On isolated singular solutions to Lane-Emden equation

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(based on joint works with H.Y.Chen, X.Huang and Z.M.Guo)

Workshop on "Singular problems associated to quasilinear equations"
in honor of 70th birthday of
Professors Marie-Françoise Bidaut-Véron and Laurent Véron

ShanghaiTech University and Masaryk University

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The case $f(t,x,u) \sim t^\beta g(u)$, $\beta > -1$
For proving equality, we use a function introduced by
Bonforte-Vazquez
which satisfies
Put
then
and, for any

Singular Lane-Emden Equation

Consider the following semilinear elliptic equation:

$$-\Delta u = u^p \quad \text{in } \Omega, \tag{2.1}$$

where Ω is a smooth domain in $\mathbb{R}^N$ with $N \geq 3$.

- $x_0 \in \Omega$, if $\lim_{x \to x_0} u(x) = +\infty$, we call that u is singular at x_0 (singular solution).

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- If $p > \frac{N}{N-2}$, there exists always a solution $w_\infty(x) \equiv c_p |x|^{-\frac{2}{p-1}}$ (slow decay solution).

Motivation

[H.Brezis, L.Véron, *Arch.Rat.Mech. and Anal.*, (1980)],
If $p \geq \frac{N}{N-2}$, $0 \in \Omega \subset \mathbb{R}^N$, $N \geq 3$, and

$$-\Delta u + |u|^{p-1}u \leq C, \quad \text{in} \;\; \Omega' = \Omega \setminus \{0\},$$

then

$$\limsup_{x \to 0} u(x) < \infty,$$

and thus if $u \in C^2(\Omega')$ satisfying

$$-\Delta u + |u|^{p-1}u = 0, \quad \text{in} \;\; \Omega',$$

then $\exists$ C^2 function in Ω which coincides with u on Ω'. That is the equation

$$-\Delta u + |u|^{p-1}u = 0$$

has the property that *any isolated singularity is "removable"*.

Motivation

[L.Véron, *Nonl. Anal.*, (1981)], Classification of isolated singularities of any solution of

$$-\Delta u + |u|^{p-1}u = 0, \quad \text{in } \Omega' = \Omega \setminus \{0\},$$

If 0 is the singular point, there exists two types of singularities when $1 < p < \frac{N}{N-2}$, and as $x \to 0$,

- either $u(x) \sim \pm c_p |x|^{-\frac{2}{p-1}}$.

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If 0 is the singular point, there exists two types of singularities when $1 < p < \frac{N}{N-2}$, and as $x \to 0$,

- either $u(x) \sim \pm c_p |x|^{-\frac{2}{p-1}}$.
- or $u(x) \sim c|x|^{2-N}$, where c is any constant.

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Singular solution in bounded domain

Example 1: Consider the Dirichlet boundary value problem:

$$-\Delta u = u^p \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega \tag{2.2}$$

where Ω is a smooth open set in $\mathbb{R}^N$ and for suitable range for the exponent p.

$u \in L^p(\Omega)$ is called a weak solution of (2.2) if the equality

$$\int_\Omega u\Delta\varphi dx + \int_\Omega u^p\varphi dx = 0$$

holds for any $\varphi \in C^2(\overline{\Omega})$ and $\varphi = 0$ on $\partial\Omega$.

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- $S \subseteq \Omega$ is called a singular set for a weak solution u of (2.1) if for any $x \in S$, u is not bounded in any neighborhood of x.

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- S is then a closed subset of Ω.

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- C.C. Chen and C.S. Lin, Existence of positive weak solutions with a prescribed singular set of semilinear elliptic equations, *J. Geom. Anal.* 9 (1999), 221-246.

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- By Variational Methods.

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Many works on the asymptotic behavior of the solutions near a isolated singularity of the equations as (2.2):

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- by Aviles when $p = \frac{n}{n-2}$, (Indiana Univ.Math. J. 1983)

$$u(x) = \frac{c_0 + o(1)}{(-|x|^2 \log |x|)^{\frac{n-2}{2}}}, \quad \text{as} x \to 0$$

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- by Caffarelli, Gidas and Spruck in the case of $p = \frac{n+2}{n-2}$ (CPAM, 1989).

