On the telescope conjecture for module categories

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Abstract

In [H. Krause, O. Solberg, Applications of cotorsion pairs, J. London Math. Soc. 68 (2003) 631–650], the Telescope Conjecture was formulated for the module category Mod R of an artin algebra R as follows: "If $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in Mod R with $\mathcal{A}$ and $\mathcal{B}$ closed under direct limits, then $\mathcal{A} = \varinjlim(\mathcal{A} \cap \text{mod } R)$". We extend this conjecture to arbitrary rings R, and show that it holds true if and only if the cotorsion pair $\mathfrak{C}$ is of finite type. Then we prove the conjecture in the case when R is right noetherian and $\mathcal{B}$ has bounded injective dimension (thus, in particular, when $\mathfrak{C}$ is any cotilting cotorsion pair). We also focus on the assumptions that $\mathcal{A}$ and $\mathcal{B}$ are closed under direct limits and on related closure properties, and detect several asymmetries in the properties of $\mathcal{A}$ and $\mathcal{B}$.
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In the late 1970s, Bousfield and Ravenel formulated a telescope conjecture for the stable homotopy category. Later on, Neeman extended it to compactly generated triangulated categories $\mathcal{T}$. In this generality, the conjecture said that any smashing localizing subcategory $\mathcal{L}$ of $\mathcal{T}$ is of finite type, cf. [10,22,26]. Keller [19] gave an example disproving the conjecture in the case when $\mathcal{T}$ is the (unbounded) derived category of the module category over a particular (non-noetherian) commutative ring. However, it appears open whether the conjecture holds true when $\mathcal{T}$ is the stable module category of a self-injective artin algebra R. In that case, the conjecture was shown to be equivalent to a certain property of cotorsion pairs of R-modules, cf. [22, Section 7]. This led Krause and Solberg to the following version of the telescope conjecture for module categories of arbitrary artin algebras:

[22, 7.9] "Let R be an artin algebra, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a complete hereditary cotorsion pair in Mod R with $\mathcal{A}$ and $\mathcal{B}$ closed under direct limits. Then $\mathcal{A} = \varinjlim(\mathcal{A} \cap \text{mod } R)$."

The latter conjecture is known to hold when $\mathfrak{C}$ is a tilting cotorsion pair by [9] (see also [18, Section 5]), when $\mathfrak{C}$ is a 1-cotilting cotorsion pair by [11], and when $\mathcal{A} \cap \text{mod } R$ is a contravariantly finite subcategory of mod R by [22].

In the present paper, we deal with the following general version of the Krause–Solberg conjecture, formulated for arbitrary rings:

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0.1. Telescope conjecture for module categories

"Let R be ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a complete hereditary cotorsion pair in Mod R with $\mathcal{A}$ and $\mathcal{B}$ closed under direct limits. Then $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$."

Here $\mathcal{A}^{<\omega} = \mathcal{A} \cap \mathrm{mod}\, R$ where mod R denotes the class of all modules possessing a projective resolution consisting of finitely generated modules.

Recall that a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is of *finite type* provided there is a set $\mathcal{C} \subseteq \mathrm{mod}\, R$ such that $\mathcal{B} = \mathrm{Ker}\, \mathrm{Ext}^1_R(\mathcal{C}, -)$. It is known that 0.1 holds for all cotorsion pairs $\mathfrak{C}$ of finite type (with $\mathcal{A}$ closed under direct limits). In Corollary 4.7, we prove the converse: any cotorsion pair $\mathfrak{C}$ for which the conclusion of 0.1 holds is necessarily of finite type.

Moreover, in Theorem 4.10, we prove that 0.1 holds for any right noetherian ring under the additional assumption that all modules in $\mathcal{B}$ have bounded injective dimensions. This yields 0.1 in the particular case when $\mathfrak{C}$ is a cotilting cotorsion pair over a right noetherian ring.

The proofs of these theorems rely on a number of results on deconstruction and completeness of cotorsion pairs from [31,8,9,28] which were essential for the recent rapid progress in infinite dimensional tilting and cotilting theory. Unfortunately, most of these preprints have not been published yet, so we supplement the original references below with quotations of the corresponding results in the recent monograph [18]. The latter was submitted for publication only in Spring 2006, but thanks to the rapid publication policy of Walter de Gruyter, it is paradoxically available in printed form much earlier then the papers submitted in 2005.

The assumptions made in 0.1 that $\mathcal{A}$ and $\mathcal{B}$ are closed under direct limits also lead us to an investigation of closure properties of cotorsion pairs, with special emphasis on tilting and cotilting cotorsion pairs (Sections 2 and 3). Finally, the last section is devoted to some asymmetries that can occur in the properties of $\mathcal{A}$ and $\mathcal{B}$. In Example 5.2(3) we exhibit an example of a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ of infinite type with $\mathcal{B}$ not being closed under coproducts and $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$. We also show that in general the validity of 0.1 does not imply $\mathcal{B} = \varinjlim \mathcal{B}^{<\omega}$, see Example 5.2(1) and Theorem 5.3.

1. Preliminaries

Notation. Let R be a ring. Denote by Mod R the category of all (right R-) modules, and by mod R the subcategory of all modules possessing a projective resolution consisting of finitely generated modules. (If R is right coherent then mod R is just the category of all finitely presented modules.)

Given an infinite cardinal κ and a class of modules $\mathcal{A}$, the symbol $\mathcal{A}^{<\kappa}$ ($\mathcal{A}^{\leq\kappa}$) denotes the subclass of $\mathcal{A}$ consisting of all modules possessing a projective resolution consisting of $<\kappa$-generated ($\leq\kappa$-generated) modules. For example, mod $R = (\mathrm{Mod}\, R)^{<\omega}$.

We denote by $\mathcal{P}$ and $\mathcal{I}$ the class of all modules of finite projective and injective dimension, respectively. For $n < \omega$, $\mathcal{P}_n$ ($\mathcal{I}_n$, $\mathcal{F}_n$) is the class of all modules of projective (injective, flat) dimension $\leq n$.

Let $\mathcal{M}$ be a subcategory of Mod R. We always assume that $\mathcal{M}$ is *full and that it is closed under direct summands and isomorphic images.*

We denote by Add $\mathcal{M}$ (respectively add $\mathcal{M}$) the subcategory of all modules isomorphic to a direct summand of a (finite) direct sum of modules of $\mathcal{M}$, and by Prod $\mathcal{M}$ the subcategory of all modules isomorphic to a direct summand of a product of modules of $\mathcal{M}$. If $\mathcal{M} = \{M\}$, we write Add M, add M, Prod M.

Furthermore, $\varinjlim \mathcal{M}$ denotes the class of all modules D such that $D = \varinjlim_{i \in I} M_i$ where $\{M_i \mid i \in I\}$ is a direct system of modules from $\mathcal{M}$. We will use repeatedly the following characterization of $\varinjlim \mathcal{M}$ due to Lenzing.

Lemma 1.1 (*[25, 2.1] (see also [18, 1.2.9])*). *Assume that $\mathcal{M}$ is an additive subcategory of* mod R. *Then the following statements are equivalent for a module A_R.*

(1) $A \in \varinjlim \mathcal{M}$.
(2) *There is a pure epimorphism $\coprod_{k \in K} X_k \to A$ for some modules X_k in $\mathcal{M}$.*
(3) *Every homomorphism $h : F \to A$ where F is finitely presented factors through a module in* add $\mathcal{M}$.

