to zero.*

Constructive proof for (**RD**): finite dimensional reduction

■ **Control of θ_2 is delicate:** We need to refine the potential term $\mathcal{V}\binom{\Lambda}{\Upsilon}$,

$$\frac{d\theta_2(s)}{ds} + \frac{2}{s}\theta_2(s) = \mathcal{O}\left(\frac{1}{s^3}\right) \overset{\tau = \ln s}{\Longrightarrow} \frac{d\theta_2(\tau)}{d\tau} = -2\theta_2(\tau) + \mathcal{O}\left(e^{-2\tau}\right),$$

which shows a negative eigenvalue $\rightsquigarrow$ θ_2 *is controllable to zero*.

■ **Control θ_0 and θ_1** (reduction to a finite dimensional problem): Consider the initial data depending on $(d_0, d_1) \in \mathbb{R}^{1+d}$:

$$\begin{pmatrix} \Lambda \\ \Upsilon \end{pmatrix}(y, s_0) = \frac{A}{s_0^2}\left[d_0 \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} + d_1 . \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}\right]\chi(y, s_0).$$

$\Longrightarrow$ A contradiction argument yields the existence a particular value $(d_0, d_1) \in \mathbb{R}^{1+d}$ such that $\theta_0(s)$ and $\theta_1(s)$ converge to zero as $s \to +\infty$. □

Idea of the stability proof for (**RD**)

- **Idea of the stability proof:** the stability directly follows from the existence proof.
  - Space-time translation invariance $\rightsquigarrow$ control of θ_0, θ_1;
  - Other directions are always controllable to zero.

- **Flexibility:** The complex Ginzburg-Landau equation by [Nouali-Zaag '18, Masmoudi-Zaag '08], other nonlinearities by [Ghoul-Ng.-Zaag '18], ...

3 - Results: Harmonic heat flow & Wave maps

$$\partial_t \Phi = \Delta \Phi + |\nabla \Phi|^2 \Phi, \qquad \textbf{(HF)}$$

$$\partial_t^2 \Phi = \Delta \Phi + \left(|\nabla \Phi|^2 - |\partial_t \Phi|^2\right) \Phi, \qquad \textbf{(WM)}$$

where $\Phi(t) : \mathbb{R}^d \to \mathbb{S}^d$.

Harmonic map Heat Flow and Wave map problems

$\Phi(t) : \mathbb{R}^d \to \mathbb{S}^d$	
Harmonic heat flow (HF)	**Wave Map (WM)**
$\partial_t \Phi = \Delta\Phi + \lvert\nabla\Phi\rvert^2\Phi$	$\partial_t^2 \Phi = \Delta\Phi + \left(\lvert\nabla\Phi\rvert^2 - \lvert\partial_t\Phi\rvert^2\right)\Phi$
$\Phi(x,t) = \left[\cos\left(u(\lvert x\rvert,t)\right), \frac{x}{\lvert x\rvert}\sin\left(u(\lvert x\rvert,t)\right)\right]^T$	
$\partial_t u = \Delta_{d,r} u - \frac{(d-1)}{2r^2}\sin(2u)$	$\partial_t^2 u = \Delta_{d,r} u - \frac{(d-1)}{2r^2}\sin(2u)$
$u_\lambda(r,t) = u\left(\frac{r}{\lambda}, \frac{t}{\lambda^2}\right)$	$u_\lambda(r,t) = u\left(\frac{r}{\lambda}, \frac{t}{\lambda}\right)$
LW-P in H^s with $s > \frac{d}{2}$ [Ref 1]	LW-P in $H^s \times H^{s-1}$ with $s > \frac{d}{2}$ [Ref 2]
Develop singularities [Ref 3]	

[Ref 1]: [Struwe, JDG'88], [Wang, ARMA'08]
[Ref 2]: [Klainerman-Selberg, CPDE'97], [Tataru, CPDE'98], [Shatah-Struwe, IMRN'02]
[Ref 3]: [Coron-Ghidaglia, CRASP'89], [Chen-Ding, IM'90], [Chang-Ding-Ye, JDG'92], [Shatah, CPAM'88].