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- Pacard: Existence and convergence of positive weak solutions of $-\Delta u = u^{\frac{n}{n-2}}$ in bounded domains of $\mathbb{R}^n$, $n \geq 3$, *Calc. Var. PDEs* 1 (1993), 243-265.

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- N.Korevaar, R.Mazzeo, F.Pacard and R. Schoen, Invent. Math. 1999.

Singular Lane-Emden Equation

Other important exponent:

The linearized operator at u_∞ is

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- Recall the Hardy inequality:

$$\int_{\mathbb{R}^N} |\nabla \phi|^2 dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{\phi^2}{|x|^2} dx + \mathcal{R}, \quad \forall\, \phi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}). \tag{2.3}$$

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- Take $pc_p^{p-1} = \frac{(N-2)^2}{4}$, there exists a unique root, called $p_c := \frac{N+2\sqrt{N-1}}{N-4+2\sqrt{N-1}} \in (\frac{N}{N-2}, \frac{N+2}{N-2})$.

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- Refs for p_c: Joseph-Lundgren; Brezis-Vazquez; Gui-Ni-Wang; Y.Li; ...etc...
Ye-Zhou (CCM, 2001, extremal solution);
Guo-Zhou (Sci. China, 2020, radial entire solutions for quasilinear elliptic equations)

Nonhomogeneous elliptic equations with the Hardy-Leray potentials

Example 2: Assume $0 \in \Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded C^2- domain.
Let $\mu \geq \mu_0 := -\frac{(N-2)^2}{4}$ and $\mathcal{L}_\mu := -\Delta + \frac{\mu}{|x|^2}$.
[Vazquez-Véron, Felmer-Quaas] What is the weakest assumption on h such that any isolated singularity of a nonnegative solution of

$$-\Delta + \frac{\mu}{|x|^2} + h(u) = 0 \quad \text{in } \Omega'$$

is "removable"? (i.e. u can be extended to a C^1-solution in $\mathcal{D}'(\Omega)$)
For solution u, we mean $u \in C^1(\Omega')$ satisfies

$$\int_\Omega (-\Delta u)\Phi dx + \int_\Omega \frac{\mu}{|x|^2} u\Phi dx + \int_\Omega h(u)\Phi dx = 0, \qquad \forall\, \Phi \in C_c^1(\Omega'), \tag{2.4}$$

Equations with the Hardy-Leray potentials

[Chen-Z.] DCDS, 2018. [Chen-Quaas-Z.] JAM, 2020.
[Chen-Quaas-Z.] PAFA, 2020.

A deeper knowledge of distributional identities allows us to draw a complete picture of the existence, non-existence and the singularities for the nonhomogeneous problem

$$\begin{cases} \mathcal{L}_\mu u = f & \text{in} \quad \Omega \setminus \{0\}, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.5}$$

The weighted Lane-Emden equation in punctured domain

We consider fast decaying solutions of weighted Lane-Emden equation in punctured domain

$$\begin{cases} -\Delta u = V(x)u^p & \text{in} \quad \mathbb{R}^N \setminus \{0\}, \\ u > 0 & \text{in} \quad \mathbb{R}^N \setminus \{0\}, \end{cases} \tag{3.6}$$

where $p > 1$, $N \geq 3$ and the potential V is a locally Hölder continuous function in $\mathbb{R}^N \setminus \{0\}$.

Nonhomogeneous potential case V

- $V(x) = |x|^{\alpha_0}(1+|x|)^{\beta-\alpha_0}$, M.Bidaut-Véron,-S.Pohozaev, 2001, nonexistence provided $\beta > -2$ and $p \leq \frac{N+\beta}{N-2}$ (also Armstrong-Sirakov, Ann. di Pisa, 2011);

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- For $p \in (\frac{N+\beta}{N-2}, \frac{N+\alpha_0}{N-2}) \cap (0, +\infty)$ with $\alpha_0 \in (-N, +\infty)$ and $\beta \in (-\infty, \alpha_0)$, Chen-Felmer-Yang (IHP, 2018) constructed the infinitely many positive solutions by dealing with the distributional solutions of

$$-\Delta u = Vu^p + \kappa\delta_0 \quad \text{in} \quad \mathbb{R}^N, \tag{3.7}$$

where $k > 0$, δ_0 is a Dirac mass at the origin.