Resolving subcategories. A class $\mathcal{S} \subseteq \mathrm{Mod}\, R$ (or $\mathcal{S} \subseteq \mathrm{mod}\, R$) is said to be a *resolving* subcategory of Mod R (respectively, of mod R) if it satisfies the following conditions:

(R1) $\mathcal{S}$ contains all (finitely generated) projective modules,
(R2) $\mathcal{S}$ is closed under extensions,
(R3) $\mathcal{S}$ is closed under kernels of epimorphisms.

Coresolving subcategories are defined by the dual conditions (CR1), (CR2), (CR3).

Orthogonal classes. For a class $\mathcal{C} \subseteq \text{Mod } R$ and for $i > 0$, we define

$$\mathcal{C}^{\perp_i} = \text{Ker Ext}^i_R(\mathcal{C}, -) \qquad {}^{\perp_i}\mathcal{C} = \text{Ker Ext}^i_R(-, \mathcal{C})$$
$$\mathcal{C}^{\perp} = \bigcap_{i>0} \mathcal{C}^{\perp_i} \qquad {}^{\perp}\mathcal{C} = \bigcap_{i>0} {}^{\perp_i}\mathcal{C}.$$

Similarly, we define the classes $\mathcal{C}^{\mathsf{T}i}$, ${}^{\mathsf{T}i}\mathcal{C}$, $\mathcal{C}^{\mathsf{T}}$, and ${}^{\mathsf{T}}\mathcal{C}$, replacing Ext by Tor.

We collect here some well-known facts often used in the sequel.

Remark 1.2. (1) If $\mathcal{S}$ is resolving, then $\mathcal{S}^{\perp_1} = \mathcal{S}^{\perp}$ and $\mathcal{S}^{\mathsf{T}1} = \mathcal{S}^{\mathsf{T}}$. Coresolving classes have the dual properties.
(2) For any $\mathcal{M} \subseteq \text{Mod } R$, the classes ${}^{\perp}\mathcal{M}$, $\mathcal{M}^{\mathsf{T}}$ are resolving, and $\mathcal{M}^{\perp}$ is coresolving.
(3) (cf. [17, 10.2.4, and 3.2.26]) If $\mathcal{C} \subseteq \text{mod } R$ and $i > 0$, then $\mathcal{C}^{\perp_i}$ and $\mathcal{C}^{\mathsf{T}i}$ are closed under direct products and direct limits.

Approximations. Let $\mathcal{M}$ be a subcategory of Mod R, and let A be a right R-module. A morphism $f \in \text{Hom}_R(A, X)$ with $X \in \mathcal{M}$ is an $\mathcal{M}$-*preenvelope* (or a left $\mathcal{M}$-approximation) of A provided that the abelian group homomorphism $\text{Hom}_R(f, M) : \text{Hom}_R(X, M) \longrightarrow \text{Hom}_R(A, M)$ is surjective for each $M \in \mathcal{M}$.

An $\mathcal{M}$-preenvelope $f \in \text{Hom}_R(A, X)$ of A is said to be *special* if f is a monomorphism and $\text{Ext}^1_R(\text{Coker } f, M) = 0$ for all $M \in \mathcal{M}$.

An $\mathcal{M}$-*envelope* of A is an $\mathcal{M}$-preenvelope $f \in \text{Hom}_R(A, X)$ which is left minimal, that is, h is an automorphism of X whenever $h \in \text{End}_R(X)$ satisfies $hf = f$. Note that $\mathcal{M}$-envelopes may not exist in general, but they are always unique up to isomorphism.

The notions of an $\mathcal{M}$-*cover* and a (*special*) $\mathcal{M}$-*precover* are defined dually.

A subcategory $\mathcal{S}$ of mod R is said to be *covariantly* (respectively, *contravariantly*) *finite* in mod R if every module in mod R has an $\mathcal{S}$-preenvelope (respectively, an $\mathcal{S}$-precover). A class of modules $\mathcal{M}$ is *definable* if it is closed under direct products, direct limits, and pure submodules. We will frequently use the following relationship between covariantly finite subcategories of mod R and definable classes.

Theorem 1.3 (*[13, 4.2], [20, 3.11]*). *Let $\mathcal{S}$ be a full additive subcategory of* mod R. *The following statements are equivalent.*
(1) *$\mathcal{S}$ is covariantly finite in* mod R.
(2) $\varinjlim \mathcal{S}$ *is closed under products.*
(3) $\varinjlim \mathcal{S}$ *is definable.*

Cotorsion pairs. Let $\mathcal{A}, \mathcal{B} \subseteq \text{Mod } R$ (or $\mathcal{A}, \mathcal{B} \subseteq \text{mod } R$) be classes of modules. Then $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a *cotorsion pair in* Mod$R$ (respectively, a *cotorsion pair in* mod R) provided $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$ (respectively, provided $\mathcal{A} = ({}^{\perp_1}\mathcal{B})^{<\omega}$ and $\mathcal{B} = (\mathcal{A}^{\perp_1})^{<\omega}$).

A cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ in Mod$R$ (or in mod R) is *complete* if every module (respectively, every module in mod R) has a special $\mathcal{A}$-precover and a special $\mathcal{B}$-preenvelope. Moreover, $\mathfrak{C}$ is *perfect* if every module (respectively, every module in mod R) has an $\mathcal{A}$-cover and a $\mathcal{B}$-envelope. Note that a complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ in Mod$R$ is perfect provided $\mathcal{A}$ is closed under direct limits [17, 7.2.6]. It is an open problem whether the converse holds true. For artin algebras, the following result was established by Auslander and Reiten.

Lemma 1.4 (*[6]*). *Let Λ be an artin algebra, and let $\mathcal{A}, \mathcal{B}$ be subcategories of* mod Λ. *The following statements are equivalent.*
(1) *$\mathcal{A}$ is a contravariantly finite subcategory of* mod Λ *satisfying conditions* (R1) *and* (R2)*, and $\mathcal{B} = (\mathcal{A}^{\perp_1})^{<\omega}$.*
(2) *$\mathcal{B}$ is a covariantly finite subcategory of* mod Λ *satisfying conditions* (CR1) *and* (CR2)*, and $\mathcal{A} = ({}^{\perp_1}\mathcal{B})^{<\omega}$.*
(3) *$(\mathcal{A}, \mathcal{B})$ is a perfect cotorsion pair in* mod Λ.

The existence of approximations cannot be omitted in the result above, as shown by the following example.

Example 1.5. Let Λ be an artin algebra such that the big (left) finitistic dimension of Λ equals $n > 1$, but its little (left) finitistic dimension is $<n$, see [32]. Consider the class $\mathcal{A}'$ of all nth syzygies of cyclic right Λ-modules, and set $\mathcal{A} = ({}^{\perp_1}(\mathcal{A}'^{\perp_1}))^{<\omega}$. Then $\mathcal{A}$ is a resolving subcategory of mod Λ, and by Baer's Lemma $\mathcal{A}^{\perp_1} = \mathcal{A}'^{\perp_1} = \mathcal{I}_n$. On the other hand, $\mathcal{B} = (\mathcal{A}^{\perp_1})^{<\omega} \subseteq \mathcal{I}_{n-1}$.

We deduce that $\mathcal{A}$ is properly contained in $({}^{\perp_1}\mathcal{B})^{<\omega}$. In fact, if we choose $M \in \Lambda\text{Mod}$ with pdim $M = n$, and $N \in \text{mod}\,\Lambda$ such that $\text{Ext}_R^n(N, D(M)) \neq 0$, then it is easy to see that the module $X = \Omega^{n-1}(N)$ is contained in $({}^{\perp_1}\mathcal{B})^{<\omega}$. On the other hand, X is not contained in $\mathcal{A}$, because $D(M) \in \mathcal{I}_n = \mathcal{A}^{\perp_1}$, and $\text{Ext}_R^1(X, D(M)) \neq 0$.

