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New conditions for Taylor varieties and CSP

Libor Barto
Department of Algebra
Charles University
Prague, Czech Republic
libor.barto@gmail.com

Marcin Kozik
TCS @ Jagiellonian
Jagiellonian University
Kraków, Poland
marcin.kozik@tcs.uj.edu.pl

***Abstract*—We provide two new characterizations for finitely generated varieties with Taylor terms. The first characterization is using "absorbing sets" and the second one "cyclic operations". These new conditions allow us to reprove the conjecture of Bang-Jensen and Hell (proved by the authors, comp. STOC'08, SICOMP'09) and the characterization of locally finite Taylor varieties using weak near-unanimity operations (proved by McKenzie and Maroti, Alg.Univ. 2009) in an elementary and self-contained way. The research is closely connected to the algebraic approach to CSP and previous results obtained by authors using similar tools [comp. STOC'08, SICOMP'09, FOCS'09 etc.].**

***Keywords*-Constraint Satisfaction Problem; Taylor conditions;**

I. Introduction

The Constraint Satisfaction Problem has been studied by computer scientists for over twenty years. It provides a common framework for many theoretical problems as well as for many real-life applications.

The results contained in this paper follow a long line of research devoted to verifying the Constraint Satisfaction Problem Dichotomy Conjecture of Feder and Vardi [14]. In general a Constraint Satisfaction Problem is a decision problem: given a number of variables and constraints imposed on them we ask whether the variables can be evaluated in such a way that all the constraints are met. The conjecture of Feder and Vardi deals with so called *non-uniform* CSP — the CSP when we ask about the complexity of the same question, but the set of allowed constrains is finite and fixed. The conjecture states that, for every finite, fixed set of constraints (a fixed *template*), the CSP defined by it is NP-complete or solvable in a polynomial time i.e. the class of CSP's exhibits a dichotomy.

The conjecture of Feder and Vardi dates back to 1993, but the first breakthrough in the research appeared in 1997 in the work of Jeavons, Cohen and Gyssens [19]. The new approach was refined later by Bulatov, Jeavons and Krokhin [12], [10]. At heart of the new approach lies a proof that the complexity of CSP, for a fixed template, depends only on the set of certain operations — polymorphisms of the template. Thus the study of templates gives way to the study of algebras associate to them.

One of the most useful tools provided by this new approach is the theorem of Bulatov, Jeavons and Krokhin [12], [10] stating that whenever an algebra associated with a core template does not lie in a Taylor variety then the CSP defined by the template is NP-complete. In the same paper authors conjecture that in all the other cases the associated CSP's are solvable in a polynomial time. This *algebraic dichotomy conjecture* has been neither confirmed nor disproved, as all the known partial results agree with this classification.

The algebraic approach provided new tools for tackling CSP's and a number of new results appeared. The result of Schaefer [21], proving dichotomy for two–element templates, have been extended by Bulatov [11] to three–element domains. The conjecture of Bang-Jensen and Hell [1] was positively verified [6], [7]. New algorithms were devised [9], [13], [18] and pre-algebraic algorithms were characterized in algebraic terms [3], [4]. In a vast majority of cases the hardness results were obtained by means of the theorem of Bulatov, Jeavons and Krokhin.

In order to prove the algebraic dichotomy conjecture one has to devise an algorithm that works for any relational structure with corresponding algebra in a Taylor variety. As the term-condition originally provided by Taylor is difficult to work with, a search for equivalent, but more elegant and useful conditions is ongoing [20], [22]. The characterization of locally finite Taylor varieties in terms of weak near-unanimity operations due to Maroti and McKenzie [20] is one of the most powerful tools in this area. The proof of this characterization is using a deep algebraic theory of Hobby and McKezie [17]. Mentioned above proof of the conjecture of Bang-Jensen and Hell hinges on this characterization; also the algebraic characterization of problems of bounded width [4] relies on (provided in [20]) similar characterization of congruence meet semi-distributive, locally finite varieties.

In this paper we provide two new characterizations of finitely generated Taylor varieties. The first characterization

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is expressed in terms of *absorbing subalgebras* developed and successfully applied by authors in [6], [7], [3], [4]. The second one uses *cyclic terms* and, in case of finitely generated varieties, is a stronger version of the weak near-unanimity condition given by Maroti and McKezie [20]. These new characterizations already proved to be useful. Not only provide they new tools for attacking the algebraic dichotomy conjecture, but allow us to present easy and elementary proofs for some of the results mentioned above.

The proofs of these new characterizations do not rely on the results of [20], [17], moreover they allow us to reprove the weak near-unanimity characterization of locally finite Taylor varieties without the overhead of the tame congruence theory. We use them to present an elementary proof of the conjecture of Bang-Jensen and Hell, and even easier proof of the theorem of Hell and Nešetřil [16]. Finally we translate the algebraic dichotomy conjecture into combinatorial terms. The results of this paper show that the tools developed for CSP can be successfully applied to algebraic questions which indicates a deep connection between CSP and universal algebra.

In section II we introduce the necessary notions concerning algebras and CSP. In section III we introduce absorbing subalgebras and state the absorbing theorem; the proofs missing in this section can be found in [2]. In section IV we use the absorbing subalgebra characterization to provide an elementary proof of the conjecture of Bang-Jensen and Hell in a slightly stronger version which is needed in section V. Finally in section V we state the characterization using cyclic terms (proven in [2]) and prove the corollaries: the theorem of Hell and Nešetřil [16] and the weak near-unanimity characterization of locally finite Taylor varieties of Maroti and McKenzie [20].

II. Preliminaries

A. Notation for sets

For a set A and a natural number n elements of A^n are the n-tuples of elements of A. We index its coordinates starting from zero, for example $(a_0, a_1, \dots, a_{n-1}) \in A^n$.

Let R be a subset of a cartesian product $A_1 \times A_2 \times \dots \times A_n$. R is called *subdirect* ($R \subseteq_S A_1 \times \dots \times A_n$) if, for every $i = 1, 2, \dots, n$, the projection of R to i-th coordinate is the whole set A_i.

Given $R \subseteq A \times B$ and $S \subseteq B \times C$, by $S \circ R$ we mean the following subset of $A \times C$:

$$S \circ R = \{(a, c) : \exists\, b \in B \quad (a, b) \in R, (b, c) \in S\}.$$

If $R \subseteq A \times A$ and n is a natural number greater than zero, then we define

$$R^{\circ n} = \underbrace{R \circ R \circ \dots \circ R}_{n}.$$

B. Algebras and varieties

A *signature* is a finite set of symbols with natural numbers (the *arities*) assigned to them. An *algebra* of a signature Σ is a pair $\mathbf{A} = (A, (t^{\mathbf{A}})_{t \in \Sigma})$, where A is a set, called the *universe* of $\mathbf{A}$, and $t^{\mathbf{A}}$ is an operation on A of arity $\mathrm{ar}(t)$, that is, a mapping $A^{\mathrm{ar}(t)} \to A$. We always use a boldface letter to denote an algebra and the same letter in a plain type to denote its universes. We often omit the superscripts of operations when the algebra is clear from the context.