Type I and Type II singularities

Harmonic heat flow (HF)	**Wave Map (WM)**
$\limsup_{t\to T} \sqrt{T-t}\|\nabla u(t)\|_{L^\infty} < \infty$	$\limsup_{t\to T} (T-t)\|\nabla u(t)\|_{L^\infty} < \infty$
Otherwise, the singularity (or blowup) is of Type II	
$d \geq 2$: Type I $\approx \phi\left(\frac{r}{\sqrt{T-t}}\right)$ [Ref 1]	$d = 2$: Blowup $\not\approx \varphi\left(\frac{r}{T-t}\right)$ [Ref 2]
$3 \leq d \leq 6$: $\exists \phi_n, \;\; \mathcal{Z}_{(0,\infty)}(\phi_n - \frac{\pi}{2}) = n$ [Ref 3]	$3 \leq d \leq 6$: $\exists \varphi_n, \;\; \mathcal{Z}_{(0,1)}(\varphi_n') = n$ [Ref 4]
$d \geq 7$: No self-similar $\to$ No Type I [Ref 5]	$d \geq 3$: $\varphi(y) = 2\arctan\left(\frac{y}{\sqrt{d-2}}\right)$ [Ref 6]

[Ref 1]: [Struwe, JDG'88]; [Ref 2]: [Struwe, CPAM'03]; [Ref 3]: [Fan, SCSA'99];
[Ref 4]: [Biernat-Bizoń-Maliborski, Nonl'17]; [Ref 5]: [Bizoń-Wasserman, IMRN'15];
[Ref 6]: [Bizon-Biernat, CMP'15].

(Non)Existence of Type I and Type II singularities

Dimension	Heat flow	Wave Map	Type
$d = 2$	NO	NO	I
	YES	YES	II
$3 \leq d \leq 6$	YES	YES	I
	Unknown?	Unknown?	II
$d \geq 7$	NO	YES	I
	YES	YES	II

- [Topping, MZ'04]; [Struwe, CPAM'03];
- [Van de Berg-Hulshof-King, SIAM'03], [Schweyer-Raphael, CPAM'13 -APDE'14], [Davila-del Pino-Wei, IM'19]; [Krieger-Schlag-Tataru, IM'08], [Carstea, CMP'10], [Rodniaski-Sterbenz, AM'10], [Raphael-Rodnianski, IHES'12];
- [Fan, SCSA'99]; [Bizon-Biernat, CMP'15], [Biernat-Bizon-Maliborski, Nonl'17];
- [Bizon-Wasserman, IMRN'15]; [Bizon-Biernat, CMP'15];

Energy identities and the stationary solution

- **Energy identities:**

$$\mathcal{E}_{(HF)}(u) = \int_{\mathbb{R}^d} \left(|\partial_r u|^2 + \frac{(d-1)}{r^2}\sin^2(u) \right), \quad \frac{d}{dt}\mathcal{E}_{(HF)}(u) = -2\int_{\mathbb{R}^d} |\partial_t u|^2 \leq 0.$$

$$\mathcal{E}_{(WM)}(u, \partial_t u) = \int_{\mathbb{R}^d} \left(|\partial_t u|^2 + |\partial_r u|^2 + \frac{(d-1)}{r^2}\sin^2(u) \right), \quad \frac{d}{dt}\mathcal{E}_{(WM)}(u, \partial_t u) = 0.$$

$$\boxed{\mathcal{E}_{(HF)}(u_\lambda) = \lambda^{d-2}\mathcal{E}_{(HF)}(u) \quad \text{and} \quad \mathcal{E}_{(WM)}(u_\lambda, \partial_t u_\lambda) = \lambda^{d-2}\mathcal{E}_{(WM)}(u, \partial_t u).}$$

$$\Longrightarrow d = 2 \text{ energy-critical, } d \geq 3 \text{ energy-supercritical.}$$

- **Stationary solution:** $Q(r) = 2\arctan(r)$ for $d = 2$;

$$d \geq 7, \quad Q(r) \sim \frac{\pi}{2} - \frac{a_0}{r^\gamma} \quad \text{for } r \gg 1,$$

with $a_0 > 0$, $\gamma = \frac{1}{2}(d - 2 - \sqrt{d^2 - 8d + 8}) \in (1, 2]$.

Type II singularity for (**HF**)

- Type II blowup: "Δ dominates ∂_t" $\rightsquigarrow$ profile, **unknown blowup rates**.

Theorem 2 (Ghoul-Ibrahim-Ng. '19]).

- Let $d \geq 7$, $\ell \in \mathbb{N}^*$ with $2\ell > \gamma$ and $\mathfrak{s} \in \mathbb{N}$ with $\mathfrak{s} = \mathfrak{s}(\ell) \to +\infty$ as $\ell \to +\infty$. There exists a smooth radially symmetric initial data $u_0 \in \mathcal{U} \subset H^{\mathfrak{s}}$ such that the solution to (HF) is of the form

$$u(r,t) = Q\left(\frac{r}{\lambda(t)}\right) + q\left(\frac{r}{\lambda(t)}, t\right),$$

where

$$\lambda(t) \sim c(u_0)(T-t)^{\frac{\ell}{\gamma}} \ll (T-t)^{\frac{1}{2}} \quad \text{as} \quad t \to T,$$

and $\lim_{t\to T} \|q(t)\|_{\dot{H}^\sigma} = 0, \quad \forall \sigma \in (d/2, \mathfrak{s}]$.