This shows that $(\mathcal{A}, \mathcal{B})$ is not a cotorsion pair in mod Λ. $\square$

We will need further terminology on cotorsion pairs.

Lemma 1.6. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in* Mod R *(or in* mod R*). Then $\mathcal{A}$ is resolving if and only if $\mathcal{B}$ is coresolving, and this is further equivalent to* $\text{Ext}_R^i(A, B) = 0$ *for all* $A \in \mathcal{A}, B \in \mathcal{B}, i \geq 2$*. In this case $(\mathcal{A}, \mathcal{B})$ is called hereditary.*

Let $\mathcal{C}$ be a class of modules. A module M is called $\mathcal{C}$-*filtered* provided there exist an ordinal σ and an increasing chain, $(M_\alpha \mid \alpha < \sigma)$, consisting of submodules of M such that $M_0 = 0$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha < \sigma$, $M = \bigcup_{\alpha<\sigma} M_\alpha$, and $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$ for each $\alpha + 1 < \sigma$.

Theorem 1.7 (*[14]*)**.** *Let $\mathcal{C}$ be a class of modules and let $\mathcal{B} = \mathcal{C}^{\perp_1}$ and $\mathcal{A} = {}^{\perp_1}(\mathcal{C}^{\perp_1})$. Then $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair, called the cotorsion pair cogenerated by $\mathcal{C}$. If the isomorphism classes of $\mathcal{C}$ form a set, then $(\mathcal{A}, \mathcal{B})$ is complete. Moreover, in this case $\mathcal{A} = {}^{\perp_1}(\mathcal{C}^{\perp_1})$ consists of all direct summands of $\mathcal{C} \cup \{R\}$-filtered modules.*

Theorem 1.8 (*[15]*)**.** *Let $\mathcal{C}$ be a class of modules and let $\mathcal{A} = {}^{\perp_1}\mathcal{C}$ and $\mathcal{B} = ({}^{\perp_1}\mathcal{C})^{\perp_1}$. Then $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair, called the cotorsion pair generated by $\mathcal{C}$. If $\mathcal{C}$ consists of pure-injective modules, then $(\mathcal{A}, \mathcal{B})$ is perfect.*

The theorems above together with Remark 1.2 apply to the following situations.

Cotorsion pairs of (co)finite type. A cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ in Mod R is of *finite type* provided it is cogenerated by a class $\mathcal{S} \subseteq \text{mod}\,R$. Then $\mathfrak{C}$ is complete and $\mathcal{B}$ is definable.

Dually, a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ in RMod is of *cofinite type* provided there is a class $\mathcal{S} \subseteq \text{mod}\,R$ such that $\mathcal{A} = \mathcal{S}^{\top_1}$. Then $\mathfrak{C}$ is perfect. Moreover, $\mathcal{A}$ is definable provided $\mathfrak{C}$ is hereditary. In fact, by the well-known Ext-Tor relation [17, 3.2.1], $(\mathcal{A}, \mathcal{B})$ is of cofinite type iff it is generated by the class $\mathcal{S}^* = \{S^* \mid S \in \mathcal{S}\}$ where S^* denotes the dual module, e. g. $S^* = \text{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z})$, or over artin algebras $S^* = D(S)$ for the usual duality D.

If $\mathfrak{C}$ is hereditary, then in both cases, we can assume w.l.o.g. that $\mathcal{S}$ is resolving.

(Co)smashing cotorsion pairs. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair in Mod R. Observe that the class $\mathcal{A}$ is always closed under coproducts, and $\mathcal{B}$ is always closed under products. We will say that $\mathfrak{C}$ is *smashing* if $\mathcal{B}$ is also closed under coproducts, and *cosmashing* if $\mathcal{A}$ is also closed under products.

For further properties of the notions defined above we refer to [18] (note however, that the terminology in [18] occasionally differs from the one used here).

2. Closure under direct limits

We start by recalling a result from [4]:

Theorem 2.1 (*[4, 2.3 and 2.4]*)**.** *Let $\mathcal{S}$ be a subcategory of* mod R *with properties* (R1) *and* (R2)*, and let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair cogenerated by $\mathcal{S}$. Then the following hold true:*
(1) $\mathcal{A} \subseteq \varinjlim \mathcal{S} = {}^{\top}(\mathcal{S}^{\top})$*, and $\mathcal{S} = \mathcal{A}^{<\omega} = (\varinjlim \mathcal{S})^{<\omega}$.*
(2) *There is a perfect cotorsion pair $(\varinjlim \mathcal{S}, \mathcal{Y})$ which is generated by the class of all pure-injective modules from $\mathcal{B}$.*

Theorem 2.1 has a number of consequences concerning conjecture 0.1.

Corollary 2.2. (1) *The (perfect) hereditary cotorsion pairs $(\mathcal{X}, \mathcal{Y})$ in* Mod R *satisfying $\mathcal{X} = \varinjlim \mathcal{X}^{<\omega}$ correspond bijectively to the resolving subcategories of* mod R*.*
The correspondence is given by the mutually inverse assignments

$$\alpha : (\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X}^{<\omega}$$
$$\beta : \mathcal{S} \mapsto (\varinjlim \mathcal{S}, \mathcal{Y}).$$

(2) *The hereditary cotorsion pairs of finite type in* Mod$R$ *correspond bijectively to the resolving subcategories of* mod R.

The correspondence is given by the mutually inverse assignments

$$\alpha : (\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A}^{<\omega}$$
$$\beta' : \mathcal{S} \mapsto (^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp}).$$

Proof. This follows by Lemma 1.6 and Theorem 2.1. □

Corollary 2.3. *Let* $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ *be a hereditary cotorsion pair of finite type in* Mod$R$. *Moreover, let* $\mathcal{B}'$ *be the class of all pure-injective modules from* $\mathcal{B}$. *Then the following statements are equivalent.*

(1) $\mathcal{A}$ *is closed under direct limits.*
(2) $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$.
(3) *The cotorsion pair* $(\varinjlim \mathcal{A}^{<\omega}, \mathcal{Y})$ *is of finite type.*
(4) *The class* $\mathcal{Y} = (^{\perp_1}\mathcal{B}')^{\perp_1}$ *is definable.*
(5) *Every pure embedding into a module* $M \in \mathcal{A} \cap \mathcal{B}$ *splits.*

Proof. First, notice that any cotorsion pair of finite type is complete by Theorem 1.7.

The equivalence of (1)–(3) follows from Theorem 2.1. Of course, (3) implies (4).

(4) $\Rightarrow$ (3): Since $\mathcal{B}' \subseteq \mathcal{Y} \subseteq \mathcal{B}$, the two definable classes $\mathcal{B}$ and $\mathcal{Y}$ contain the same pure-injective modules, and so they coincide.

To prove the equivalence of (2) and (5), we generalize an argument from [4, 4.2]:

First, if $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$ then $\mathcal{A}$ is closed under pure-epimorphic images by Lemma 1.1. Since $\mathfrak{C}$ is of finite type, $\mathcal{B}$ is closed under pure submodules. So, if $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ is a pure-exact sequence with $M \in \mathcal{A} \cap \mathcal{B}$, then $P \in \mathcal{A}$ and $N \in \mathcal{B}$, and the sequence splits.