A *term* in a signature Σ is a formal expression using variables and compositions of symbols in Σ. In this paper we introduce a special notation for a particular case of composition of terms: given a k-ary term t_1 and an l-ary term t_2 we define

$$t_1 \prec t_2(x_0, x_1, \dots, x_{kl-1})$$

to be

$$t_1(t_2(x_0, \dots, x_{l-1}), t_2(x_l, \dots), \dots, t_2(\dots, x_{kl-1})).$$

For an algebra $\mathbf{A}$ and a term h in the same signature Σ, $h^{\mathbf{A}}$ has the natural meaning in $\mathbf{A}$ and is called a *term operation* of $\mathbf{A}$. Again, we usually omit the superscripts of term operations when the algebra is clear from the context. The set of all term operations of $\mathbf{A}$ is called the *clone of term operations* of $\mathbf{A}$ and it is denoted $\mathrm{Clo}(\mathbf{A})$.

For a pair of terms s, t over a signature Σ, we say that an algebra $\mathbf{A}$ in the signature Σ *satisfies the identity* $s \approx t$ if the term operations $s^{\mathbf{A}}$ and $t^{\mathbf{A}}$ are the same.

There are three fundamental operations on algebras of a fixed signature Σ: forming subalgebras, factoralgebras and products. A subset B of the universe of an algebra $\mathbf{A}$ is called a *subuniverse*, if it is closed under all operations (equivalently term operations) of $\mathbf{A}$. Given a subuniverse B of $\mathbf{A}$ we can form the algebra $\mathbf{B}$ by restricting all the operations of $\mathbf{A}$ to the set B. In this situation we write $B \leq \mathbf{A}$ or $\mathbf{B} \leq \mathbf{A}$. We call the subuniverse B (or the subalgebra $\mathbf{B}$) *proper* if $\emptyset \neq B \neq A$. The smallest subalgebra of $\mathbf{A}$ containing a set $B \subseteq A$ is called the subalgebra *generated by* B and will be denoted by $\mathrm{Sg}_{\mathbf{A}}(B)$. It can be equivalently described as the set of elements which can be obtained by applying term operations of $\mathbf{A}$ to elements of B.

Given a family of algebras $\mathbf{A}_i, i \in I$ we define its product $\prod_{i \in I} \mathbf{A}_i$ to be the algebra with the universe equal to the cartesian product of A_i's and with operations computed coordinatewise. The product of algebras $\mathbf{A}_1, \dots, \mathbf{A}_n$ will be denoted by $\mathbf{A}_1 \times \dots \times \mathbf{A}_n$ and a product of n copies of an algebra $\mathbf{A}$ — $\mathbf{A}^n$. $\mathbf{R}$ is a *subdirect subalgebra* of $\mathbf{A}_1 \times \mathbf{A}_2 \times \dots \times \mathbf{A}_n$ if R is subdirect in $A_1 \times A_2 \cdots A_n$ and, in such a case, we write $\mathbf{R} \leq_S \mathbf{A}_1 \times \dots \times \mathbf{A}_n$.

An equivalence relation $\sim$ on the universe of an algebra $\mathbf{A}$ is a *congruence*, if it is a subalgebra of $\mathbf{A}^2$. The corresponding *factor algebra* $\mathbf{A}/\sim$ has, as the universe, the set of $\sim$-blocks and the operations are defined using (arbitrarily chosen) representatives.

A *variety* is a class of algebras of the same signature closed under forming isomorphic copies, subalgebras, factoralgebras and products. For a pair of terms s, t over a signature Σ, we say that a class of algebras $\mathcal{V}$ in the signature Σ *satisfies the identity* $s \approx t$ if every algebra in the class does. By Birkhoff's theorem, a class of algebras is a variety if and only if there exists a set of equations E such that the members of $\mathcal{V}$ are precisely those algebras which satisfy all the equations from E.

A variety $\mathcal{V}$ is called *locally finite*, if every finitely generated algebra contained in $\mathcal{V}$ is finite. $\mathcal{V}$ is called *finitely generated*, if there exists a finite set $\mathcal{K}$ of finite algebras such that $\mathcal{V}$ is the smallest variety containing $\mathcal{K}$. In such a case $\mathcal{V}$ is actually generated by a single, finite algebra, the product of members of $\mathcal{K}$. Every finitely generated variety is locally finite, and if a variety is generated by a single algebra then all identities satisfied in this algebra are also satisfied in the variety.

C. Taylor varieties

A term s is *idempotent* in a variety (or an algebra), if it satisfies the identity

$$s(x, x, \dots, x) \approx x.$$

An algebra (a variety) is idempotent if all its terms are.

A term t of arity at least 2 is called a *weak near-unanimity* term of a variety (or an algebra), if t is idempotent and satisfies

$$t(y, x, x, \dots, x) \approx t(x, y, x, x, \dots, x) \approx \cdots \\ \cdots \approx t(x, x, \dots, y, x) \approx t(x, x, \dots, x, y).$$

A term t of arity at least 2 is called a *cyclic* term of a variety (or an algebra), if t is idempotent and satisfies

$$t(x_0, x_1, \dots, x_{k-1}) \approx t(x_1, x_2, \dots, x_{k-1}, x_0).$$

Finally a term t of arity k is called a *Taylor term* of a variety (or an algebra), if t is idempotent and for every $j < k$ it satisfies an identity of the form

$$t(\Box_0, \Box_1, \dots, \Box_{k-1}) \approx t(\triangle_0, \triangle_1, \dots, \triangle_{k-1}),$$

where all $\Box_i$'s and $\triangle_i$'s are substituted with either x or y, but $\Box_j$ is x while $\triangle_j$ is y.

Definition II.1. *An idempotent variety $\mathcal{V}$ is called* Taylor *if it has a Taylor term.*

Study of Taylor varieties has been a recurring subject in universal algebra for many years. One of the first characterizations is due to Taylor [23]

Theorem II.2 (Taylor [23])**.** *Let $\mathcal{V}$ be an idempotent variety. The following are equivalent.*

- *$\mathcal{V}$ is a Taylor variety.*
- *$\mathcal{V}$ doesn't contain a two-element algebra whose every (term) operation is a projection.*

Further research led to discovery of other equivalent conditions [22], [20]. One of the most important ones is the result of Maroti and McKenzie [20]

Theorem II.3 (Maroti and McKenzie [20])**.** *Let $\mathcal{V}$ be an idempotent, locally finite variety. The following are equivalent.*

- *$\mathcal{V}$ is a Taylor variety.*
- *$\mathcal{V}$ has a weak near-unanimity term.*

This result, together with a similar characterization provided in the same paper for congruence meet semi-distributive varieties, found deep applications in CSP [6], [7], [4].

D. Relational structures and CSP

Let Σ be a signature as in Section II-B. A *relational structure* of the signature Σ is a pair $\mathbb{A} = (A, (R^{\mathbb{A}})_{R \in \Sigma})$, where A is a set, called the *universe* of $\mathbb{A}$, and $R^{\mathbb{A}}$ is a relation on A of arity $\mathrm{ar}(R)$, that is, a subset of $A^{\mathrm{ar}(R)}$.

Let $\mathbb{A}, \mathbb{B}$ be relational structures of the same signature. A mapping $f : A \to B$ is a *homomorphism* from $\mathbb{A}$ to $\mathbb{B}$, if it preserves R for all $R \in \Sigma$, that is, $(f(a_0), f(a_1), \dots, f(a_{\mathrm{ar}(R)-1})) \in R^{\mathbb{B}}$ for any $(a_0, \dots, a_{\mathrm{ar}(R)-1}) \in R^{\mathbb{A}}$. A finite relational structure $\mathbb{A}$ is a *core*, if every homomorphism from $\mathbb{A}$ to itself is bijective.