- The constructed solution is $(\ell - 1)$ codimension stable.

Remark: Type II blowup for $d = 2$ constructed by [Raphäel-Schweyer APDE'14 & CPAM'13], and [Davila-del Pino-Wei IM'19].

Type II Singularity for (**WM**)

- Type II blowup: "Δ dominates ∂_t^2" $\rightsquigarrow$ profile, **unknown blowup rates**.

Theorem 3 (Ghoul-Ibrahim-Ng. '19]).

- Let $d \geq 7$, $\ell \in \mathbb{N}^*$ with $\ell > \gamma$ and $\mathfrak{s} \in \mathbb{N}$ with $\mathfrak{s} \gg 1$. There exists $(u_0, u_1) \in H^{\mathfrak{s}} \times H^{\mathfrak{s}-1}$ such that the solution to (**WM**) is of the form

$$u(r,t) = Q\left(\frac{r}{\lambda(t)}\right) + \varepsilon\left(\frac{r}{\lambda(t)}, t\right),$$

where

$$\lambda(t) \sim c(u_0)(T-t)^{\frac{\ell}{\gamma}}, \quad (\text{"Type II blowup"})$$

and $\lim_{t\to T} \|\vec{\varepsilon}(t)\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} = 0$ for all $\sigma \in (d/2, \mathfrak{s}]$.

- The constructed solution is $(\ell - 1)$ codimension stable.

Remark: - Type I blowup: $u(r,t) = 2\arctan\left(\frac{r}{(T-t)\sqrt{d-2}}\right)$ for $d \geq 3$, [Bizon-Biernat '15].

Type II blowup for $d = 2$: [Krieger-Schlag-Tataru IM'08, Raphäel-Rodnianski IHES'12]. (KS)

Constructive proof for (**WM**): computing an approximate solution

- **Renormalized variables:** $\vec{w}(y,s) = \binom{u_1}{\lambda u_2}(r,t), \quad y = \dfrac{r}{\lambda(t)}, \quad \dfrac{ds}{dt} = \dfrac{1}{\lambda(t)},$

$$\boxed{\partial_s \vec{w} + b_1 \Lambda \vec{w} = \vec{F}(\vec{w}),}$$

$$b_1 = -\frac{\lambda_s}{\lambda}, \quad \Lambda \vec{w} = \begin{pmatrix} y\partial_y w_1 \\ w_2 + y\partial_y w_2 \end{pmatrix}, \quad \vec{F}(\vec{u}) = \begin{pmatrix} u_2 \\ \Delta_{r,d} u_1 - \frac{(d-1)}{2r^2}\sin(2u_1) \end{pmatrix}.$$

- **The approximate solution:** Let $L \gg 1$, $b = (b_1, \cdots, b_L)$, $\vec{Q} = \binom{Q}{0}$,

$$\boxed{\vec{Q}_{b(s)}(y) = \vec{Q}(y) + \sum_{i=1}^{L} b_i \vec{T}_i(y) + \sum_{i=2}^{L+2} \vec{S}_i(y,b),}$$

where

- $\mathscr{H} = \begin{pmatrix} 0 & -1 \\ \mathscr{L} & 0 \end{pmatrix}, \quad \mathscr{L} = -\Delta_{r,d} + \dfrac{d-1}{r^2}\cos(2Q);$
- $\mathscr{H}\vec{T}_{k+1} = -\vec{T}_k, \quad \vec{T}_0 = \Lambda \vec{Q}, \quad \rightsquigarrow \ |\vec{T}_k(y)| \sim c_k y^{k-\gamma}$ for $y \gg 1$.