For the converse, note first that $\mathfrak{C}$ being of finite type implies $\mathcal{A} \subseteq \varinjlim \mathcal{A}^{<\omega}$ by Theorem 2.1. Next, we claim $\mathcal{A} \cap \mathcal{B} = \varinjlim \mathcal{A}^{<\omega} \cap \mathcal{B}$. Let $N \in \varinjlim \mathcal{A}^{<\omega} \cap \mathcal{B}$. Since $\mathfrak{C}$ is complete, there is a special $\mathcal{A}$-precover $\mathcal{E} : 0 \longrightarrow B \longrightarrow A \longrightarrow N \longrightarrow 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then $A \in \mathcal{A} \cap \mathcal{B}$. Moreover, since $N \in \varinjlim \mathcal{A}^{<\omega}$ is a pure-epimorphic image of an element of $\mathcal{A}$, $\mathcal{E}$ is pure exact. So by (5), $\mathcal{E}$ splits, proving our claim.

Let us now take an arbitrary module $N \in \varinjlim \mathcal{A}^{<\omega}$ and a special $\mathcal{B}$-preenvelope $0 \longrightarrow N \longrightarrow B' \longrightarrow A' \longrightarrow 0$ with $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$. Then A' and therefore also B' belong to $\varinjlim \mathcal{A}^{<\omega}$. So, by the claim above, $B' \in \mathcal{A} \cap \mathcal{B}$, which yields $N \in \mathcal{A}$ as $\mathcal{A}$ is resolving. This shows that $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$, so (2) holds. □

In particular, we infer that all cotorsion pairs of finite type with $\mathcal{A}$ closed under direct limits satisfy conjecture 0.1. In Section 4, we will prove that also the converse is true in the sense that any cotorsion pair satisfying 0.1 is necessarily of finite type.

Corollary 2.4. *Let* $(\mathcal{A}, \mathcal{B})$ *be a hereditary cotorsion pair of finite type in* Mod R. *Then the following statements are equivalent.*

(1) $\mathcal{A}$ *is definable.*
(2) $\mathcal{A}$ *is closed under direct limits, and* $\mathcal{A}^{<\omega}$ *is covariantly finite in* mod R.

Proof. By Theorem 1.3 and Corollary 2.3. □

3. Closure properties of tilting and cotilting cotorsion pairs

Before we continue our discussion of conjecture 0.1, let us apply the considerations above to tilting theory, which will be a source of interesting examples in Section 5.

Let $n < \omega$. A module T is *n-tilting* provided

(T1) $T \in \mathcal{P}_n$,

(T2) $\mathrm{Ext}_R^i(T, T^{(I)}) = 0$ for each $i \geq 1$ and all sets I, and

(T3) there exist $r \geq 0$ and a long exact sequence $0 \longrightarrow R \longrightarrow T_0 \longrightarrow \cdots \longrightarrow T_r \longrightarrow 0$ such that $T_i \in \mathrm{Add}\, T$ for each $0 \leq i \leq r$.

Every n-tilting module T induces a complete hereditary smashing cotorsion pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{B} = T^{\perp}$ and $\mathcal{A} \subseteq \mathcal{P}_n$, see [1]. Such cotorsion pairs are called *n-tilting cotorsion pairs*. By [9] (see also [18, Section 5]), tilting cotorsion pairs are always of finite type.

Dually, a module C is *n-cotilting* provided that

(C1) $C \in \mathcal{I}_n$,

(C2) $\mathrm{Ext}_R^i(C^I, C) = 0$ for each $i \geq 1$ and all sets I, and

(C3) there exist $r \geq 0$ and an exact sequence $0 \longrightarrow C_r \longrightarrow \cdots \longrightarrow C_0 \longrightarrow W \longrightarrow 0$ where W is an injective cogenerator for $\mathrm{Mod}\,R$ and $C_i \in \mathrm{Prod}\, C$ for each $0 \leq i \leq r$.

Every n-cotilting module C is pure-injective by [30], and so it induces a perfect hereditary cosmashing cotorsion pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A} = {}^{\perp}C$ and $\mathcal{B} \subseteq \mathcal{I}_n$, see [1]. Such cotorsion pairs are called *n-cotilting cotorsion pairs*. Cotilting cotorsion pairs are not always of cofinite type [7], however, the class $\mathcal{A}$ is always definable.

Finally, we recall from [21] that a module M with $\mathrm{Add}\, M$ being closed under products is said to be *product-complete*. Note that M is product-complete iff $\mathrm{Add}\, M = \mathrm{Prod}\, M$. Moreover, every product-complete module is Σ-pure-injective.

Proposition 3.1. *Let T be a tilting module with corresponding tilting cotorsion pair $(\mathcal{A}, \mathcal{B})$. Then the following statements are equivalent.*

(1) *$\mathcal{A}$ is definable.*

(2) *$(\mathcal{A}, \mathcal{B})$ is cosmashing.*

(3) *T is product-complete.*

If R has finite global dimension, then (1)–(3) *are further equivalent to*

(4) *T is a cotilting module such that $\mathcal{A} = {}^{\perp}T$.*

Proof. For the equivalence of (1)–(3), we generalize an argument from [4, 4.3], which we include for the reader's convenience. Clearly, (1) $\Rightarrow$ (2). Moreover, we know from [1, 2.4] that $\mathrm{Add}\, T = \mathcal{A} \cap \mathcal{B}$, and that $\mathcal{A}$ consists of the modules A having a long exact sequence $0 \to A \to T_0 \to \cdots \to T_n \to 0$ with $T_0, \ldots, T_n \in \mathrm{Add}\, T$. We then deduce that $\mathcal{A}$ is closed under direct products iff so is $\mathrm{Add}\, T$, which means that (2) and (3) are equivalent. Moreover, under the assumption (3), the module T is Σ-pure-injective. Then every pure submodule of a module $M \in \mathcal{A} \cap \mathcal{B}$ is a direct summand of M, and thus $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$ by Corollary 2.3. So (3) $\Rightarrow$ (1) holds by Theorem 1.3.

Assume now (4). Then we have from [1, 2.4] that $\mathrm{Add}\, T = \mathcal{A} \cap \mathcal{B} = \mathrm{Prod}\, T$, so (3) holds true.

Conversely, if (3) holds true and $\mathrm{gldim}\, R < \infty$, then T satisfies conditions (C1) and (C2) in the definition of a cotilting module. Moreover, if W is an injective cogenerator for $\mathrm{Mod}\,R$, then $W \in \mathcal{B}$, and since W has finite projective dimension, we deduce from [1, 2.4] that there is a long exact sequence $0 \to T_m \to \cdots \to T_0 \to W \to 0$ with $T_0, \ldots, T_m \in \mathrm{Add}\, T = \mathrm{Prod}\, T$. So, also (C3) is satisfied, and T is a cotilting module. Furthermore, $T \in \mathcal{B}$ implies that $\mathcal{A} \subseteq {}^{\perp}T$.