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- For $V \equiv 1$ and $p \ge \frac{N+2}{N-2}$, study by
  Davila-Del Pino-MussoWei, Calc. Var., 2008.
  Dancer-Du-Guo, JDE, 2011 ($p > \frac{N+2}{N-2}$, exterior domain).
  Mazzeo-Pacard, Duke J. 1999.

etc...

Conditions on potential V

Assume that the potential function V is Hölder continuous and satisfies the following conditions:

$(\mathcal{V}_0)$ (i) near the origin,

$$|V(x)-1| \le c_0|x|^{\tau_0} \quad \text{for} \;\; x\in B_1(0), \tag{3.8}$$

for some $c_0>0$ and $\tau_0>0$;

(ii) global control,

$$0 \le V(x) \le c_\infty(1+|x|)^\beta \quad \text{for} \quad |x|>0, \tag{3.9}$$

where $c_\infty \ge 1$ and $\beta \in \mathbb{R}$.

Existence of fast decaying solutions

Theorem

(Chen-Huang-Z., Adv. Nonlinear Stud. 2020) Let $p \in \left(\frac{N}{N-2}, p_c\right)$, *$V$ satisfies* $(\mathcal{V}_0)$ *with* τ_0, β *verifying* ($A := \frac{4}{p-1} - N + 2$)

$$\tau_0 > \tau_p^* := \frac{1}{2}\left(A - \sqrt{A^2 - 8(N - 2 - \frac{2}{p-1})}\right) > 0 \qquad (3.10)$$

and

$$\beta < (N-2)p - N. \qquad (3.11)$$

Then $\exists\, \nu_0 > 0$ *s.t. for any* $\nu \in (0, \nu_0]$, *(3.6) has a* ν*-fast decaying solution* u_ν, *which has singularity at the origin as*

$$\lim_{|x| \to 0} u_\nu(x)|x|^{\frac{2}{p-1}} = c_p, \qquad (3.12)$$

Existence of fast decaying solutions

Theorem

(continued) Furthermore, the mapping $\nu \in (0, \nu_0] \mapsto u_\nu$ is increasing, continuous and satisfies that

$$\lim_{\nu \to 0} \|u_\nu\|_{L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})} = 0. \tag{3.13}$$

Some remarks

- Here, $\nu \in (0, \nu_0]$ = an interval (parametrization)!
  Main difficulty: V breaks the scaling invariance of the equation. No ODE's method, No variational method.

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  Main difficulty: V breaks the scaling invariance of the equation. No ODE's method, No variational method.
- In general, the monotonicity (increasing) of mapping $k \mapsto \tilde{\nu}_k$ fails.

When $\nu_0 = +\infty$?

$(\mathcal{V}_1)$ (I) $V \geq 1$ in $\mathbb{R}^N \setminus \{0\}$ and there exist $\alpha_1 \geq 0$, $l_1 > 1$ such that

$$V(l_1^{-1}x) \geq l_1^{-\alpha_1} V(x), \qquad \forall x \in \mathbb{R}^N \setminus \{0\}; \tag{3.14}$$

(II) $V \leq 1$ in $\mathbb{R}^N \setminus \{0\}$ (i.e. $c_\infty = 1$, $\beta = 0$ in $(\mathcal{V}_0)$) and there exist $\alpha_2 \leq 0$, $l_2 > 1$ such that

$$V(l_2^{-1}x) \leq l_2^{-\alpha_2} V(x), \qquad \forall x \in \mathbb{R}^N \setminus \{0\}. \tag{3.15}$$

Theorem

Assume that V verifies $(\mathcal{V}_0)$ with τ_0, β verifying (3.10) and (3.11) respectively and $p \in \left(\frac{N}{N-2}, p_c\right)$.
If $(\mathcal{V}_1)$ part (I) or part (II) holds, then for any $\nu \in (0, +\infty)$, (3.6) has a ν-fast decaying solution u_ν, which has singularity at $\{0\}$ verifying (3.12) and the mapping $\nu \in (0, \infty) \mapsto u_\nu$ is increasing, continuous and (3.13) holds true.

