# Dirac operator in several variables and combinatorial identities


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**Abstract.** The Dolbeault sequence is a fundamental tool for many problems in the function theory of several complex variables. A lot of attention was paid in the last decades to its analogue in the function theory of several Clifford variables. The first operator in this resolution is the Dirac operator in several variables. The complete description is known in dimension 4 (i.e., in the case of quaternionic variables, see [1, 6, 4]). Much less is known in higher dimensions. The case of three variables was described completely (see [18]). The full description of the complex for all dimensions is not known at present. Even the case of the stable range (i.e., when the number of variables is less or equal to the half of dimension) is still not fully understood.

There are two different approaches to the stable range case, one based on classical algebraic geometry (the Hilbert syzygy theory, see [8]), the other one on representation theory (differential invariants in certain parabolic geometries, see [14, 20]). Differential operators in these resolutions are acting on vector-valued functions. Such spaces of functions are quite complicated in general and the first problem in the description of the resolution is to understand their dimensions. Both the approaches mentioned above suggest an answer to this question, although such answers look quite different. The aim of the paper is to compare these two results and to show that they lead to complicated combinatorial identities.



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## THE ANALOGUE OF THE DOLBEAULT SEQUENCE

The Dirac operator $\partial_{\underline{x}} = e_1\partial_{x_1} + \cdots + e_m\partial_{x_m}$ generalizes the well known Cauchy-Riemann operator $\partial_{\bar{z}} = \partial_x + i\partial_y$ from complex analysis to hypercomplex analysis. It acts on functions defined on an open subset in $\mathbb{R}^m$, we shall consider here values in an irreducible $Spin(m)$ representation $\mathbb{S}_m$. A natural problem in Clifford analysis is to study the space of monogenic functions, which constitutes the kernel of the Dirac operator (for details, see [12]). A natural generalization of the theory of several complex variables is then the study of solutions of the Dirac operator $\mathscr{D}_k := (\partial_{\underline{x}_1}, \ldots, \partial_{\underline{x}_k})$ in several Clifford variables. The operator $\mathscr{D}_k$ is now acting on functions defined on open subsets in $(\mathbb{R}^m)^k$ with values in the spinor representation $\mathbb{S}_m$.

We would like to describe compatibility conditions for the image of $\mathscr{D}_k^0 = \mathscr{D}_k$. More precisely, we want to characterize $k$-tuples $(g_1, \ldots, , g_k)$ on the right hand side of the system

$$\left\{ \begin{array}{rcl} \partial_{\underline{x}_1} f & = & g_1 \\ \vdots & & \\ \partial_{\underline{x}_k} f & = & g_k \end{array} \right. , \tag{1}$$

as a kernel of a suitable differential operator $\mathscr{D}_k^1$. Here the functions $g_j$ are of the same type as the function $f$. Continuing in the same way with the system associated to $\mathscr{D}_k^1$, we are looking for a sequence $\mathscr{D}_k^j$, $j = 1, \ldots, t$ of differential operators which form a resolution of the first operator $\mathscr{D}_k^0$. If $\mathbb{W}_j$ denotes the values of the functions at each step and $\mathscr{C}^\infty(\mathbb{W}_j)$ denotes smooth maps on $(\mathbb{R}^m)^k$ with values in $\mathbb{W}_j$, we want to find a complex

$$0 \longrightarrow \mathscr{C}^\infty(\mathbb{W}_0) \xrightarrow{\mathscr{D}_k^0} \mathscr{C}^\infty(\mathbb{W}_1) \xrightarrow{\mathscr{D}_k^1} \ldots \longrightarrow \mathscr{C}^\infty(\mathbb{W}_{t-1}) \xrightarrow{\mathscr{D}_k^{t-1}} \mathscr{C}^\infty(\mathbb{W}_t) \longrightarrow 0, \tag{2}$$

which is (locally) exact. We will present in turn two different approaches to the construction of such resolution.

## BETTI NUMBERS

The first method is to apply (at least for small values of $k$ and $m$) the algebraic techniques described in [8] and to construct a free resolution of the module $coker(P^t)$, where $P$ is the symbol matrix of the operator $\mathscr{D}_k := (\partial_{\underline{x}_1}, \ldots, \partial_{\underline{x}_k})$. Dualizing it, one obtains a complex of polynomial maps and free modules

$$0 \longrightarrow R^{\beta_0} \xrightarrow{P} R^{\beta_1} \xrightarrow{P_1} \ldots \longrightarrow R^{\beta_{t-1}} \xrightarrow{P_{t-1}} R^{\beta_t} \longrightarrow 0. \tag{3}$$