It remains to prove that every module $X \in {}^{\perp}T$ belongs to $\mathcal{A}$. To this end, we consider a special $\mathcal{B}$-preenvelope $0 \longrightarrow X \longrightarrow B \longrightarrow A \longrightarrow 0$. Then since $A \in \mathcal{A}$ belongs to ${}^{\perp}T$, we have $B \in \mathcal{B} \cap {}^{\perp}T$. As above, we consider a long exact sequence $0 \to T_m \xrightarrow{f} T_{m-1} \to \cdots \to T_0 \to B \to 0$ with $T_0, \ldots, T_m \in \mathrm{Add}\, T = \mathrm{Prod}\, T$, and we choose it to be of minimal length m. Assume $m > 0$. Since $B, T_0, \ldots, T_m$ all belong to the resolving subcategory ${}^{\perp}T$, it follows that $\mathrm{Coker}\, f$ also belongs to ${}^{\perp}T$. But then $\mathrm{Ext}_R^1(\mathrm{Coker}\, f, T_m) = 0$, so $\mathrm{Coker}\, f$ even belongs to $\mathrm{Add}\, T$, contradicting the minimality of m. We conclude that $m = 0$, that is, that B belongs to $\mathrm{Add}\, T \subseteq \mathcal{A}$. Since $\mathcal{A}$ is resolving, this completes the proof. $\square$

Dually, one obtains the following result for cotilting cotorsion pairs, see also [12, 3.4].

Proposition 3.2. *Let C be a cotilting module with corresponding cotilting cotorsion pair* $(\mathcal{A}, \mathcal{B})$. *Then the following statements are equivalent.*

(1) $(\mathcal{A}, \mathcal{B})$ *is smashing.*
(2) C *is* Σ-*pure-injective.*
(3) *There is a product-complete cotilting module* C' *such that* $\mathcal{A} = {}^{\perp}C'$.

If R has finite global dimension, then (1)–(3) *are further equivalent to*

(4) *There is a tilting (and cotilting) module* C' *such that* $\mathcal{B} = C'^{\perp}$.

Proof. With arguments dual to those used in 3.1, we see that $\mathcal{B}$ is closed under direct sums iff so is Prod C. Since C is pure-injective, the latter implies that the module C is Σ-pure-injective. So, we have (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1). Moreover, (4) $\Rightarrow$ (1) because tilting cotorsion pairs are smashing, and if gldim $R < \infty$, then the module C' in (3) satisfies (4) dually to Proposition 3.1.

It remains to prove (2) $\Rightarrow$ (3): Assume that C is Σ-pure-injective. By [24, 8.1], there is a cardinal κ such that every product of copies of C is a direct sum of modules of cardinality at most κ. Of course, the isomorphism classes of all κ-generated modules lying in Prod C form a set $\mathcal{K}$. Let C' be the direct sum of all modules in $\mathcal{K}$, and P the direct product of all modules in $\mathcal{K}$. We then have Prod $C \subseteq$ Add C'. Moreover, $P \in$ Prod C is Σ-pure-injective. Hence the pure submodule C' of P is a direct summand of P. This proves Prod $C' \subseteq$ Prod C, and further, by Σ-pure-injectivity, Add $C' \subseteq$ Prod C'. We then conclude that Add $C' =$ Prod $C' =$ Prod C, so C' is a product-complete cotilting module such that $\mathcal{A} = {}^{\perp}C'$. $\square$

Corollary 3.3. *Let R be right noetherian and hereditary, and let* $(\mathcal{A}, \mathcal{B})$ *be a cotorsion pair. The following statements are equivalent.*

(1) $\mathcal{A}$ *and* $\mathcal{B}$ *are definable.*
(2) $\mathcal{A}$ *and* $\mathcal{B}$ *are closed under direct limits.*
(3) $(\mathcal{A}, \mathcal{B})$ *is smashing and cosmashing.*
(4) *There is a product-complete tilting module* M *such that* $\mathcal{B} =$ Gen M.
(4′) *There is a product-complete cotilting module* M *such that* $\mathcal{A} =$ Cogen M.
(5) *There is a tilting and cotilting module* M *such that* $\mathcal{A} =$ Cogen M *and* $\mathcal{B} =$ Gen M.

Proof. Clearly, (1) implies (2). Moreover, by [1, 4.1 and 4.2] we know that $(\mathcal{A}, \mathcal{B})$ is (co)smashing iff it is a (co)tilting cotorsion pair. So, (2) implies that $(\mathcal{A}, \mathcal{B})$ is smashing, and therefore tilting, thus a hereditary cotorsion pair of finite type. Further, $\mathcal{A}^{<\omega}$ is a resolving subcategory of $\mathcal{P}_1^{<\omega}$. As R is right noetherian, it follows from [2, 2.5] that $\mathcal{A}^{<\omega}$ is covariantly finite in mod R. Since $\mathcal{A}$ is closed under direct limits, we then conclude from 2.4 that $\mathcal{A}$ is definable. In particular, $(\mathcal{A}, \mathcal{B})$ is cosmashing, so we have shown (2) $\Rightarrow$ (3). The implication (5)$\Rightarrow$(1) follows from the fact that (co)tilting classes are always definable. The remaining implications hold by Propositions 3.1 and 3.2. $\square$

Example 3.4. Let Λ be a tame hereditary artin algebra (w.l.o.g. basic indecomposable). Reiten and Ringel have shown in [27] that there is a cotorsion pair $(\mathcal{C}, \mathcal{D})$ in ModΛ which is generated by the class **q** of all indecomposable preinjective modules and is cogenerated by the class **t** of all indecomposable regular modules. In other words, $(\mathcal{C}, \mathcal{D})$ is a hereditary cotorsion pair of finite and cofinite type. In particular, $\mathcal{C}$ and $\mathcal{D}$ are definable. Now let $S_\lambda, \lambda \in \mathbb{P}$, be a complete irredundant set of quasi-simple modules and let $S_\lambda[\infty], \lambda \in \mathbb{P}$, be the corresponding Prüfer modules. Let further G be the generic module. Then $W = \bigoplus_{\lambda \in \mathbb{P}} S_\lambda[\infty] \oplus G$ is a tilting and cotilting module such that $\mathcal{C} =$ Cogen W and $\mathcal{D} =$ Gen W; for details see [27].

Remark 3.5. The additional hypothesis in Propositions 3.1 and 3.2 is necessary. In fact, if there is an n-tilting–cotilting cotorsion pair $(\mathcal{A}, \mathcal{B})$, then every module M has a long exact sequence $0 \to A_m \xrightarrow{f} \cdots \to A_0 \to M \to 0$ where $A_0, \ldots, A_m \in \mathcal{A}$ and $m \leq n$, and moreover, $\mathcal{A} \subseteq \mathcal{P}_n$, see [1]. But then gldim $R \leq 2n$.

4. The telescope conjecture for module categories

In this section, we deal in detail with conjecture 0.1. We have seen in Corollary 2.3 that 0.1 holds for any cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ of finite type such that $\mathcal{A}$ is closed under direct limits. Our first main result shows that the converse is also true, that is, the cotorsion pairs satisfying 0.1 must be of finite type. We start with some preliminary results.

Proposition 4.1. *Let R be a ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair cogenerated by a class $\mathcal{C}$ of countably presented modules. Assume that $B^{(\omega)} \in \mathcal{B}$ whenever $B \in \mathcal{B}$. Then*

(1) $\mathfrak{C}$ *is smashing, and $\mathcal{B}$ is closed under pure submodules.*
(2) *If $\mathfrak{C}$ is hereditary then $\mathcal{B}$ is definable.*
(3) *If $\mathcal{C} \subseteq \varinjlim \mathcal{A}^{<\omega}$ then $\mathfrak{C}$ is of finite type.*

Proof. (1) First, our assumption on the class $\mathcal{B}$ implies that $\mathcal{B}$ is closed under pure submodules by [8, Theorem 2.5] (see also [18, 5.2.16]). Since $\mathcal{B}$ is closed under arbitrary direct products, and direct sums are pure submodules in direct products, we infer that $\mathfrak{C}$ is smashing.