The limit of $\{u_\nu\}_\nu$ as $\nu \to +\infty$

$(\mathcal{V}_\infty)$ Assume that V is radially symmetric, decreasing w.r.t. $|x|$ and

$$\frac{1}{\gamma}|x|^\alpha \le V(x) \le \gamma|x|^\alpha \quad \text{for} \quad |x| > 1, \tag{3.16}$$

where $\gamma > 1$ and

$$(N-2)p_c - N - 2 < \alpha \le 0. \tag{3.17}$$

Theorem

Assume that $p \in \left(\frac{N}{N-2}, p_c\right)$, V verifies $(\mathcal{V}_0)$ part (i) with τ_0 satisfying (3.10), $(\mathcal{V}_1)$ part (II) and $(\mathcal{V}_\infty)$. Then the limit $u_\infty := \lim_{\nu\to+\infty} u_\nu$ exists and is a solution of (3.6) verifying (3.12) and

$$\frac{1}{c_1} \le u_\infty(x)|x|^{\frac{2+\alpha}{p-1}} \le c_1, \quad |x| \ge 1, \tag{3.18}$$

where $c_1 > 1$.

$V \equiv 1$, Lane-Emden equation

- When $p \in (\frac{N}{N-2}, \frac{N+2}{N-2})$, positive isolated singular solutions have the following structure:
  (i) *either a k-fast decaying solution* w_k *with* $k > 0$ *such that*

$$\lim_{|x| \to 0^+} w_k(x)|x|^{\frac{2}{p-1}} = c_p \quad \textit{and} \quad \lim_{|x| \to +\infty} w_k(x)|x|^{N-2} = k.$$

  (ii) or *a slow decaying solution* $w_\infty(x) = c_p|x|^{-\frac{2}{p-1}}$ *and* $w_\infty = \lim_{k \to +\infty} w_k$.
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$$\lim_{|x|\to +\infty} u(x)|x|^{N-2} = +\infty.$$

- Conversely, for any $k > 0$, there exists an unique k-fast decaying solution w_k.

$V \equiv 1$, Lane-Emden equation

Furthermore, the fast decaying solution w_k could be written by

$$w_k(x) = |x|^{-\frac{2}{p-1}} \bar{w}_p(-\ln|x| + b_p^{-1}(\ln k - \ln d_0)),$$

where $b_p = N - 2 - \frac{2}{p-1} > 0$, $c > 0$ is independent of k and $\bar{w}_p(\cdot)$ is a positive and bounded function independently of k. Assume that $t = -\ln|x| + b_p^{-1}(\ln k - \ln d_0)$, then the function $\bar{w}_p$ satisfies

$$\begin{cases} \bar{w}_p'' - \left(N - 2 - \frac{4}{p-1}\right)\bar{w}_p' - c_p^{p-1}\bar{w}_p + \bar{w}_p^p = 0 \quad \text{in} \quad \mathbb{R}, \\ \bar{w}_p(-\infty) = 0 \quad \text{and} \quad \bar{w}_p(+\infty) = c_p. \end{cases} \tag{3.19}$$

$V \equiv 1$, Lane-Emden equation

- $\bar{w}_p$ is increasing for $p \in (\frac{N}{N-2}, \ , p_c]$ and for $p \in (p_c, \frac{N+2}{N-2})$, $\bar{w}_p$ is oscillating as $t \to +\infty$.

Some preliminary results

Proposition

(i) For $p \in (\frac{N}{N-2}, p_c]$, *we have that*

$$p \cdot \sup_{t\in\mathbb{R}} \bar{w}_p^{p-1} \leq \frac{(N-2)^2}{4}, \tag{3.20}$$

where $'='$ *holds only for* $p = p_c$.
(ii) For $p \in (p_c, \frac{N+2}{N-2})$, *we have that*

$$p \cdot \sup_{t\in\mathbb{R}} \bar{w}_p^{p-1} > \frac{(N-2)^2}{4}. \tag{3.21}$$