Each map in the complex can be viewed as the symbol of a differential operator. The condition $P_{i+1} \circ P_i = 0$ means that $P_{i+1}$ encodes the compatibility conditions for the system of equations associated to $P_i$ (see [8]). Suppose that we are in the stable range, i.e. $m \geq 2k$ where $n$ denotes the dimension of the spinor space $\mathbb{S}_m$. Let us introduce the ring $R := \mathbb{C}[x_{11}, \ldots, x_{km}]$ and let the matrix $P \in \mathrm{Mat}_{kn,n}(R)$ represent the symbol of the operator $D$ acting on spinor valued functions on $(\mathbb{R}^m)^k$. Then the associated module $\mathscr{M} := coker(P^t) = R^n/im(P^t)$ has a finite free resolution

$$0 \longrightarrow \oplus_j R(-j)^{\beta_{tj}} \longrightarrow \oplus_j R(-j)^{\beta_{t-1j}} \longrightarrow \ldots \longrightarrow \oplus_j R(-j)^{\beta_{1j}} \xrightarrow{P} \oplus_j R(-j)^{\beta_{0j}} \longrightarrow 0. \tag{4}$$

The integer $\beta_{ij}$ is the $i^{th}$ graded Betti number in degree $j$. Another important invariant for the module is its Hilbert series $\mathscr{H}_{\mathscr{M}}(z) = \sum_j dim_{\mathbb{C}}(\mathscr{M}_j) z^j$ where $\mathscr{M}_j$ is the $j^{th}$ graded component of $\mathscr{M}$. Computational evidence in this case shows [10] that the Hilbert series is

$$\mathscr{H}_{\mathscr{M}}(z) = \frac{n(1+z)^{\binom{k}{2}}(1-z)^{\binom{k+1}{2}}}{(1-z)^{mk}} = \frac{HN(z)}{(1-z)^{km}}.$$

The relation between the Hilbert numerator $HN(z)$ and the graded Betti numbers is (see [13])

$$HN(z) = \sum_{j=0}^{d} \sum_{i=0}^{t} (-1)^i \beta_{ij} z^j. \tag{5}$$

One defines

$$\mathbf{B_j} := \frac{1}{j!} \frac{d^j}{dz^j} [(1+z)^{\binom{k}{2}}(1-z)^{\binom{k+1}{2}}]_{|_{z=0}},$$

so the numerator is $HN(z) = n \sum_{j=0}^{d} \mathbf{B_j} z^j$ and we have that

$$\mathbf{B_j} = \sum_{t=\max(0,\mathbf{j}-\binom{k}{2})}^{\min(\mathbf{j},\binom{k+1}{2})} (-1)^t \binom{\binom{k}{2}}{t} \binom{\binom{k+1}{2}}{j-t}. \tag{6}$$

Unfortunately, it is only when the resolution is "pure", i.e. for all $i$ there exists exactly one $j$such that $\beta_{ij} \neq 0$, that the Hilbert numerator actually determines the Betti numbers. In our case, this happens only when $k = 2$ [8].

## INVARIANT DIFFERENTIAL OPERATORS, WEYL DIMENSION FORMULA

The Dirac operator is invariant with respect to the group of orthogonal transformations but it is invariant also with respect to a bigger group of conformal transformation. The Dirac operator $\mathscr{D}_k$ in several variables is clearly invariant with respect to the product $SL_k \times Spin(m)$ (the second factor acts on each variable separately by the corresponding rotations, while the first one permutes them; at the same time, elements in $Spin(m)$ act by left multiplication on the values of the functions). Moreover, the operator $\mathscr{D}_k$ is invariant (in a suitable interpretation, for details see [14]) with respect to a (bigger) parabolic subgroup $P$ of the group $G = Spin(k+1, m+1)$.

The main tool used in the second method for the construction of the resolution is to use this bigger invariance of the first operator $\mathscr{D}_k$ and to build the resolution from differential operators having the same invariance properties. Some steps in this direction were made in [14] (using the dual formulation in terms of Verma module homomorphisms) and in [15] (using the Penrose transform techniques, as explained in [3, 2]). As a result of this approach, one gets a clear description of the spaces in which the functions take values, expressed in terms of the so called Hasse graph.

The spinor representation $\mathbb{S}_m$ can be considered as a representation of the Levi factor $G_0 \simeq \mathbb{R}_* \times SL(k) \times Spin(m)$, which can be characterized by its highest weight for a Cartan subalgebra of $G$. The Hasse graph is defined to be the orbit of the affine action of the Weyl group $W$ of $G$. The Hasse graph was computed in [14], where it is possible to find more details. We would like to compute dimensions of the corresponding modules, hence we can consider the spaces $\mathbb{V}_j, j = 1,\dots,t$ as modules over the semi-simple part $SL(k) \times Spin(m)$ of $G_0$.