$\Longrightarrow$ **$\vec{S}_i$ has a better behavior than $\vec{T}_i$ in the blowup zone $\frac{1}{\sqrt{b_1}}$.**

Constructive proof for (**WM**): formal law of λ

- **The leading dynamical system driving the law:**

$$\boxed{(b_k)_s + (k-\gamma)b_1 b_k - b_{k+1} = 0 \quad \text{for } 1 \le k \le L} \quad \big(b_{L+1} = 0\big) \tag{Sys-b}$$

- **The explicit solution:** Fix $\ell \in \mathbb{N}^*$ with $\ell > \gamma$,

$$\bar{b}_k\Big|_{1\le k\le \ell} = \frac{c_k}{s^k}, \quad \bar{b}_k\Big|_{k\ge \ell+1} = 0, \quad c_1 = \frac{\ell}{\ell-\gamma}.$$

$$\boxed{b_1 \sim -\frac{\lambda_s}{\lambda} = -\lambda_t \quad \rightsquigarrow \quad \lambda(t) \sim (T-t)^{\frac{\ell}{\gamma}}.}$$

- **Linearizing** (Sys-b) **around** $\bar{b}$ **displays** $(\ell - 1)$ **unstable directions:**

$$b_k = \bar{b}_k + \frac{\beta_k}{s^k} \Longrightarrow s\beta_s = A_\ell \beta + \mathcal{O}(|\beta|^2) \overset{(\tau = \ln s)}{\Longrightarrow} \beta_\tau = A_\ell \beta + \mathcal{O}(|\beta|^2),$$

$$A_\ell = P_\ell^{-1} D_\ell P_\ell, \quad D_\ell = \mathrm{diag}\left\{-1, \frac{2\gamma}{\ell-\gamma}, \cdots, \frac{\ell\gamma}{\ell-\gamma}\right\}.$$

Constructive proof for (**WM**): modulation equations

- **The linearized problem:** $\vec{w}(y,s) = \vec{Q}_b(y) + \vec{q}(y,s)$,

$$\partial_s \vec{q} + b_1 \Lambda \vec{q} + \mathscr{H}\vec{q} = -\vec{E}_b + \vec{\mathrm{Mod}} + L(\vec{q}) + N(\vec{q}),$$

$$\text{where} \quad \vec{\mathrm{Mod}} \approx \sum_{k=1}^{L} \left[(b_k)_s + (k-\gamma) b_1 b_k - b_{k+1}\right] \vec{T}_k.$$

- **The modulation equations:** Projecting onto suitable directions yields

$$\boxed{\sum_{k=1}^{L} |(b_k)_s + (k-\gamma) b_1 b_k - b_{k+1}| \lesssim \sqrt{\mathcal{E}_{\Bbbk}} + b_1^{L+1+\nu}}$$

where $\Bbbk = L + 1 + \hbar$, $\hbar = \hbar(d) \geq 0$, $\nu \in (0,1)$,

$$\mathcal{E}_{\Bbbk} = \int_{\mathbb{R}^d} \left(q_1 \mathscr{L}^{\Bbbk} q_1 + q_2 \mathscr{L}^{\Bbbk-1} q_2\right) \gtrsim \|\vec{q}\|^2_{\dot{H}^{\Bbbk} \times \dot{H}^{\Bbbk-1}}.$$

under suitable orthogonality conditions.

Constructive proof for (**WM**): finite dimensional reduction

■ **Control $\mathcal{E}_{\Bbbk}$:** Local Morawetz control + Coercivity of $\mathscr{L}^{\Bbbk}$,

$$\frac{d}{ds}\left(\frac{\mathcal{E}_{\Bbbk}}{\lambda^{2\Bbbk-d}}\right) \lesssim \frac{b_1}{\lambda^{2\Bbbk-d}}\left(b_1^{L+\nu}\sqrt{\mathcal{E}_{\Bbbk}} + b_1^{2L+2\nu}\right) \Longrightarrow \mathcal{E}_{\Bbbk}(s) \lesssim b_1^{2L+2\nu}.$$

■ **A technical issue** (sharp control for b_L):

$$\left|(b_L)_s + (L-\gamma)b_L b_1 + \frac{d}{ds}\left(\frac{\left(\mathscr{L}^L\vec{q}, \chi_{B_0}\Lambda\vec{Q}\right)}{\left(\Lambda\vec{Q}, \chi_{B_0}\Lambda\vec{Q}\right)}\right)\right| \lesssim \frac{\sqrt{\mathcal{E}_{\Bbbk}}}{B_0^{2\nu}} + b_1^{L+1+\nu}.$$

■ **Finite dimensional reduction:** Control unstable directions $(P_\ell\beta)_k$ for $2 \le k \le \ell$ by a contradiction argument. $\rightsquigarrow$ The constructed solution is $(\ell-1)$-codimension stable.