For a fixed relational structure $\mathbb{A}$ of a signature Σ $\mathrm{CSP}(\mathbb{A})$ is the following decision problem:

INPUT:	A rel. str. $\mathbb{X}$ of the signature Σ.
QUESTION:	Does $\mathbb{X}$ map homomorphically to $\mathbb{A}$?

It is easy to see that if $\mathbb{A}'$ is a core of $\mathbb{A}$ (i.e. a core which is contained in $\mathbb{A}$ and such that $\mathbb{A}$ can be mapped homomorphically into it) then $\mathrm{CSP}(\mathbb{A})$ and $\mathrm{CSP}(\mathbb{A}')$ are identical.

The celebrated conjecture of Feder and Vardi [14] states that the class of CSP's exhibits a dichotomy:

The dichotomy conjecture of Feder and Vardi. *For any relational structure $\mathbb{A}$ the problem $\mathrm{CSP}(\mathbb{A})$ is solvable in a polynomial time, or NP-complete.*

E. Algebraic approach to CSP

A mapping $f : A^n \to A$ is *compatible* with an m-ary relation R on A if the tuple

$$\left(f(a_0^0, a_0^1, \dots, a_0^{n-1}), \dots, f(a_{m-1}^0, a_{m-1}^1, \dots, a_{m-1}^{n-1})\right)$$

belongs to R whenever $(a_0^i, \dots, a_{m-1}^i) \in R$ for all $i < n$. A mapping compatible with all the relations in a relational structure $\mathbb{A}$ is a *polymorphism* of this structure.

For a given relational structure $\mathbb{A} = (A, (R^{\mathbb{A}})_{R \in \Sigma})$ we define an algebra $\mathrm{IdPol}(A)$ (often denoted by just $\mathbf{A}$). This algebra $\mathbf{A}$ has an underlying set equal to A and the operations of $\mathbf{A}$ are the idempotent polymorphisms of $\mathbb{A}$ (we formally define a signature of $\mathbf{A}$ to be identical with the set of it's operations).

It follows from an old result [8], [15] that a relation R of arity k is a subuniverse of $\mathrm{IdPol}(\mathbb{A})^k$ if and only if R can be positively primitively defined from relations in $\mathbb{R}$ and singleton unary relations identifying every element of A. That is, R can be defined by a formula which uses relations in R, singleton unary relations on A, the equality relation on A, conjunction and existential quantification.

Already the first results on the algebraic approach to CSP [19], [12], [10] show that whenever a relational structure $\mathbb{A}$ is a core then $\mathrm{IdPol}(\mathbb{A})$ fully determines the computational complexity of $\mathrm{CSP}(\mathbb{A})$. Moreover Bulatov, Jeavons and Krokhin showed [12], [10] that

Theorem II.4 (Bulatov, Jeavons and Krokhin [12], [10])**.** *Let $\mathbb{A}$ be a finite relational structure which is a core. If $\mathrm{IdPol}(\mathbb{A})$ does not lie in a Taylor variety, then $\mathrm{CSP}(\mathbb{A})$ is NP-complete.*

In the same paper they conjectured that these are the only cases of finite cores which give rise to NP-complete CSP's.

The dichotomy conjecture of Bulatov, Jeavons and Krokhin. *Let $\mathbb{A}$ be a finite relational structure which is a core. If $\mathrm{IdPol}(\mathbb{A})$ does not lie in a Taylor variety, then $\mathrm{CSP}(\mathbb{A})$ is NP-complete. Otherwise is it solvable in a polynomial time.*

This conjecture is supported by many partial results on the complexity of CSPs [11], [6], [7], [4], [18] and it renewed interest in properties of finitely generated Taylor varieties.

III. Absorbing subalgebras and absorption theorem

In this section we introduce the concept of an absorbing subalgebra, state the absorption theorem and prove some of its corollaries. We begin with the definition of an absorbing subalgebra.

Definition III.1. *Let $\mathbf{A}$ be an algebra and $t \in \mathrm{Clo}(\mathbf{A})$. We say that a subalgebra $\mathbf{B}$ of $\mathbf{A}$ is an* absorbing subalgebra of $\mathbf{A}$ with respect to t *if, for any $k < \mathrm{ar}(t)$, any choice of $a_i \in A$ such that $a_i \in B$ for all $i \neq k$ we have $t(a_0, \ldots, a_{\mathrm{ar}(t)-1}) \in B$.*

We say that $\mathbf{B}$ is an absorbing subalgebra of $\mathbf{A}$*, or that* $\mathbf{B}$ absorbs $\mathbf{A}$ *(and write $\mathbf{B} \triangleleft \mathbf{A}$), if there exists $t \in \mathrm{Clo}(\mathbf{A})$ such that $\mathbf{B}$ is an absorbing subalgebra of $\mathbf{A}$ with respect to t.*

We also speak about absorbing subuniverses i.e. universes of absorbing subalgebras. Absorbing subalgebras are closed under taking intersection and $\triangleleft$ is a transitive relation:

Proposition III.2. *Let $\mathbf{A}$ be an algebra.*

- *If $\mathbf{C} \triangleleft \mathbf{B} \triangleleft \mathbf{A}$, then $\mathbf{C} \triangleleft \mathbf{A}$.*
- *If $\mathbf{B} \triangleleft \mathbf{A}$ and $\mathbf{C} \triangleleft \mathbf{A}$, then $B \cap C \triangleleft \mathbf{A}$.*

Proof: We start with a proof of the first item. Assume that $\mathbf{B}$ absorbs $\mathbf{A}$ with respect to t (of arity m) and that $\mathbf{C}$ absorbs $\mathbf{B}$ with respect to s (of arity n). We will show that $\mathbf{C}$ is an absorbing subalgebra of $\mathbf{A}$ with respect to $s \prec t$. Indeed, take any tuple $(a_0, \ldots, a_{mn-1}) \in A^{mn}$ such that $a_i \in C$ for all but one index, say j, and consider the evaluation of $s \prec t(a_0, \ldots, a_{mn-1})$. Every evaluation of the term t appearing in $s \prec t$ is of the form

$$t(a_{im}, \ldots, a_{im+m-1})$$

and therefore whenever j does not fall into the interval $[im, im+m-1]$ the result of it falls in C (as C is a subuniverse of $\mathbf{A}$). In the case when j is in the interval we have a term t evaluated on the elements of C (and therefore elements of B) in all except one coordinate. The result of such an evaluation falls in B (as $\mathbf{B}$ absorbs $\mathbf{A}$ with respect to t). Thus s is applied to a tuple consisting of elements of C on all but one position, and on this position the argument comes from B. Since $\mathbf{C}$ absorbs $\mathbf{B}$ with respect to s the results falls in C and the first part of the proposition is proved.