Some preliminary results

Lemma

Let $p \in (\frac{N}{N-2}, p_c]$ *and* $b_p = N - 2 - \frac{2}{p-1}$, *then*

$$w_k(x) = |x|^{-\frac{2}{p-1}} \bar{w}_p(-\ln|x| + b_p^{-1}(\ln k - \ln d_0)), \tag{3.22}$$

and for any $r \in (0, 1]$, *there exists* $k_r = r^{b_p}$ *such that for* $0 < k \le k_r$, $\forall\, x \in \mathbb{R}^N \setminus \{0\}$,

$$w_k(x) \le ckr^{2-N}(1+|x|)^{2-N}\chi_{\mathbb{R}^N \setminus B_r(0)}(x) + c_p|x|^{-\frac{2}{p-1}}\chi_{B_r(0)}(x), \tag{3.23}$$

where $c > 0$ *is independent of* k, r.

Some preliminary results

Lemma

Assume that $\alpha \in (0, N)$, f *is a nonnegative function satisfying that*

$$|f(x)| \le |x|^{-\theta}(1+|x|)^{\theta-\tau} \quad \text{for} \quad |x| > 0$$

with $\alpha < \theta < N$ *and* $\tau > N$. *Then there exists* $c > 0$ *such that*

$$\int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\alpha}} \le c|x|^{-\theta+\alpha}(1+|x|)^{-N+\tau}, \quad \forall\, x \in \mathbb{R}^N \setminus \{0\}. \tag{3.24}$$

Some preliminary results

A comparison principle for general Hardy operator:

Lemma

Let W be Hölder continuous locally in $\bar{\Omega} \setminus \{0\}$ s.t.
$\lim\limits_{|x|\to 0} W(x)|x|^2 = \mu$ *with* $\mu \in \left(0, \frac{(N-2)^2}{4}\right)$ *and*
$W(x) \le \frac{(N-2)^2}{4}|x|^{-2}$ *in* Ω. *Then* $\mathcal{L}_W w := -\Delta w - Ww$ *verifies:*

$$f_1 \ge f_2 \quad \text{in} \quad \Omega \setminus \{0\} \quad \text{and} \quad g_1 \ge g_2 \quad \text{on} \quad \partial\Omega.$$

imply that: If $\liminf\limits_{x\to 0} u_1(x)|x|^{-\tau_-(\mu)} \ge \limsup\limits_{x\to 0} u_2(x)|x|^{-\tau_-(\mu)}$ *holds, then* $u_1 \ge u_2$ *in* $\Omega \setminus \{0\}$, *where* u_i *($i = 1, 2$) are the classical solutions of*

$$\begin{cases} \mathcal{L}_W u = f_i & \text{in } \Omega \setminus \{0\}, \\ u = g_i & \text{on } \partial\Omega. \end{cases}$$

Proof of Theorem for singular solution

Main idea: using the Schauder fixed point theorem to construct a solution v_k of the problem

$$-\Delta v = V(w_k + v)_+^p - w_k^p \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{4.25}$$

for $k > 0$ sufficiently small and w_k is the k-fast decaying solution for $V \equiv 1$, then a $\tilde{\nu}_k$-fast decaying solution $\tilde{u}_{\nu_k} := v_k + w_k$ of (3.6) is derived.

Proposition

Assume that $p \in \left(\frac{N}{N-2},\, p_c\right)$, V verifies $(\mathcal{V}_0)$ with τ_0, β verifying (3.10) and (3.11) respectively. Then $\exists\, k^* > 0$ s.t. $\forall\, k \in (0,\, k^*)$, (4.25) has a classical solution v_k such that

$$|v_k(x)| \le ck|x|^{-\theta_0}(1 + |x|)^{2-N+\theta_0}, \quad \forall\, x \in \mathbb{R}^N \setminus \{0\}, \tag{4.26}$$

for some $\theta_0 \in [\frac{N-2}{2}, \frac{2}{p-1})$ is well determined. and τ_p^* is from the assumption (3.10).