3 - Results: The 2D Keller-Segel system

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \\ 0 = \Delta \Phi_u + u. \end{cases} \quad \text{in } \mathbb{R}^2.$$

Modeling features:

- Describing the *chemotaxis* in biology, [Patlak '53], [Keller-Segel '70], [Nanjundiah '73], [Hillen-Painter '09]; interacting stochastic many-particles system, [Othmer-Stevens '90], [Stevens '00]), [Chavanis '08], [Hillen-Painter '08]; as a diffusion limit of a kinetic model [Chalub-Markowich-Perthame-Schmeiser '04];
- Competition between dispersion of cells (diffusion) and aggregation;
- Rich model from mathematical point of view, [Horstman '03 & '04];

Basis features

$$\boxed{\partial_t u = \nabla \cdot (\nabla u - u\nabla\Phi_u)}$$

The 2D Keller-Segel equation:

$$\partial_t u = \nabla \cdot \big(u\nabla(\ln u - \Phi_u)\big), \quad \Phi_u = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|u(y)dy.$$

- mass conservation: $M = \int_{\mathbb{R}^2} u_0(x)dx = \int_{\mathbb{R}^2} u(x,t)dx;$
- L^1-scaling invariance: $\forall\gamma > 0, \;\; u_\gamma(x,t) = \frac{1}{\gamma^2}u\left(\frac{x}{\gamma}, \frac{t}{\gamma^2}\right), \quad \int_{\mathbb{R}^2} u_\gamma = \int_{\mathbb{R}^2} u;$
- free energy functional: $\mathcal{F}(u) = \int_{\mathbb{R}^2} u\left(\ln u - \frac{1}{2}\Phi_u\right), \quad \frac{d}{dt}\mathcal{F}(u) \leq 0;$
- stationary solution: $Q_{\gamma,a}(x) = \frac{1}{\gamma^2}Q\left(\frac{x-a}{\gamma}\right)$, where

$$Q(x) = \frac{8}{(1+|x|^2)^2}, \quad \int_{\mathbb{R}^2} Q = 8\pi.$$

Diffusion vs. Aggregation

$$\boxed{\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)}$$

- If $M < 8\pi$: global existence + spreading, [Blanchet-Dolbeault-Perthame '06]. The proof mainly relies on the free energy functional $\mathcal{F}(u)$ and the Log HLS inequality.
- If $M = 8\pi$ and $\int_{\mathbb{R}^2} |x|^2 u < +\infty$: **blowup in infinite time**, [Blanchet-Carrillo-Masmoudi '08]. Constructive approaches by [Ghoul-Masmoudi '18] (radial), [Davila-del Pino-Dolbeault-Musso-Wei '20] (full nonradial):

$$\|u(t)\|_{L^\infty} \sim c_0 \log t \quad \text{as} \quad t \to +\infty.$$

- If $M > 8\pi$: **blowup in finite time**, [Childress-Percus '81], [Jager-Luckhaus '92], [Nagai-Senba '98], [Senba-Suzuki '03]:

$$\text{(virial identity)} \quad \frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 u(x,t)dx = \frac{M}{2\pi}(8\pi - M).$$

  Constructive approaches in the radial setting by [Herrero-Velázquez '96], [Raphaël-Schweyer '14]:

$$\|u(t)\|_{L^\infty} \sim C_0 \frac{e^{\sqrt{2|\log(T-t)|}}}{T-t} \quad \text{as} \quad t \to T.$$

A numerical simulation of finite time singularity

A numerical simulation of blowup for the 2D Keller-Segel system

$$\partial_t u = \Delta u - \nabla.(u \nabla \Phi_u), \quad -\Delta \Phi_u = u.$$

Finite time blowup for the 2DKS

$$\boxed{\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)}$$

- Type I does not exist, [Senba-Suzuki '11]: $\partial_t u = \Delta u - \nabla u . \nabla \Phi_u + u^2$.
- Type II: "Δ dominates ∂_t" $\rightsquigarrow$ profile, **unknown blowup rates**.

Theorem 4 ([Collot-Ghoul-Masmoudi-Ng., 2020).

- There exists a set $\mathcal{O} \subset L^1 \cap \mathcal{E}$, where $\mathcal{E} = \{u : \sum_{k=0}^{2} \|\langle x \rangle^k \nabla^k u\|_{L^2} < +\infty\}$, of initial data u_0 (not necessary radially symmetric) such that

$$u(x,t) = \frac{1}{\lambda^2(t)} \left[Q\left(\frac{x - a(t)}{\lambda(t)} \right) + \varepsilon(x,t) \right],$$

where $a(t) \to \bar{a} \in \mathbb{R}^2$ and $\sum_{k=0}^{1} \|\langle y \rangle^k \nabla^k \varepsilon(t)\|_{L^2} \to 0$ as $t \to T$, and λ is given by either

$$\lambda(t) \sim 2e^{-\frac{\gamma+2}{2}} \sqrt{T-t} \exp\left(-\frac{\sqrt{|\log(T-t)|}}{\sqrt{2}} \right), \qquad \textbf{(C1)}$$

or

$$\lambda(t) \sim c(u_0)(T-t)^{\frac{\ell}{2}} |\log(T-t)|^{-\frac{\ell+1}{2(\ell-1)}}, \quad \ell \geq 2 \text{ integer}. \qquad \textbf{(C2)}$$

- Case (**C1**) is stable and Case (**C2**) is $(\ell - 1)$-codimension stable.