For the second part we consider $\mathbf{B} \triangleleft \mathbf{A}$ and $\mathbf{C} \triangleleft \mathbf{A}$; it follows easily that $B \cap C \triangleleft \mathbf{C}$. Now it is enough to apply the first part. ■

Among the absorbing subuniverses of a fixed algebra we distinguish those minimal with respect to inclusion:

Definition III.3. *If $\mathbf{B} \triangleleft \mathbf{A}$ and no proper subalgebra of $\mathbf{B}$ absorbs $\mathbf{A}$, we call $\mathbf{B}$ a* minimal absorbing subalgebra *of $\mathbf{A}$ (and write $\mathbf{B} \triangleleft\!\triangleleft\, \mathbf{A}$).*

Alternatively, we can say that $\mathbf{B}$ is a minimal absorbing subalgebra of $\mathbf{A}$, if $\mathbf{B} \triangleleft \mathbf{A}$ and $\mathbf{B}$ has no proper absorbing subalgebras. Equivalence of these definitions follows from transitivity of $\triangleleft$. Observe also that two minimal absorbing subuniverses of $\mathbf{A}$ are either disjoint or coincide, but the union of all minimal absorbing subuniverses need not be the whole set A.

Let $\mathbf{A}$, $\mathbf{B}$ be algebras of the same type (often members of a Taylor variety) and R be a subuniverse of $\mathbf{A} \times \mathbf{B}$. In such a case we use the following notation: for $X \subseteq A$ and $Y \subseteq B$ we put

$$\begin{aligned} X^{+R} &= \{b \in B : \exists\, a \in X \quad (a,b) \in R\} \\ Y^{-R} &= \{a \in A : \exists\, b \in Y \quad (a,b) \in R\} \end{aligned}$$

When R is clear from the context we write just X^+ and Y^-. The following lemma shows that these operations preserve (absorbing) subalgebras.

Lemma III.4. *Let $R \leq \mathbf{A} \times \mathbf{B}$, where $\mathbf{A}, \mathbf{B}$ are algebras of the same signature. If $X \leq \mathbf{A}$ and $Y \leq \mathbf{B}$, then $X^+ \leq \mathbf{B}$ and $Y^- \leq \mathbf{A}$. Moreover, if $R \leq_S \mathbf{A} \times \mathbf{B}$ and $X \triangleleft \mathbf{A}$ and $Y \triangleleft \mathbf{B}$, then $X^+ \triangleleft \mathbf{B}$ and $Y^- \triangleleft \mathbf{A}$.*

Proof: Suppose $X \leq \mathbf{A}$ and take any term t, say of arity j, in the given signature. Let $b_0, \ldots, b_{j-1} \in X^+$ be arbitrary.

From the definition of X^+ we can find $a_0, \ldots, a_{j-1} \in X$ such that $(a_i, b_i) \in R$ for all $0 \leq i < j$. Since R is a subuniverse of $\mathbf{A} \times \mathbf{B}$, the pair $(t(a_0, \ldots, a_{j-1}), t(b_0, \ldots, b_{j-1}))$ is in R. But $t(a_0, \ldots, a_{j-1}) \in X$ as X is a subuniverse of $\mathbf{A}$. Therefore $t(b_0, \ldots, b_{j-1}) \in X^+$ and we have shown that X^+ is closed under all term operations of $\mathbf{B}$ i.e. $X^+ \leq \mathbf{B}$.

Suppose X absorbs $\mathbf{A}$ with respect to a term t of arity j. Let $0 \leq k < j$ be arbitrary and let $b_0, \ldots, b_j \in B$ be elements such that $b_i \in X^+$ for all $i \neq k$. Then, for every $i, i \neq k$, we can find $a_i \in X$ such that $(a_i, b_i) \in R$. Also, since the projection of R to the second coordinate is B, we can find $a_k \in A$ such that $(a_k, b_k) \in R$. We again have $(t(a_0, \ldots, a_{j-1}), t(b_0, \ldots, b_{j-1})) \in R$ and $t(a_0, \ldots, a_{j-1}) \in X$ (as X absorbs $\mathbf{A}$ with respect to t). It follows that $t(b_0, \ldots, b_{j-1})) \in X^+$ and that $X^+ \triangleleft \mathbf{B}$ with respect to t.

The remaining two statements are proved in an identical way. ∎

When $R \leq A \times B$ it is helpful to draw this situation as a bipartite undirected graph in the following sense: the vertex set is the disjoint union of A (draw it on the left) and B (on the right) and two elements $a \in A$ from the left side and $b \in B$ from the right side are adjacent if $(a, b) \in R$. Note that $R \leq_S \mathbf{A} \times \mathbf{B}$ if and only if every vertex in this graph has a neighbor. Such a relation R is *linked* if the corresponding graph is connected:

Definition III.5. *Let $R \subseteq A \times B$ and let $a, a' \in A$. We say that $a, a' \in A$ are* linked *in R via $c_0, \ldots, c_{2n}$, if $a = c_0, c_{2n} = a'$ and $(c_{2i}, c_{2i+1}) \in R$ and $(c_{2i+2}, c_{2i+1}) \in R$ for all $i = 0, 1, \ldots, n-1$.*

We say that $R \subseteq_S A \times B$ is linked*, if a, a' are R-linked for any $a, a' \in A$.*

These definitions and basic properties allow us to state the absorption theorem which is the first main result of the paper.

Theorem III.6. *Let $\mathcal{V}$ be an idempotent, locally finite variety, then TFAE:*

- *$\mathcal{V}$ is a Taylor variety;*
- *for any finite $\mathbf{A}, \mathbf{B} \in \mathcal{V}$ and any linked $\mathbf{R} \leq_S \mathbf{A} \times \mathbf{B}$:*
  - *$\mathbf{R} = \mathbf{A} \times \mathbf{B}$ or*
  - *$\mathbf{A}$ has a proper absorbing subuniverse or*
  - *$\mathbf{B}$ has a proper absorbing subuniverse.*

A proof of this theorem can be found in [2]. The proof is self-contained and elementary. In section IV we use Theorem III.6 to reprove a stronger version of the "smooth theorem" [6], [7] which, in turn, will be used to prove Theorem V.3. This approach simplifies significantly the known proof of "smooth theorem", and does not rely on the involved algebraic results results from [20].

IV. New proof of the "smooth theorem"

The "smooth theorem" classifies the computational complexity of CSP's generated by smooth digraphs (digraphs, where every vertex has at least one incoming and at least one outgoing edge). This classification was conjectured by Bang-Jensen and Hell [1] and confirmed by the authors in [6], [7]. The proof presented in those papers heavily relied on the results of McKenzie and Maroti [20] which characterized the locally finite Taylor varieties in terms of weak near-unanimity operations. We present an alternative proof of the "smooth theorem" which depends only on Theorem III.6. The "smooth theorem" states:

Theorem IV.1. *Let $\mathbb{H}$ be a smooth digraph. If each component of the core of $\mathbb{H}$ is a circle, then* $\mathrm{CSP}(\mathbb{H})$ *is polynomially decidable. Otherwise* $\mathrm{CSP}(\mathbb{H})$ *is NP-complete.*

A. Basic digraph notions

A *digraph* is a pair $\mathbb{G} = (V, E)$, where V is a finite set of vertices and $E \subseteq V \times V$ is a set of edges. If the digraph is fixed we write $a \to b$ instead of $(a, b) \in E$. A *loop* is an edge of the form (a, a). $\mathbb{G}$ is said to be *smooth* if every vertex has an incoming and an outgoing edge, in other words, $\mathbb{G}$ is smooth, if E is a subdirect product of V and V. A *smooth part* of a digraph $\mathbb{G}$ is the largest induced subdigraph of $\mathbb{G}$ which is smooth (the subdigraph can be empty).