(2) Since $\mathcal{B}$ is coresolving, (1) also implies that $\mathcal{B}$ is closed under pure-epimorphic images, thus in particular under direct limits. This shows that $\mathcal{B}$ is definable.

(3) It suffices to verify that $\mathcal{B} = (\mathcal{A}^{<\omega})^{\perp_1}$. Clearly $\mathcal{B} \subseteq (\mathcal{A}^{<\omega})^{\perp_1}$. For the reverse inclusion, we first show that the classes $\mathcal{B}$ and $(\mathcal{A}^{<\omega})^{\perp_1}$ contain the same pure-injective modules. Indeed, for any pure-injective module I, the functor $\operatorname{Ext}^1_R(-, I)$ takes direct limits into inverse limits by [5]. So, the assumption $\mathcal{C} \subseteq \varinjlim \mathcal{A}^{<\omega}$ implies that any pure-injective module $I \in (\mathcal{A}^{<\omega})^{\perp_1}$ belongs to $\mathcal{B}$. Now, let $M \in (\mathcal{A}^{<\omega})^{\perp_1}$, and let P be the pure-injective envelope of M. Since the class $(\mathcal{A}^{<\omega})^{\perp_1}$ is definable, $P \in (\mathcal{A}^{<\omega})^{\perp_1}$. But then $P \in \mathcal{B}$, and thus $M \in \mathcal{B}$ since M is a pure submodule of P. This proves that $\mathcal{B} = (\mathcal{A}^{<\omega})^{\perp_1}$. $\square$

Remark 4.2. Let R be a right $\aleph_0$-noetherian ring. Then for each $n < \omega$, the cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^\perp)$ is cogenerated by a class of countably presented modules (see e.g. [17, Section 7.4]).

Denote by $(\mathcal{A}, \mathcal{B})$ the cotorsion pair (of finite type) cogenerated by $\mathcal{P}^{<\omega}$. Let $\operatorname{fdim}(R)$ and $\operatorname{Fdim}(R)$ denote the little and the big finitistic dimensions of R, respectively.

Clearly, $\mathfrak{C}_F = (\mathcal{P}, \mathcal{P}^\perp)$ is a cotorsion pair iff $\operatorname{Fdim}(R) < \infty$. Like in [3], we infer from Proposition 4.1 and [9, Theorem 4.2] (see also [18, 5.2.20]) that $\mathcal{A} = \mathcal{P}$ iff $\mathfrak{C}_F$ is a tilting cotorsion pair iff $\mathfrak{C}_F$ is a cotorsion pair of finite type iff $\operatorname{Fdim}(R) < \infty$ and $B^{(\omega)} \in \mathcal{P}^\perp$ whenever $B \in \mathcal{P}^\perp$.

By Theorem 1.7, the condition $\mathcal{A} = \mathcal{P}$ is also equivalent to (i) $\operatorname{Fdim}(R) < \infty$ and (ii) each module of finite projective dimension is a direct summand in a $\mathcal{P}^{<\omega}$-filtered module, see [3, 3.2].

Of course, (ii) implies $\operatorname{fdim}(R) = \operatorname{Fdim}(R)$ (but the converse fails, even when $\operatorname{fdim}(R) = \operatorname{Fdim}(R) = 1$, for the IST-algebra R from [23], see [4]).

Note that this is the way the equality $\operatorname{fdim}(R) = \operatorname{Fdim}(R)$ was proved for artin algebras with $\mathcal{P}^{<\omega}$ contravariantly finite in [3], and for all Iwanaga–Gorenstein rings in [2].

In view of Proposition 4.1, our strategy will consist in proving that every cotorsion pair $(\mathcal{A}, \mathcal{B})$ satisfying 0.1 is cogenerated by the class of countably presented modules from $\mathcal{A}$. To this end, we need results which enable us to filter modules from $\mathcal{A}$ by "smaller" modules which still belong to $\mathcal{A}$. The following two lemmas are the first step in this direction.

Lemma 4.3. *Let C be an injective cogenerator in $\operatorname{Mod} R$. Define $F(X) = C^{\operatorname{Hom}_R(X,C)}$ and $F(\varphi)(f) = f(- \circ \varphi)$, for all $X, Y \in \operatorname{Mod} R$, every $\varphi \in \operatorname{Hom}_R(X, Y)$ and $f \in F(X)$. Then F is an endofunctor of $\operatorname{Mod} R$ preserving monomorphisms. Moreover, the family $\iota = (\iota_X \mid X \in \operatorname{Mod} R)$ consisting of canonical embeddings $\iota_X : X \to F(X)$ is a natural transformation from the identity functor to F.*

Proof. It is straightforward to check that F is a functor and ι is a natural transformation. ι_X is an embedding since C is a cogenerator, and the injectivity of C implies that F preserves monomorphisms. $\square$

Lemma 4.4. *Let R be an arbitrary ring, and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathcal{B}$ is closed under direct limits. Let λ be a regular uncountable cardinal, $\kappa \geq \lambda$, $A \in \mathcal{A}$ a κ-presented module, and X a subset of A with $\operatorname{card} X < \lambda$. Then there exists a $<\lambda$-presented module $\bar{X}$ such that $X \subseteq \bar{X} \subseteq A$. Moreover, $\bar{X}$ can be taken to be of the form $\pi(R^{(I)})$ where $\pi : R^{(\kappa)} \to A$ is an epimorphism and I is a subset of κ of cardinality $<\lambda$.*

Proof. By assumption, A has a presentation

$$0 \longrightarrow K \xrightarrow{\subseteq} R^{(\kappa)} \xrightarrow{\pi} A \longrightarrow 0$$

with gen(K) $\leq \kappa$, and there is $I_0 \subseteq \kappa$ of cardinality $<\lambda$ such that $X \subseteq \pi(R^{(I_0)})$. Let $\mathcal{L}$ be the set consisting of all $<\lambda$-generated submodules of K. We claim that $K \cap R^{(I_0)} \subseteq L_0$ for some $L_0 \in \mathcal{L}$.

Let $\mathcal{D} = \{\langle L', L\rangle \in \mathcal{L} \times \mathcal{L} \mid L \nsubseteq L'\}$. Using the notation from Lemma 4.3, for each $\langle L', L\rangle \in \mathcal{D}$, we define $\tau_{\langle L', L\rangle} : L \to F((L + L')/L')$ as the composition of the canonical projection $L \to (L + L')/L'$ with the embedding $\iota_{(L+L')/L'}$. For $L \in \mathcal{L}$, put $\mathcal{L}_L = \{L' \in \mathcal{L} \mid \langle L', L\rangle \in \mathcal{D}\}$. Note that for every $L, \tilde{L} \in \mathcal{L}$, $L \subseteq \tilde{L}$ implies $\mathcal{L}_L \subseteq \mathcal{L}_{\tilde{L}}$.