Comments

$$\boxed{\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)}$$

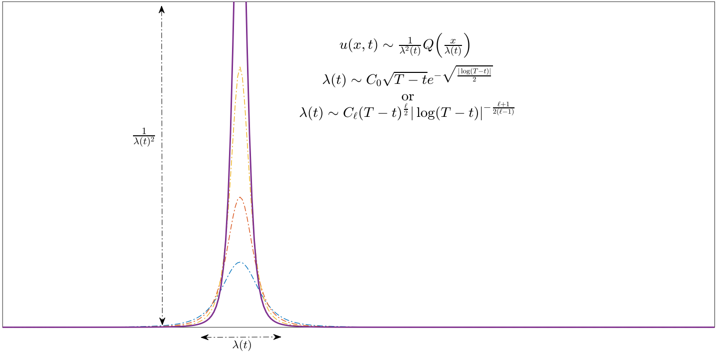


Fig 1: The form of single-point finite time blowup solutions.

Comments

$$\boxed{\partial_t u = \nabla \cdot \left(\nabla u - u \nabla \Phi_u\right)}$$

■ Existing results: formal level (numerical observation, formal matching asymptotic expansions) and in the radial setting to remove the nonlocal structure difficulty, i.e. $u(x,t) = u(r,t)$,

$$m(r) = \int_0^r u(\zeta)\zeta d\zeta, \quad u(r) = \frac{\partial_r m(r)}{r}, \quad \partial_r \Phi_u(r) = -\frac{m(r)}{r}, \quad r = |x|,$$

$$\partial_t u = \frac{1}{r}\partial_r \left(r\partial_r u - ru\partial_r \Phi_u\right) \quad \Longrightarrow \quad \boxed{\partial_t m = \partial_r^2 m - \frac{\partial_r m}{r} + \frac{\partial_r m^2}{2r}}$$

Refs: [Herrero-Velazquez '96 & '97], [Velazquez '02], [Schweyer-Raphael '14], [Dyachenko-Lushnikov-Vladimirova '13], ...

■ The new result: full nonradial setting, refined description of the stable blowup mechanism, new (unstable) blowup dynamics, a nature approach via spectral analysis/robust energy-type method.

Strategy of the new constructive proof

$$\boxed{\partial_t u = \nabla \cdot \big(\nabla u - u \nabla \Phi_u\big)}$$

- Self-similar variables:

$$u(x,t) = \frac{1}{T-t} w(z,\tau), \quad z = \frac{x}{\sqrt{T-t}}, \quad \tau = -\log(T-t),$$

$$\boxed{\partial_\tau w = \nabla \cdot (\nabla w - w \nabla \Phi_w) - \frac{1}{2}\nabla \cdot (zw)}$$

- Linearized problem: $w(z,\tau) = Q_\nu(z) + \eta(z,\tau)$, where $Q_\nu(z) = \frac{1}{\nu^2} Q\big(\frac{z}{\nu}\big)$ and η solves

$$\partial_\tau \eta = \mathscr{L}^\nu \eta + \left(\frac{\nu_\tau}{\nu} - \frac{1}{2}\right) \nabla \cdot (zQ_\nu) - \nabla \cdot \Big(\eta \Phi_\eta\Big), \qquad \nu \to 0 \ \text{ unknown},$$

$$\boxed{\mathscr{L}^\nu \eta = \underbrace{\nabla \cdot \Big(\nabla \eta - \eta \nabla \Phi_{Q_\nu} - Q_\nu \nabla \Phi_\eta\Big)}_{\equiv \mathscr{L}_0^\nu \eta} - \frac{1}{2}\nabla \cdot (z\eta)}$$

- Structure of $\mathscr{L}_0^\nu$:

$$\mathscr{L}_0^\nu \eta = \nabla \cdot \Big(Q_\nu \nabla \mathscr{M}^\nu \eta\Big), \quad \mathscr{M}^\nu \eta = \frac{\eta}{Q_\nu} - \Phi_\eta.$$

($\mathscr{M}^\nu$ comes from the linearization of the energy functional $\mathcal{F}$ around Q_ν).