An *oriented path* is a digraph $\mathbb{P}$ with vertex set $P = \{p_0, \ldots, p_k\}$ and edge set consisting of k edges — for all $i < k$ either (p_i, p_{i+1}), or (p_{i+1}, p_i) is an edge of $\mathbb{P}$. An *initial segment* of such a path is any path induced by $\mathbb{P}$ on vertices $\{p_0, \ldots, p_i\}$ for some $i < k$. We denote the oriented path consisting of k edges pointing forward by $\cdot \xrightarrow{k} \cdot$ and, similarly the oriented path consisting of k edges pointing backwards by $\cdot \xleftarrow{k} \cdot$. The concatenation of paths is performed in the natural way. A (k, n)*-fence* (denoted by $\mathbb{F}[k, n]$) is the oriented path consisting of $2kn$ edges, k forward edges followed by k backward edges, n times i.e.:

$$\underbrace{\cdot \xrightarrow{k} \cdot \xleftarrow{k} \cdot \cdots \cdot \xrightarrow{k} \cdot \xleftarrow{k} \cdot}_{n}$$

The *algebraic length* of an oriented path is the number of forward edges minus the number of backward edges (and thus all the fences have algebraic length zero). Let $\mathbb{G}$ be a digraph, let $\mathbb{P}$ be an oriented path with vertex set $P = \{p_0, \ldots, p_k\}$, and let a, b be vertices of $\mathbb{G}$. We say that a is *connected to* b via $\mathbb{P}$, if there exists a homomorphism $f : \mathbb{P} \to \mathbb{G}$ such that $f(p_0) = a$ and $f(p_k) = b$. We sometimes write $a \xrightarrow{k} b$ when a is connected to b via $\cdot \xrightarrow{k} \cdot$. If $a \xrightarrow{k} a$ (for some k) then a is *in a cycle* and any image of the path $\cdot \xrightarrow{k} \cdot$ with the same initial and final vertex is *a cycle*. *A circle* is a cycle which has no repeating vertices and no chords.

The relation "a is connected to b via some path" is an equivalence, its blocks (or sometimes the corresponding induced subdigraphs) are called *weak components* of $\mathbb{G}$. The vertices a and b are in the same *strong component* if $a \xrightarrow{k} b \xrightarrow{k'} a$ for some k, k'. For a subset B of A and an oriented path $\mathbb{P}$ we set

$$B^{\mathbb{P}} = \{c : \exists b \in B \ \ b \text{ is connected to } c \text{ via } \mathbb{P} \ \}.$$

Note that $B^{\cdot \xrightarrow{k} \cdot}$ is formally equal to $B^{+E^{\circ k}}$ but we prefer the first notation.

Finally, $\mathbb{G}$ has *algebraic length* k, if there exists a vertex a of $\mathbb{G}$ such that a is connected to a via a path of algebraic length k and k is the minimal positive number with this property. The following proposition summarizes easy results concerning reachability via paths:

Proposition IV.2. *Let $\mathbb{G}$ be a smooth digraph, then:*

- *for any vertices a, b in $\mathbb{G}$ if a is connected to b via $\cdot \xrightarrow{k} \cdot$ then a is connected to b via every path of algebraic length k;*
- *for any vertex a and any path $\mathbb{P}$ there exists a vertex b and a path $\mathbb{Q}$ which is an initial segment of some fence such that $\{a\}^{\mathbb{P}} \subseteq \{b\}^{\mathbb{Q}}$;*
- *if $H \subseteq G$ is such that $H^{\cdot \to \cdot} \supseteq H$ or $H^{\cdot \leftarrow \cdot} \supseteq H$ then the digraph $\mathbb{G}$ restricted to H contains a cycle (i.e.* the smooth part *of H is non-empty)*

Proof: The first item of the proposition follows directly form the definition of a smooth digraph.

We prove the second item by induction on the length of $\mathbb{P}$. If the length is zero there is nothing to prove. Therefore we take an arbitrary path $\mathbb{P}$ of length n and arbitrary $a \in A$. The proof splits into two cases depending on the direction of the last edge in $\mathbb{P}$ We consider the case when the last edge of $\mathbb{P}$ points forward first and set $\mathbb{P}'$ to be $\mathbb{P}$ take away last edge. The inductive assumption for a and $\mathbb{P}'$ provides vertex b and path $\mathbb{Q}'$ (an initial fragment of a fence $\mathbb{F}[k, l]$). If algebraic length of $\mathbb{Q}'$ strictly smaller than k we put $\mathbb{Q}'''$ to be a path such that the concatenation of $\mathbb{Q}'$ and $\mathbb{Q}'''$ is an initial fragment of a fence $\mathbb{F}[k, l+1]$ and such that the algebraic length of $\mathbb{Q}'''$ is one; then the concatenation of $\mathbb{Q}'$ and $\mathbb{Q}'''$ proves the second item of the proposition (as, by the first item of the proposition, every element reachable from $\{b\}^{\mathbb{Q}'}$ by $\cdot \to \cdot$ is also reachable by $\mathbb{Q}'''$). If the algebraic length of $\mathbb{Q}'$ equals k we consider a path $\mathbb{Q}''$ obtained from $\mathbb{Q}'$ by substituting each subpath of shape $\cdot \to \cdot \leftarrow \cdot$ with $\cdot \xrightarrow{2} \cdot \xleftarrow{2} \cdot$. Path $\mathbb{Q}''$ is an initial fragment of $\mathbb{F}[k+1, l]$ we have $\{b\}^{\mathbb{Q}'} \subseteq \{b\}^{\mathbb{Q}''}$ (as the digraph is smooth) – now we can find $\mathbb{Q}'''$ as in the previous case.

If the last edge of $\mathbb{P}$ points backwards we proceed with dual reasoning. If algebraic length of $\mathbb{Q}'$ is greater than zero we obtain $\mathbb{Q}'''$ of algebraic length -1 as before and the proposition is proved. If algebraic length of $\mathbb{Q}'$ is zero we substitute b with any vertex b' such that $b' \to b$ and alter $\mathbb{Q}'$ by substituting each $\cdot \leftarrow \cdot \to \cdot$ with $\cdot \xleftarrow{2} \cdot \xrightarrow{2} \cdot$. The new path is an initial fragment of $\mathbb{F}[k+1, l]$ and we can proceed as in previous case.

For the third item of the proposition. Without loss of generality we can assume the first possibility and choose an arbitrary $b_0 \in H$. As $H \subseteq H^{\cdot \to \cdot}$ there is an element $b_1 \in H$ such that $b_1 \to b_0$. Repeating the same reasoning for $b_1, b_2, \ldots$ we obtain a sequence of vertices in H such that $b_{i+1} \to b_i$. As H is finite, we obtain a cycle in H and the last item of the proposition is proved. ∎

B. Reduction of the problem

We present a number of elementary reductions of Theorem IV.1. Bang-Jensen and Hell [1] showed that if digraph $\mathbb{H}$ has a core which is a disjoint union of circles then $\mathrm{CSP}(\mathbb{H})$ is solvable in a polynomial time. On the other hand, using Theorem II.4 and the fact that CSPs of a relational structure and its core are the same, it suffices to prove that:

Theorem IV.3. *If a smooth digraph admits a Taylor polymorphism then it retracts onto the disjoint union of circles.*

Finally Theorem IV.3 reduces to the theorem below and an elementary proof of this reduction can be found in [6], [7].