There is a one-to-one correspondence between irreducible representations of the product of two groups and tensor products of irreducible representations of individual factors. In our case, the second factor in the tensor product will always be an irreducible $Spin(m)$ module $\mathbb{S}_m$, hence an irreducible module for the product $SL(k) \times Spin(m)$ will be denoted by a highest weight $\lambda = (\lambda_1,\dots,\lambda_k), \lambda_i \in \mathbb{Z}$ for $SL(k)$. Given a weight $\lambda$, let us denote by $\lambda'$ its conjugate, i.e. $\lambda'_j = \sharp\{i|\lambda_i \geq j\}$. Note that if we use the Young (or Ferrers) diagrams instead of highest weights, conjugation corresponds to a reflection of the diagram along the diagonal. For our purposes, we shall need only weights of the form $\lambda = (\lambda_1,\dots,\lambda_k)$ with $k \geq \lambda_1 \geq \dots \geq \lambda_k \geq 0$ and $\lambda = \lambda'$. Let us denote the set of such weights by the symbol $\mathscr{A}$.

We define a function $\sigma$ on $\mathscr{A}$ by $\sigma(\lambda) = \sum_1^k(\lambda_j - j + 1)^+$, where $(\alpha)^+, \alpha \in \mathbb{R}$ denotes $max(0,\alpha)$. For example, $\sigma((3,2,1)) = 3+1+0 = 4$. In terms of diagrams, $\sigma(\lambda)$ equals to the number of boxes below or on the diagonal. Values of $\sigma(\lambda)$ clearly belong to the interval $< 0, \binom{k+1}{2} >$, the maximum being achieved for $\lambda = (k,\dots,k)$.

We can now compute dimensions of the spaces $\mathbb{W}_j$ appearing in the resolution by invariant operators in the following way. The length of the resolutions will be $t = \binom{k+1}{2}$ and

$$\mathbb{W}_j \simeq \oplus_{\lambda\in\mathscr{A},\sigma(\lambda)=j} \mathbb{V}_\lambda \otimes \mathbb{S}_m,\ j = 0\dots,\binom{k+1}{2}.$$

The dimension of the module $\mathbb{S}_m$ equals to $n$ and the dimension of $\mathbb{V}_\lambda$ can be computed by the formula

$$\mathbf{d}^{\mathbf{k}}_{\lambda} = \frac{(\lambda_1+k-1)!(\lambda_2+k-2)!\cdots\lambda_k!}{(k-1)!(k-2)!\cdots 2!\prod(h)} \tag{7}$$

where the term $\prod(h)$ is the product of all the hook lengths associated to the Ferrer diagram $\lambda$. Hence we have an explicit formula for the dimension of the spaces $\mathbb{W}_j$.

## THE CONJECTURE

We can now compare the results coming from both approaches. We get in such a way some nontrivial relations involving various combinatorial quantities. The form of the invariant resolution shows that the graded Betti numbers can be expressed easily using the dimensions of individual irreducible representations appearing in the invariant resolution. In particular, $\beta_{ij}$ equals to $n$ times the sum of dimensions of all $\mathbb{V}_\lambda$ over the set of all weights $\alpha$ with $\sigma(\lambda) = j$ and $|\lambda| = i$. Using the relation with the Hilbert series numerator, we get the following conjecture.

**Conjecture**

*For every nonnegative integer j, the j-th coefficient of the Hilbert numerator associated to the Dirac operator in k variables is*

$$\mathbf{B_j} = \sum_{\lambda=\lambda',|\lambda|=j} (-1)^{\sigma(\lambda)}\,\mathbf{d}^k_\lambda.$$

In low dimensions, the relations can be checked explicitly. We have checked them for $k \leq 6$ also using CoCoA [5]. For example, for $k = 4$, we get

$$HN(z) = z^{16} - 4z^{15} + 20z^{13} - 20z^{12} - 36z^{11} + 64z^{10} + 20z^9 - 90z^8 + 20z^7 + 64z^6 - 36z^5 - 20z^4 + 20z^3 - 4z + 1.$$

On the other hand, 16 (nontrivial) graded Betti numbers calculated using Weyl dimension formula (7) are given by

$$\beta_{00} = 1, \beta_{11} = 4, \beta_{23} = 20, \beta_{34} = 20, \beta_{35} = 36, \beta_{46} = 64, \beta_{58} = 45, \beta_{69} = 20,$$

$$\beta_{47} = 20, \beta_{48} = 45, \beta_{6,10} = 64, \beta_{7,11} = 36, \beta_{7,12} = 20, \beta_{8,13} = 20, \beta_{9,15} = 4, \beta_{10,16} = 1.$$

Using the relation (5) , we see that the resulting coefficients in the Hilbert series coincides, hence both resolutions have the same dimensions of the spaces $\mathbb{W}_j, j = 1,\dots,t$.

It is probable that the conjecture above will be proved soon by further development of methods used for both approaches to the construction of analogues of the Dolbeault resolution.

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