Proof of Proposition

- The choice of θ_0 shows that

$$pc_p^{p-1} < \theta_0(N-2-\theta_0) \le (\frac{N-2}{2})^2. \tag{4.27}$$

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- Let $q_0 \in \left(\frac{N}{N-1}, \frac{N}{\theta_0+1}\right)$, and

$$\mathcal{D}_\epsilon := \left\{ v \in L^{q_0}(\mathbb{R}^N) : \, |v(x)| \leq \epsilon |x|^{-\theta_0}(1+|x|)^{2-N+\theta_0}, \; \forall\, x \neq 0 \right\}, \tag{4.28}$$

and

$$\mathcal{T} v := \Gamma * \left(V\left(w_k + v\right)_+^p - w_k^p \right), \quad \forall\, v \in \mathcal{D}_\epsilon, \tag{4.29}$$

where Γ is the fundamental solution of $-\Delta$ in $\mathbb{R}^N$.

Proof of Proposition

- $\mathcal{T}\mathcal{D}_\epsilon \subset \mathcal{D}_\epsilon$ for $\epsilon,\ k > 0$ small suitably.
  2 cases: $p \in \left(\frac{N}{N-2}, p_c\right) \cap (1,2]$ (for $N \geq 5$) and
  $$p \in (\tfrac{N}{N-2}, p_c) \cap (2,+\infty) \text{ (for } N = 3,4).$$
  We have used
  - $\delta_0 := 1 - \frac{p c_p^{p-1}}{\theta_0(N-2-\theta_0)} \max_{B_{r^*}(0)} V > 0.$

Proof of Proposition

- $\mathcal{TD}_\epsilon \subset \mathcal{D}_\epsilon$ for $\epsilon,\ k > 0$ small suitably.
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  We have used
  - $\delta_0 := 1 - \frac{p c_p^{p-1}}{\theta_0(N-2-\theta_0)} \max_{B_{r^*}(0)} V > 0.$
  - $\beta < p(N-2) - N$
  - $\epsilon = c_{\delta_0} k^p$
- $\mathcal{T}\mathcal{D}_\epsilon \subset W^{1,q_0}(\mathbb{R}^N) \cap \mathcal{D}_\epsilon$ and $\mathcal{T}$ is compact.

Proof of Proposition

Remark: Under the assumption of Proposition 4.1,
if $V \le 1$, we can take

$$\mathcal{D}_\epsilon^- := \left\{ w \in L^{q_0}(\mathbb{R}^N) : \, -\epsilon |x|^{-\theta_0}(1+|x|)^{2-N+\theta_0} \le w(x) \le 0, \; \forall\, x \ne 0 \right\}; \tag{4.30}$$

if $V \ge 1$, we can take

$$\mathcal{D}_\epsilon^+ := \left\{ w \in L^{q_0}(\mathbb{R}^N) : \, 0 \le w(x) \le \epsilon |x|^{-\theta_0}(1+|x|)^{2-N+\theta_0}, \; \forall\, x \ne 0 \right\}. \tag{4.31}$$

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- $\mathcal{D}_\epsilon$ is a closed and convex, applying Schauder fixed point theorem, then $\exists\ v_k \in \mathcal{D}_\epsilon$ such that

$$\mathcal{T} v_k = v_k$$

which is a classical solution of (4.25).

Proof of Theorem

The proof of Theorem is based on

Theorem

(Existence when V is comparable to 1) Under assumptions of Theorem 9, let $V \geq 1$. Then $\exists\ \nu_0 \in (0, +\infty]$ such that $\forall\ \nu \in (0, \nu_0)$, (3.6) has a ν-fast decaying solution u_ν, which has singularity at 0 as (3.12) and the mapping $\nu \in (0, \nu_0) \mapsto u_\nu$ is increasing, continuous and (3.13) holds. Moreover, if (3.14) holds for some $\alpha_1 \geq 0$ and $l_1 > 1$, then $\nu_0 = +\infty$.

and the following: Let

$$V_1 = 1 - (V-1)_- \quad \text{and} \quad V_2 = 1 + (V-1)_+,$$

then $V = V_2 V_1$ in $\mathbb{R}^N \setminus \{0\}$ and consider

$$v_n = \Gamma * (V_2 V_1 v_{n-1}^p) \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$

with the initial data $v_0 := u_{\nu,1}$.