Theorem IV.4. *If a smooth digraph has algebraic length one and admits a Taylor polymorphism then it contains a loop.*

In fact, in the reminder of this section, we prove a stronger version of Theorem IV.4:

Theorem IV.5. *Let $\mathbf{A}$ be a finite algebra in a Taylor variety and let $\mathbb{G} = (A, E)$ be a smooth digraph of algebraic length one such that E is a subuniverse of $\mathbf{A}^2$. Then $\mathbb{G}$ contains a loop; moreover if there exists $I \triangleleft \mathbf{A}$ such that I is contained in a weak component of $\mathbb{G}$ of algebraic length* 1*, then the loop can be found in some J such that $J \triangleleft\triangleleft \mathbf{A}$.*

C. The proof

Our proof of Theorem IV.5 proceeds by induction on the size of the vertex set of $\mathbb{G} = (A, E)$. If $|A| = 1$ there is nothing to prove (as the only smooth digraph on such a set contains a loop); for the induction step we assume that Theorem IV.5 holds for all smaller digraphs.

Claim IV.5.1. Let H be a weak component of $\mathbb{G}$ of algebraic length one, then there exists $a \in H$ and a path $\mathbb{P}$ such that $\{a\}^{\mathbb{P}}$ contains a cycle.

Proof: We choose $a \in H$ to be the element of the component H such that there is a path $\mathbb{Q}$ of algebraic length one connecting a to a. We define the sequence of sets $B_0 = \{a\}$ and $B_i = B_{i-1}^{\mathbb{Q}}$ recursively. As a is connected to a via $\mathbb{Q}$ we have $B_0 \subseteq B_1$ and therefore $B_i \subseteq B_{i+1}$ for any i (as by definition $B_{i-1} \subseteq B_i$ implies that $B_{i-1}^{\mathbb{Q}} \subseteq B_i^{\mathbb{Q}}$ i.e. $B_i \subseteq B_{i+1}$). As $\mathbb{Q}$ is of algebraic length one we can use Proposition IV.2 to infer that $\{a\}^{\cdot \to \cdot} \subseteq B_1$ and further that

$\{a\}^{\cdot \xrightarrow{k} \cdot} \subseteq B_k$ for any k. These facts together imply that

$$\bigcup_{i=0}^{k} \{a\}^{\cdot \xrightarrow{i} \cdot} \subseteq B_k$$

and, as the digraph is finite, we can find a cycle in one of the B_k's. Take $\mathbb{P}$ to be the $\mathbb{Q}$ concatenated with itself sufficiently many times to witness the claim. ■

Claim IV.5.2. Let H be a weak component of $\mathbb{G}$ of algebraic length one, then there exists $a \in H$ and a fence $\mathbb{F}$ such that $\{a\}^{\mathbb{F}} = H$.

Proof: Let us choose $a \in H$ and $\mathbb{P}'$ as provided by Claim IV.5.1. Set B to be the set of elements of $\{a\}^{\mathbb{P}'}$ which belong to some cycle fully contained in $\{a\}^{\mathbb{P}'}$. Proposition IV.2 implies that $B^{\mathbb{F}[|A|,1]}$ contains all elements reachable by $\cdot \xrightarrow{i} \cdot$ or $\cdot \xleftarrow{i} \cdot$ (for any i), from any element of B. Indeed if such a c is reachable from $b \in B$ by $\cdot \xleftarrow{i} \cdot$ then it is reachable by $\cdot \xleftarrow{|A|} \cdot$ from some $b' \in B$ and further by $\mathbb{F}[|A|, 1]$ from some $b'' \in B$. In the other case $b \xrightarrow{i} c$ for some $b \in B$. There obviously exists d such that $d \xleftarrow{|A|} c$ and since $b \xrightarrow{i} c \xrightarrow{|A|} d$ we have some $j \leq |A|$ and $b \xrightarrow{j} d$. Thus there exists $b' \in B$ with $b' \xrightarrow{|A|} d$ and c is reachable by $\mathbb{F}[|A|, 1]$ from b'.

In a similar way we can show that $B^{\mathbb{F}[|A|,|A|]} = H$. Thus, for an appropriate path $\mathbb{P}$ we have a connected to every element of H by $\mathbb{P}$. The second item of Proposition IV.2 provides b and an initial segment $\mathbb{Q}$ of a fence $\mathbb{F}$ such that b is connected to every element from H by $\mathbb{Q}$. Obviously b is connected to every element of H by $\mathbb{F}$ as well and the claim is proved. ■

The remaining part of the proof splits into two cases: in the first case the algebra $\mathbf{A}$ has an absorbing subuniverse in a weak component of algebraic length one and in the second it doesn't. Let us focus on the first case and define $I \triangleleft \mathbf{A}$ contained in a weak component (denoted by H) of algebraic length one of $\mathbb{G}$.

Claim IV.5.3. There is a fence $\mathbb{F}$ such that $I^{\mathbb{F}} = H$.

Proof: Let a and $\mathbb{F}'$ be provided by Claim IV.5.2. We put $\mathbb{F}$ to be a concatenation of $\mathbb{F}'$ with itself. Since $a \in I^{\mathbb{F}'}$, then $I^{\mathbb{F}} = H$. ■

Let $\mathbb{P}$ be the longest initial segment of $\mathbb{F}$ (provided by Claim IV.5.3) such that $I^{\mathbb{P}} \neq H$. Put $S = I^{\mathbb{P}}$. By multiple application of Lemma III.4 we infer that S is a subuniverse of $\mathbf{A}$ and that $S \triangleleft \mathbf{A}$. The following lemma will be of use in both of the cases:

Lemma IV.6. *Let $\mathbf{A}$ be a finite algebra and let $\mathbb{G} = (A, E)$ be a smooth digraph such that E is a subuniverse of $\mathbf{A}^2$. If B is a subuniverse of $\mathbf{A}$ (an absorbing subuniverse of $\mathbf{A}$) then the smooth part of $(B, E \cap (B \times B))$ forms a subuniverse of $\mathbf{A}$ (an absorbing subuniverse of $\mathbf{A}$ respectively).*

Proof: Note that if the smooth part of $(B, E \cap (B \times B))$ is empty then the lemma holds. Assume it is non-empty and let $\mathbf{A}$, $\mathbb{G}$, B be as in the statement of the lemma. We put $B_1 \subseteq B$ to be the set of all the vertices in B with at least one outgoing and at least one incoming edge in $E \cap (B \times B)$ (i.e. an outgoing edge and an incoming edge to elements of B). As $B_1 = B \cap B^{+E} \cap B^{-E}$ Lemma III.4 implies that B_1 is a subuniverse (absorbing subuniverse resp.) of $\mathbf{A}$. We put $B_2 = B_1 \cap B_1^{+E} \cap B_1^{-E}$ and continue the reasoning. Since $\mathbf{A}$ is finite we obtain some k such that $B_k = B_{k+1}$. Thus $\mathbb{G}$ restricted to B_k has no sources and no sinks – this proves the lemma. ■

The definition of S implies that $S^{\cdot \rightarrow \cdot} = H \supseteq S$ or $S^{\cdot \leftarrow \cdot} = H \supseteq S$, and therefore, by Proposition IV.2, S contains a cycle. Thus the smooth part of $\mathbb{G}$ restricted to S, denoted by S', is non-empty and, by Lemma IV.6, it absorbs $\mathbf{A}$. If the digraph $\mathbb{G}$ restricted to S' has algebraic length one and is weakly connected, then we use the inductive assumption:

- either $\mathbb{G}$ restricted to S' has no absorbing subuniverses in a weak component of algebraic length one; in such a case, as it is weakly connected, it has no absorbing subuniverses at all — therefore $S' \triangleleft\!\triangleleft \mathbf{A}$ and the inductive assumption provides a loop in S', or
- $\mathbb{G}$ restricted to S' has an absorbing subuniverse; then it has a loop in $J \triangleleft\!\triangleleft S'$ and, as $J \triangleleft\!\triangleleft \mathbf{A}$, the theorem is proved.