Now for each $L \in \mathcal{L}$, we put

$$G(L) = \prod_{L' \in \mathcal{L}_L} F((L + L')/L'),$$

notice that $G(L) \in \mathcal{I}_0$, and for every $\varepsilon : L \subseteq \tilde{L}$ ($\in \mathcal{L}$), we define

$$G(\varepsilon) = \prod_{L' \in \mathcal{L}_L} F(\varepsilon_{L'})$$

where $\varepsilon_{L'}$ is the inclusion $(L + L')/L' \subseteq (\tilde{L} + L')/L'$. Then G is a functor from the small category $\mathcal{L}$, morphisms of which are just inclusions, to Mod R. Moreover, G preserves monomorphisms (since F does), and there is the natural transformation $\tau = (\tau_L \mid L \in \mathcal{L})$ from the canonical embedding $\mathcal{L} \hookrightarrow$ Mod R to G where τ_L is a fibred product of $(\tau_{\langle L', L\rangle} \mid L' \in \mathcal{L}_L)$: it is routine to check that the square

$$
\begin{array}{ccc}
\tilde{L} & \xrightarrow{\tau_{\tilde{L}}} & G(\tilde{L}) \\
\varepsilon \uparrow & & G(\varepsilon) \uparrow \\
L & \xrightarrow{\tau_L} & G(L)
\end{array}
$$

commutes for each $L, \tilde{L} \in \mathcal{L}$ and $\varepsilon : L \subseteq \tilde{L}$ (one needs the fact that ι is a natural transformation).

Let E be a direct limit of the directed system $G(\mathcal{L})$. For every $L \in \mathcal{L}$, denote by ν_L the colimit injection $G(L) \hookrightarrow E$. Since K is a directed union of its $<\lambda$-generated submodules, it follows from the preceding paragraph that there exists the unique homomorphism $f : K \to E$ such that $f \upharpoonright L = \nu_L \tau_L$ for all $L \in \mathcal{L}$. Note that $\mathcal{L}$ is λ-directed since λ is a regular cardinal, so $G(\mathcal{L})$ has the same property.

Using the assumption put on $\mathcal{B}$, we have $E \in \mathcal{B}$, which allows us to extend f to some $g : R^{(\kappa)} \to E$. Since card $I_0 < \lambda$ and $G(\mathcal{L})$ is λ-directed, there exists $L_0 \in \mathcal{L}$ such that $g \upharpoonright R^{(I_0)}$ factorizes through ν_{L_0}. We deduce then that $K \cap R^{(I_0)} \subseteq L_0$; if not, there exist $x \in K \cap R^{(I_0)}$ and $L \in \mathcal{L}$ such that $x \in L \setminus L_0$, whence $\tau_{\langle L_0, L+L_0\rangle}(x) \neq 0 \neq \tau_{L+L_0}(x)$ contradicting $f \upharpoonright (K \cap R^{(I_0)})$ being factorized through ν_{L_0}. Our claim is proved.

Since L_0 is a $<\lambda$-generated module, $L_0 \subseteq R^{(I_1)}$ for some $I_0 \subseteq I_1 \subseteq \kappa$ with card $I_1 < \lambda$. Iterating this construction, we obtain a set $I = \bigcup_{n<\omega} I_n$ such that $K \cap R^{(I)} = L$ for some $L \in \mathcal{L}$, and $\bar{X} = \pi(R^{(I)}) \cong R^{(I)}/L$ has the desired properties. $\square$

Lemma 4.5. *Let* $(\mathcal{A}, \mathcal{B})$ *be a cotorsion pair such that* $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$. *Let* λ *be a regular uncountable cardinal,* $\kappa \geq \lambda$, $A \in \mathcal{A}$ *a* κ-*presented module, and* X *be a subset of* A *of cardinality* $<\lambda$. *Assume that either* (i) R *is a right* $\aleph_0$-*noetherian ring, or* (ii) $\mathcal{B}$ *is closed under direct limits. Then there is a* $<\lambda$-*presented module* $A' \in \mathcal{A}$ *such that* $X \subseteq A' \subseteq A$.

Proof. *Step* 1: For any $<\lambda$-presented submodule B of A, we construct a $<\lambda$-generated submodule B' of A containing B with the property that any homomorphism of the form $D \xrightarrow{h} B \subseteq B'$ with D finitely presented factors through a module in $\mathcal{A}^{<\omega}$.

To this end, we fix a pure-exact sequence $0 \longrightarrow \text{Ker } \pi \longrightarrow \bigoplus_{i \in I} D_i \xrightarrow{\pi} B \longrightarrow 0$ with D_i finitely presented for all $i \in I$. Since B is $<\lambda$-presented, we will w.l.o.g. assume that I has cardinality $<\lambda$. For F a non-empty finite subset of I, let $D_F = \bigoplus_{i \in F} D_i$, and $\pi_F = \pi \upharpoonright D_F$. By induction on card($F$), we define finitely generated modules $A_F \in \mathcal{A}^{<\omega}$ and $C_F \subseteq A$ such that there is a commutative diagram

$$
\begin{array}{ccc}
D_F & \xrightarrow{\pi_F} & B \\
f_F \downarrow & & \subseteq \downarrow \\
A_F & \xrightarrow{g_F} & A
\end{array}
$$

and $\pi(D_F) \subseteq C_F = \operatorname{Im} g_F$. Hereby we proceed as follows:

If card$(F) = 1$, then the existence of A_F and C_F follows immediately from Lemma 1.1 since $A \in \varinjlim \mathcal{A}^{<\omega}$. If card$(F) > 1$, we take $M = D_F \oplus \bigoplus_{\emptyset \neq G \subsetneq F} A_G$ and let $g = \pi_F \oplus \bigoplus_{\emptyset \neq G \subsetneq F} g_G$. By Lemma 1.1, there exist $A_F \in \mathcal{A}^{<\omega}, h_F : M \to A_F$ and $g_F : A_F \to A$ such that $g = g_F h_F$, and we put $C_F = \operatorname{Im} g_F$ and $f_F = h_F \upharpoonright D_F$. Note that C_F contains C_G for each $\emptyset \neq G \subsetneq F$.

Now let B' be the union of all C_F where F runs through all non-empty finite subsets of I. This is a directed union of $<\lambda$-many finitely generated submodules of A, so B' is a $<\lambda$-generated submodule of A containing B. Moreover, if $h : D \to B$ is a homomorphism with D finitely presented, then there is a factorization f of h through the pure epimorphism π. But then $\operatorname{Im} f \subseteq D_F$ for a non-empty finite subset $F \subseteq I$, and $D \xrightarrow{h} B \subseteq B'$, which equals $g_F f_F f$, factors through $A_F \in \mathcal{A}^{<\omega}$ as required.

Step 2: Consider now the presentation of A from Lemma 4.4. We will define A' as the union of an increasing chain $(B_n \mid n < \omega)$ of $<\lambda$-presented submodules in A of the form $\pi(R^{(J_n)})$ for some J_n of cardinality $<\lambda$ (where $J_0 \subseteq J_1 \subseteq \cdots$). The chain will be defined by induction on n:

Take $B_0 = \pi(R^{(J_0)}) < \lambda$-presented and such that $X \subseteq B_0$ (this is clearly possible in case (i), and it is possible by Lemma 4.4 in case (ii)). If B_n is defined, there is a $<\lambda$-generated submodule B'_n of A containing B_n constructed as in Step 1. Let $B_{n+1} = \pi(R^{(J_{n+1})})$ be a $<\lambda$-presented submodule of A containing B'_n (again, obtained using the $\aleph_0$-noetherian property of R in case (i), and Lemma 4.4 in case (ii)).