Properties of the linearized operator $\mathscr{L}^\nu = \nabla.(Q_\nu \nabla \mathscr{M}^\nu \cdot) - \frac{1}{2}\nabla.(z\cdot)$

- In the radial sector, the (nonlocal) operator $\mathscr{L}^\nu$ becomes a local operator through the partial mass setting, i.e. $\zeta = |z|$, $m_f(\zeta) = \int_0^\zeta f(r)rdr$,

$$\mathscr{L}^\nu f = \frac{1}{\zeta}\partial_\zeta\left(\mathscr{A}^\nu m_f\right), \quad \boxed{\mathscr{A}^\nu \phi = \zeta\partial_\zeta\left(\frac{\partial_\zeta\phi}{\zeta}\right) - \frac{\partial_\zeta(m_{Q_\nu}\phi)}{\zeta} - \frac{1}{2}\zeta\partial_\zeta\phi} \equiv \mathscr{A}_0^\nu\phi - \frac{1}{2}\zeta\partial_\zeta\phi.$$

- [Collot-Ghoul-Masmoudi-Ng., '20]: $\mathscr{A}^\nu$ is self-adjoint in $L^2_{\frac{\omega_\nu}{\zeta}}$, its eigenvalues are

$$\boxed{\mathrm{spec}(\mathscr{A}^\nu) = \left\{\alpha_{n,\nu} = 1 - n + \frac{1}{2\ln\nu} + \mathcal{O}\left(\frac{1}{|\ln\nu|^2}\right), \quad n \in \mathbb{N}\right\}} \quad \omega_\nu = \frac{e^{-\frac{\zeta^2}{4}}}{Q_\nu}.$$

The eigenfunction $\phi_{n,\nu}$ solving $\mathscr{A}^\nu\phi_{n,\nu} = \alpha_{n,\nu}\phi_{n,\nu}$ is defined by

$$\phi_{n,\nu}(\zeta) = \sum_{j=0}^n c_{n,j}\nu^{2j-2}T_j\left(\frac{\zeta}{\nu}\right) + lot, \quad \mathscr{A}_0^\nu T_{j+1} = -T_j, \quad T_0 = \xi\partial_\xi m_Q.$$

Proof: Schrödinger type operator $\rightsquigarrow$ *discreteness*, Sturm comparison principle $\rightsquigarrow$ *uniqueness*, matching asymptotic expansions + implicit function theorem $\rightsquigarrow$ $(\alpha_{n,\nu}, \phi_{n,\nu})$.

Properties of the linearized operator $\mathscr{L}^\nu = \nabla.(Q_\nu \nabla \mathscr{M}^\nu \cdot) - \frac{1}{2}\nabla.(z\cdot)$

- First expression:

$$\mathscr{L}^\nu f = \mathscr{L}_0^\nu f - \frac{1}{2}\nabla.(zf) \quad \text{with} \quad \mathscr{L}_0^\nu f = \nabla \cdot (Q_\nu \nabla \mathscr{M}^\nu f) \text{ and } \mathscr{M}^\nu f = \frac{f}{Q_\nu} - \Phi_f,$$

The operator $\mathscr{L}_0^\nu$ is self-adjoint in L^2 with respect to the inner product

$$\langle f, g \rangle_{\mathscr{M}^\nu} = \int_{\mathbb{R}^2} f \mathscr{M}^\nu g \, dz, \quad \text{(positivity)} \quad \langle f, f \rangle_{\mathscr{M}^\nu} \sim \int_{\mathbb{R}^2} \frac{f^2}{Q_\nu} \, dz.$$

- Second expression:

$$\mathscr{L}^\nu f = \mathscr{H}^\nu f - \nabla Q_\nu \cdot \nabla \Phi_f \quad \text{with} \quad \mathscr{H}^\nu f = \frac{1}{\omega_\nu} \nabla \cdot \Big(\omega_\nu \nabla f\Big) + (2Q_\nu - 2)f.$$

The operator $\mathscr{H}^\nu$ is self-adjoint in $L^2_{\omega_\nu}$ with $\omega_\nu = \frac{e^{-\frac{|z|^2}{4}}}{Q_\nu}$.