Therefore to conclude the first case of the theorem it remains to prove

Claim IV.6.1. $\mathbb{G}$ restricted to S' is a weakly connected digraph of algebraic length 1.

Proof: Assume that S' absorbs $\mathbf{A}$ with respect to t of arity k and let m, n be natural numbers such that every two vertices of H are connected via the (m, n)-fence (implied by Claim IV.5.2) denoted by $\mathbb{F}$. We will show that any two vertices $a, b \in S'$ are connected via the (m, nk)-fence in the digraph $\mathbb{G}$ restricted to S'.

As the digraph $\mathbb{G}$ restricted to S' is smooth, a is connected to a via $\mathbb{F}$ and b is connected to b via $\mathbb{F}$ (by Proposition IV.2). Let $f : \mathbb{F} \rightarrow S'$ and $g : \mathbb{F} \rightarrow S'$ be the corresponding digraph homomorphisms. Moreover, a is connected to b via $\mathbb{F}$ in the digraph $\mathbb{G}$ and we take the corresponding homomorphism $h : \mathbb{F} \rightarrow \mathbb{G}$. For every $i = 0, 1, \ldots, k-1$ we consider the following matrix with k rows and $2nm + 1$ columns: To the first $(k - i - 1)$ rows we write f-images of the vertices of $\mathbb{F}$, to the $(k - i)$th row we write h-images, and to the last i rows we write g-images. We apply the term operation t to columns of this matrix. Since $E \leq \mathbf{A}^2$ we obtain a homomorphism from $\mathbb{F}$ to $\mathbb{G}$ which realizes a connection from

$$t(\underbrace{a, a, \ldots, a}_{(k-i)}, \underbrace{b, b, \ldots, b}_{i})$$

to

$$t(\underbrace{a, a, \ldots, a}_{(k-i-1)}, \underbrace{b, b, \ldots, b}_{(i+1)}).$$

Moreover, since all but one member of each column are elements of S' and $S' \lhd \mathbf{A}$, we actually get a homomorphism $\mathbb{F} \to S'$. By joining these homomorphisms for $i = 0, 1, \ldots, k-1$ we obtain that $a = t(a, a, \ldots, a)$ is connected to $b = t(b, b, \ldots, b)$ via the (m, nk)-fence in S'.

As $S' \subseteq H$ all the elements of S' are connected in H, and, using the paragraph above, also in S'. Moreover we can take two elements $a, b \in S'$ such that $a \to b$. As a is connected to b via a (m, nk)-fence in S' the algebraic length of $\mathbb{G}$ restricted to S' is one. ■

It remains to prove the case of Theorem IV.5 when there is no absorbing subuniverse in any weak component of $\mathbb{G}$ of algebraic length one. We choose such a component and call it H. By Claim IV.5.2 there is an $a \in H$ and $\mathbb{F}$ such that $H = \{a\}^{\mathbb{F}}$. Since $\{a\}$ is a subuniverse the reasoning similar to the one in the first case shows that H is a subuniverse as well. If $H \subsetneq A$ we are done by the inductive assumption. Therefore $H = A$ and there is no absorbing subuniverse in $\mathbf{A}$. Let k be minimal such that there exists m and $a \in A$ with $\{a\}^{\mathbb{F}[k,m]} = A$. This implies that $E^{\circ k} \leq_S A \times A$ is linked and, as there is no absorbing subuniverse in $\mathbf{A}$, Theorem III.6 implies that $E^{\circ k} = A \times A$. In particular the digraph $\mathbb{G}$ is strongly connected. Choose any $a \in A$ and consider the fence $\mathbb{F}[k-1, m']$ for m' large enough so that $B = \{a\}^{\mathbb{F}[k-1,m']} = \{a\}^{\mathbb{F}[k-1,m'+1]}$. B is a proper subuniverse of $\mathbf{A}$ (by yet again the same reasoning) and it suffices to prove that the smooth part of $\mathbb{G}$ restricted to B (which is going to be a subuniverse by Lemma IV.6) has algebraic length 1.

Claim IV.6.2. The smooth part of B, denoted by B', is non-empty and has algebraic length one.

Proof: Note that, by definition of B, $B^{\mathbb{F}[k-1,1]} = B$. Choose an arbitrary $b \in B$ and let $a \in A$ be such that $b \xrightarrow{k-1} a$ (such an a exists as $\mathbb{G}$ is smooth). Since $E^{\circ k} = A \times A$ we get $b \xrightarrow{k} a$. Consider the first element b_1 on this path: $b \to b_1$ and $b_1 \in B$ as $b \xrightarrow{k-1} a \xleftarrow{k-1} b_0$. Therefore $b \to b_1$ in $E \cap (B \times B)$ and thus $B^{\cdot\leftarrow\cdot} \supseteq B$. By Proposition IV.2 the smooth part of B is non-empty.

It is easy to see that for any $b \in B'$ and any $a \xrightarrow{i} b$ whenever $i \leq k-1$ then $a \in B$ (as $B^{\mathbb{F}[k-1,1]} = B$). Let us choose $b, b' \in B'$ such that $b \xrightarrow{k-1} b'$ in B'. As $E^{\circ k} = A \times A$ we have $b \xrightarrow{k} b'$ in $\mathbf{A}$. By the remark at the beginning of this paragraph all the vertices on the path $b \xrightarrow{k} b'$ are in B, and, since b, b' are in B', the whole path falls in B'. This gives a path of algebraic length one connecting b to b in B' which proves the claim. ■

V. Cyclic terms in Taylor varieties

Our approach to cyclic terms hinges on the following definition:

Definition V.1. *An n-ary relation R on a set A is called* cyclic*, if for all $a_0, \ldots, a_{n-1} \in A$*

$$(a_0, a_1, \ldots, a_{n-1}) \in R \quad \Rightarrow \quad (a_1, a_2, \ldots, a_{n-1}, a_0) \in R.$$

The following easy consequence of the definition can be found in [5].

Lemma V.2. *Let $\mathbf{A}$ be a finite, idempotent algebra then TFAE:*

- *$\mathbf{A}$ has a k-ary cyclic term;*
- *every cyclic subalgebra of $\mathbf{A}^k$ contains a constant tuple.*

Our proof of the theorem below is based on this fact and uses Theorem III.6 and Theorem IV.5. The details of the proof are to be found in [2].