Proof of Theorem

- $$\tilde{u}_{\nu_k} = w_k + v_k \geq w_k \quad \text{and} \quad \tilde{\nu}_k = c_N \int_{\mathbb{R}^N} V \tilde{u}_{\nu_k}^p dx,$$

  then $k \leq \tilde{\nu}_k \leq k + c_{\delta_0} k^p$ and $\tilde{u}_{\nu_k}$ satisfies then

  $$\lim_{|x|\to 0^+} \tilde{u}_{\nu_k}(x)|x|^{\frac{2}{p-1}} = c_p \quad \text{and} \quad \lim_{|x|\to +\infty} \tilde{u}_{\nu_k}(x)|x|^{N-2} = \tilde{\nu}_k, \tag{4.32}$$

Proof of Theorem

- $$\tilde u_{\nu_k} = w_k + v_k \ge w_k \quad \text{and} \quad \tilde\nu_k = c_N \int_{\mathbb{R}^N} V \tilde u_{\nu_k}^p dx,$$
  then $k \le \tilde\nu_k \le k + c_{\delta_0} k^p$ and $\tilde u_{\nu_k}$ satisfies then
  $$\lim_{|x|\to 0^+} \tilde u_{\nu_k}(x)|x|^{\frac{2}{p-1}} = c_p \quad \text{and} \quad \lim_{|x|\to +\infty} \tilde u_{\nu_k}(x)|x|^{N-2} = \tilde\nu_k, \tag{4.32}$$
- *Existence by iteration method:* Take $v_0 := w_k$ and v_n the unique solution of
  $$v_n = \Gamma * (V v_{n-1}^p) \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \tag{4.33}$$
  that is,
  $$\begin{cases} -\Delta v_n = V v_{n-1}^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \\ \lim_{|x|\to 0} v_n(x)|x|^{N-2} = 0. \end{cases}$$
  $\tilde u_{\nu_k}$ is an upper bound for $\{v_n\}_n$ for $k \in (0, k^*)$.

Further results

(Chen-Guo-Z., 2019, Preprint)
Consider singular positive solutions of Lane-Emden equation

$$\begin{cases} -\Delta u = u^p & \text{in} \quad \mathbb{R}^N \setminus \Sigma, \\ u > 0 & \text{in} \quad \mathbb{R}^N \setminus \Sigma. \end{cases} \tag{5.34}$$

where $\Sigma \subset \mathbb{R}^N$ is closed.

Theorem

Let p_c be as before, with $N \geq 3$, $p \in \left(\frac{N}{N-2}, p_c\right]$, the set $\Sigma \subset \mathbb{R}^N$ be closed and contain at most finite accumulation points $\{A_i\}_{i \in I}$ with $I \subset \mathbb{N}$. Assume that u is a nonnegative slow decaying solution of (5.34), then there exists $i_0 \in I$ such that

$$u(x) \equiv c_p |x - A_{i_0}|^{-\frac{2}{p-1}}, \qquad \forall\, x \in \mathbb{R}^N \setminus \Sigma.$$

Further results

On the contrary, the fast decaying solution with two points blowing up for $p \in (\frac{N}{N-2},\, p_c)$ and our existence result is stated as follows.

Theorem

Let $N \geq 3$, $p \in \left(\frac{N}{N-2},\, p_c\right)$, $\Sigma = \{A_1,\, A_2\}$ satisfy that $d = |A_1 - A_2| - 1 > 0$. Then there exists $d_p^* > 0$ depending on p such that for $d > d_p^*$, there are $k^* > 0$ and a mapping $k \in (0,\, k^*) \mapsto \nu_k$ is continuous, increasing, $\nu_k \geq 2k$ and $\lim\limits_{k\to 0} \nu_k = 0$, problem (5.34) has a solution u_{ν_k} with ν_k-fast decaying at infinity and singularities at Σ, that is,

$$u_{\nu_k}(x) \geq w_{A_1,k}(x) + w_{A_2,k}(x), \quad \forall\, x \in \mathbb{R}^N \setminus \Sigma \tag{5.35}$$

and

$$\lim_{|x|\to+\infty} u_{\nu_k}(x)|x|^{N-2} = \nu_k. \tag{5.36}$$

Bon Anniversaire!

Marie-Françoise et Laurent