- The well-adapted scalar product and coercivity:

$$\boxed{\int_{\mathbb{R}^2} \mathscr{L}^\nu(f\sqrt{\rho}) \mathscr{M}^\nu(f\sqrt{\rho}) \leq -c \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{Q_\nu} \rho dz} \quad \rho = e^{-\frac{|z|^2}{4}}.$$

Approximate solution

$$\boxed{\partial_\tau w = \nabla \cdot (\nabla w - w\nabla\Phi_w) - \frac{1}{2}\nabla\cdot(zw)}$$

- The approximate solution: for $\ell \geq 1$ integer,

$$w^{app}(z,\tau) = Q_\nu(z) + \underbrace{a_\ell(\tau)\Big[\varphi_{\ell,\nu}(|z|) - \varphi_{0,\nu}(|z|)\Big]}_{\text{modification driving the law of blowup}} \quad \text{with} \quad \varphi_{n,\nu} = \frac{\partial_\zeta \phi_{n,\nu}}{\zeta}.$$

A suitable projection onto $\varphi_{\ell,\nu}$ and compatibility condition, we obtain the leading ODE

$$(\ell = 1,\ \text{stable}) \quad \frac{\nu_\tau}{\nu} = \frac{1}{4\ln\nu} + \frac{e_2}{|\ln\nu|^2} \quad \Longrightarrow \quad \boxed{\nu = C_0 e^{-\sqrt{\frac{\tau}{2}}}}$$

$$(\ell \geq 2,\ \text{unstable}) \quad \frac{\nu_\tau}{\nu} = \frac{1-\ell}{2} + \frac{\ell+1}{4\ln\nu} \quad \Longrightarrow \quad \boxed{\nu = C_\ell e^{\frac{(1-\ell)\tau}{2}}\tau^{\frac{1+\ell}{2(1-\ell)}}}$$

- The linearized equation: $\varepsilon = w - w^{app}$,

$$\partial_\tau \varepsilon = \mathscr{L}^\nu \varepsilon + \mathit{Error} + \mathit{SmallLinear} + \mathit{Nonlinear}.$$

Control of the remainder

$$\boxed{\partial_\tau \varepsilon = \mathscr{L}^\nu \varepsilon + Error + \cdots}$$

- Decomposition: $\varepsilon = \varepsilon^0 + \varepsilon^\perp$, $\varepsilon^0(\zeta) = \frac{\partial_\zeta m_\varepsilon}{\zeta}$,

$$\partial_\tau m_\varepsilon = \mathscr{A}^\nu m_\varepsilon + m_E + \cdots, \quad \partial_\tau \varepsilon^\perp = \mathscr{L}^\nu \varepsilon^\perp + \cdots .$$

- For the radial part, we use the spectral gap

$$\langle m_\varepsilon, \mathscr{A}^\nu m_\varepsilon \rangle_{L^2_{\frac{\omega_\nu}{\zeta}}} \le \alpha_{N+1,\nu} \|m_\varepsilon\|^2_{L^2_{\frac{\omega_\nu}{\zeta}}} \quad \text{for} \quad m_\varepsilon \perp \phi_{n,\nu}, \quad n = 0, ..., N.$$

$$\Longrightarrow \quad \boxed{\frac{d}{d\tau}\|m_\varepsilon\|^2_{L^2_{\frac{\omega_\nu}{\zeta}}} \le -\|m_\varepsilon\|^2_{L^2_{\frac{\omega_\nu}{\zeta}}} + C\frac{\nu^2}{|\ln \nu|^2}}$$

- For the nonradial part, we use the coerivity of $\mathscr{L}^\nu$ and the well-adapted norm

$$\|\varepsilon^\perp\|_0^2 = \int_{\mathbb{R}^2} \varepsilon^\perp \sqrt{\rho}\mathscr{M}^\nu(\varepsilon^\perp\sqrt{\rho})\, dz \sim \int_{\mathbb{R}^2} \frac{|\varepsilon^\perp|^2}{Q_\nu}\rho\, dz.$$

$$\Longrightarrow \quad \boxed{\frac{d}{d\tau}\|\varepsilon^\perp\|_0^2 \le -c\|\varepsilon^\perp\|_0^2 + Ce^{-2\kappa\tau}} \quad 0 < \kappa \ll 1.$$

- Nonlinear analysis: ...

4 - Conclusion & Perspectives

Conclusion and Perspectives

■ Existence/Stability of blowup solutions via constructive approaches:

- Spectral analysis: (non)-variational problems whose spectrum of the linearized operator is fairly understood. $\rightsquigarrow$ **Classification** of blowup dynamics.
- Energy methods: *robust* for various problems (parabolic/hyperbolic) with variational structures, but gives no answers to the classification question.
- A combination of the two methods is effective for complicated problems.

■ Adaptability and Flexibility for studying singularity formation, asymptotic stability, dynamical classification, stability and instability of steady states, long-time asymptotic, ...

■ Interesting problems:

- multiple-collapse phenomena/ interaction-collision of multi-solitons;
- *classification* of blowup dynamics (rates & profiles);

Multiple collapse phenomena

A multiple-collapse phenomenon in the 2D Keller-Segel system