Theorem V.3. *Let $\mathcal{V}$ be an idempotent variety generated by a finite algebra $\mathbf{A}$ then TFAE:*

- *$\mathcal{V}$ is a Taylor variety;*
- *$\mathcal{V}$ (equivalently the algebra $\mathbf{A}$) has a cyclic term;*
- *$\mathcal{V}$ (equivalently the algebra $\mathbf{A}$) has a cyclic term of arity p, for every prime $p > |A|$.*

We provide more information on the arities of possible cyclic terms in the algebra. Let $\mathbf{A}$ be a finite algebra and let $C(\mathbf{A})$ be the set of arities of cyclic operations of $\mathbf{A}$ i.e.:

$$C(\mathbf{A}) = \{n : \mathbf{A} \text{ has a cyclic term of arity } n\}.$$

It have been shown in [5]

Proposition V.4 ([5]). *Let $\mathbf{A}$ be a finite algebra let m, n be natural numbers. Then the following are equivalent.*

(i) *$m, n \in C(\mathbf{A})$;*
(ii) *$mn \in C(\mathbf{A})$.*

This implies that $C(\mathbf{A})$ is fully determined by its prime elements. The assumption "finitely generated' cannot be relaxed to "locally finite" [5] and there are algebras in Taylor varieties with no cyclic terms of arities smaller than their size [5]. However the following simple corollary provides, under special circumstances, additional elements in $C(\mathbf{A})$.

Corollary V.5. *Let $\mathbf{A}$ be a finite, idempotent algebra and α be a congruence of $\mathbf{A}$. If $\mathbf{A}/\alpha$ and every α-block in A have cyclic operation of arity k then so does $\mathbf{A}$.*

Proof: To apply Lemma V.2 we focus on an arbitrary cyclic subalgebra $\mathbf{B}$ of $\mathbf{A}^k$. Our first objective is to find a tuple in $\mathbf{B}$ with all elements congruent to each other modulo α. Let us choose any tuple $(a_0, \ldots, a_{k-1}) \in B$ and let $c(x_0, \ldots, x_{k-1})$ be the operation of $\mathbf{A}$ which gives rise to the cyclic operation of $\mathbf{A}/\alpha$. Therefore

$c(a_0,\dots,a_{k-1}), c(a_1,\dots,a_{k-1},a_0),\dots$ all lie in one congruence block of α (as the results of these evaluations are equal in $\mathbf{A}/\alpha$). Now we apply the term $c(x_0,\dots,x_{k-1})$ in $\mathbf{B}$ to $(a_0,\dots,a_{k-1}),(a_1,\dots,a_{k-1},a_0),\dots$ and obtain the tuple $(c(a_0,\dots,a_{k-1}), c(a_1,\dots,a_{k-1},a_0),\dots)$ in $\mathbf{B}$ with all coordinates in the same congruence block.

Let C be a congruence block of α such that $C^k\cap B\neq\emptyset$. It is easy to see that in such a case $C^k\cap B$ is a cyclic subalgebra of $\mathbf{C}^k$. As the block C has a cyclic operation of arity k then, again by Lemma V.2, we obtain a constant in $C^k\cap B$ and the corollary is proved. ■

And this leads to the following easy observation.

Corollary V.6. *Let* $\mathbf{A}$ *be a finite, idempotent algebra in Taylor variety. Let* $0_A=\alpha_0\subseteq\cdots\subseteq\alpha_n=1_A$ *be an increasing sequence of congruences on* $\mathbf{A}$*. If* p *is a prime number such that, for every* $i\geq 1$*, every class of* α_i *splits into less than* p *classes of* α_{i-1} *then* $\mathbf{A}$ *has a* p*-ary cyclic term.*

A. Consequences of the Theorem V.3

We present two immediate consequences. First we reprove a theorem of Hell and Nešetřil [16]. It follows immediately from the smooth theorem of Section IV, but the following proof is an elegant way of presenting it.

Corollary V.7 (Hell and Nešetřil [16])**.** *Let* $\mathbb{G}$ *be an undirected graph without loops. If* $\mathbb{G}$ *is bipartite then* $\mathrm{CSP}(\mathbb{G})$ *is solvable in a polynomial time. Otherwise it is NP-complete.*

Proof: Without loss of generality we can assume that $\mathbb{G}$ is a core. If the graph $\mathbb{G}$ is bipartite then it is a single edge and $\mathrm{CSP}(\mathbb{G})$ is solvable in a polynomial time. Assume now that $\mathbb{G}$ is not bipartite — therefore there exists a cycle $a\xrightarrow{2k+1}a$ of odd length in $\mathbb{G}$. As vertex a is in a 2-cycle (i.e. an undirected edge) therefore we can find a path $a\xrightarrow{i(2k+1)+j2}a$ for any non-negative numbers numbers i and j. Thus, for any number $l\geq 2k$ we have $a\xrightarrow{l}a$. Let p be any prime greater than $\max\{2k,|A|\}$ and t be any p-ary polymorphism of $\mathbb{G}$. Let $a=a_0\to a_1\to\cdots\to a_{p-1}\to a$. Then

$$t(a_0,\dots,a_{p-1})\to t(a_1,\dots,a_{p-1},a_0)$$

and, if t were a cyclic operation we would have

$$t(a_0,\dots,a_{p-1})=t(a_1,\dots,a_{p-1},a_0)$$

which implies a loop in $\mathbb{G}$. This contradiction shows that $\mathbb{G}$ has no cyclic polymorphism for some prime greater than the size of the vertex set which, by Theorem V.3, implies that the associated variety is not Taylor and therefore by Theorem II.4 $\mathrm{CSP}(\mathbb{G})$ is NP-complete. ■

A second consequence of Theorem V.3 is a proof of a result of Maroti and McKenzie [20]

Corollary V.8 (Maroti and McKenzie [20])**.** *Let* $\mathcal{V}$ *be a locally finite idempotent variety then TFAE:*

- $\mathcal{V}$ *is a Taylor variety;*
- $\mathcal{V}$ *has weak near-unanimity term.*

Proof: Every weak near-unanimity term is a Taylor term, and therefore only one implication is interesting. Let $\mathcal{V}$ be a Taylor variety. Let $\mathbf{F}$ be a free algebra in $\mathcal{V}$ on two generators. As $\mathcal{V}$ is locally finite the algebra $\mathbf{F}$ is finite, it generates a Taylor variety and thus, by Theorem V.3 has a cyclic operation c. Since in $\mathbf{F}$ we have $c(x_0,\dots,x_{p-1})\approx c(x_1,\dots,x_{p-1},x_0)$ then

$$c(y,x,x,\dots,x)\approx c(x,y,x,x,\dots,x)\approx\cdots$$
$$\cdots\approx c(x,x,\dots,y,x)\approx c(x,x,\dots,x,y)$$

holds in $\mathcal{V}$ and the corollary is proved. ■

Finally we are able to restate the algebraic dichotomy conjecture of Bulatov, Jeavons and Krokhin:

The dichotomy conjecture of Bulatov, Jeavons and Krokhin. *Let* $\mathbb{A}$ *be a relational core. Let* p *be a prime number greater than the size o the universe of* $\mathbb{A}$*. If every positively primitively defined cyclic relation in* A^p *has a constant then* $\mathrm{CSP}(\mathbb{A})$ *is solvable in a polynomial time. Otherwise it is NP-complete.*

This statement is equivalent to the original algebraic dichotomy conjecture by Theorem V.3 and Lemma V.2.

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