It remains to prove that $A' \in \mathcal{A}$. By Lemma 1.1, it suffices to show that every R-homomorphism $h : D \to A'$ with D finitely presented has a factorization through a module in $\mathcal{A}^{<\omega}$. However, $\operatorname{Im} h \subseteq B_n$ for some $n < \omega$, and the claim then follows by construction of B'_n in Step 1. $\square$

Theorem 4.6. *Let R be a ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a smashing cotorsion pair such that $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$. Assume that either* (i) *R is a right $\aleph_0$-noetherian ring, or* (ii) *$\mathcal{B}$ is closed under direct limits. Then $\mathfrak{C}$ is of finite type.*

Proof. We denote by $\mathcal{A}_0$ the class of all countably presented modules in $\mathcal{A}$. Let $A \in \mathcal{A}$, and let $\kappa \geq \aleph_0$ be such that A is a κ-presented module. By induction on κ, we will prove that A is $\mathcal{A}_0$-filtered. There is nothing to prove for $\kappa = \aleph_0$.

If κ is a regular uncountable cardinal then Lemma 4.5 yields a κ-filtration, $\mathcal{F} = (A_\alpha \mid \alpha < \kappa)$, of A such that $A_\alpha \in \mathcal{A}$ is $<\kappa$-presented for each $\alpha < \kappa$. By [31, Theorem 8] (see also [18, 4.3.2]), there is a subfiltration, $\mathcal{G}$, of $\mathcal{F}$ such that all successive factors in $\mathcal{G}$ are $<\kappa$-presented modules from $\mathcal{A}$, so they are $\mathcal{A}_0$-filtered by inductive premise. Hence A is $\mathcal{A}_0$-filtered.

If κ is singular, we use Shelah's Singular Compactness Theorem [16, IV.3.7] as follows: first, call a module M "free" if M is $\mathcal{A}_0$-filtered. For each regular uncountable cardinal $\lambda < \kappa$, we let S_λ denote the set of all $<\lambda$-presented submodules $A' \subseteq A$ with $A' \in \mathcal{A}$. Clearly, $0 \in S_\lambda$, and S_λ is closed under unions of well-ordered chains of length $<\lambda$ since $\mathcal{A}$ is closed under arbitrary direct limits. By Lemma 4.5, each subset of A of cardinality $<\lambda$ is contained in an element of S_λ. By inductive premise, S_λ consists of "free" modules for all regular $\omega < \lambda < \kappa$, so A is "free" by [16, IV.3.7]. This proves that each $A \in \mathcal{A}$ is $\mathcal{A}_0$-filtered.

So, we infer from the Eklof Lemma [16, XII.1.5] (see also [18, 3.1.2]) that $\mathcal{B} = (\mathcal{A}_0)^{\perp_1}$. Finally, Proposition 4.1 shows that $\mathfrak{C}$ is of finite type. $\square$

Corollary 4.7. *Let R be an arbitrary ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair such that $\mathcal{A}$ and $\mathcal{B}$ are closed under direct limits. Then $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$ if and only if $\mathfrak{C}$ is of finite type.*

Now, we are going to prove a particular case of conjecture 0.1 for arbitrary right noetherian rings. This will imply the validity of 0.1 in the particular case that $\mathfrak{C}$ is a cotilting cotorsion pair over a right noetherian ring.

By Theorem 4.6, the proof of conjecture 0.1 amounts to showing that $\mathfrak{C}$ is of finite type. First, we need a lemma which is implicit already in [7], and the dual version of which appears in [28]:

Lemma 4.8. *Let R be a ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a smashing cotorsion pair cogenerated by a class $\mathcal{C}$ such that $\mathcal{C}^\perp$ contains all direct sums of injective modules. Then $\mathcal{C}^{\perp_n}$ is closed under arbitrary direct sums for each $n \geq 1$.*

Proof. By induction on n. The case of $n = 1$ is clear since $\mathfrak{C}$ is smashing. Let $(M_\alpha \mid \alpha < \kappa)$ be a family of modules in $\mathcal{C}^{\perp_{n+1}}$. Consider short exact sequences

$$0 \longrightarrow M_\alpha \longrightarrow I_\alpha \longrightarrow C_\alpha \longrightarrow 0$$

with I_α injective for each $\alpha < \kappa$. Since $0 = \mathrm{Ext}_R^{n+1}(A, M_\alpha) \cong \mathrm{Ext}_R^n(A, C_\alpha)$ for all $A \in \mathcal{C}$, the inductive premise gives $\bigoplus_{\alpha < \kappa} C_\alpha \in \mathcal{C}^{\perp_n}$, so our assumption on $\mathcal{C}^\perp$ yields $\bigoplus_{\alpha < \kappa} M_\alpha \in \mathcal{C}^{\perp_{n+1}}$. $\square$

Proposition 4.9. *Let R be a right coherent ring. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a smashing hereditary cotorsion pair cogenerated by a class $\mathcal{C} \subseteq (\mathrm{Mod}\,R)^{\le \omega}$ and such that $\mathcal{B} \subseteq \mathcal{I}_n$ for some $n \ge 0$. Then $\mathfrak{C}$ is of finite type.*

Proof. We will construct cotorsion pairs $\mathfrak{C}_i = (\mathcal{A}_i, \mathcal{B}_i)$, $1 \le i \le n+1$, such that

$$\mathcal{B} = \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \cdots \subseteq \mathcal{B}_n \subseteq \mathcal{B}_{n+1}$$

and by reverse induction on i, we will show that $\mathfrak{C}_i$ is of finite type for each $1 \le i \le n+1$.

Let us start with the cotorsion pair $\mathfrak{C}_{n+1} = (\mathcal{A}_{n+1}, \mathcal{B}_{n+1})$ cogenerated by the class $\mathcal{S}_{n+1}$ of all modules that are kth syzygies of modules from $\mathrm{mod}\,R$ for some $k \ge n$. Then $\mathcal{B}_{n+1} = \bigcap_{k \ge n+1}(\mathrm{mod}\,R)^{\perp_k}$, and we claim that $\mathcal{B}_{n+1} \subseteq \bigcap_{k \ge n+1} \mathcal{A}^{\perp_k}$. In fact, $(\mathrm{mod}\,R)^{\perp_1}$ coincides with the class of all pure submodules of injective modules since R is right coherent. Moreover, $\mathrm{mod}\,R$ is resolving, so $(\mathrm{mod}\,R)^{\perp_1} = (\mathrm{mod}\,R)^\perp$ by Remark 1.2. Since $\mathcal{B}$ is definable by Proposition 4.1(2), we deduce that $(\mathrm{mod}\,R)^\perp \subseteq \mathcal{B}$, and our claim follows by dimension shifting.

We now set $\mathcal{B}_i = \mathcal{B}_{n+1} \cap \bigcap_{k \ge i} \mathcal{A}^{\perp_k}$ for $1 \le i \le n$. Then, as $\mathcal{B} \subseteq \mathcal{I}_n \subseteq \mathcal{B}_{n+1}$, we have $\mathcal{B} = \mathcal{B}_1$. Moreover, all $\mathcal{B}_i$ are obviously coresolving. Further, applying Lemma 4.8 to $\mathfrak{C}$ (which is possible because $\mathfrak{C}$ is cogenerated by $\mathcal{A}$ and $\mathcal{A}^\perp = \mathcal{B}$ contains all direct sums of injective modules), we infer that all $\mathcal{B}_i$ are closed under direct sums.

For each $1 \le i \le n$, we thus obtain a hereditary smashing cotorsion pair $\mathfrak{C}_i = (\mathcal{A}_i, \mathcal{B}_i)$ which is cogenerated by a class of countably presented modules, namely by $\mathcal{S}_i = \mathcal{S}_{n+1} \cup \mathcal{C}_i$, where $\mathcal{C}_i$ denotes the class of all modules that are kth syzygies of modules from $\mathcal{C}$ for some $k \ge i - 1$.

Of course, $\mathfrak{C}_{n+1}$ is of finite type. Let $1 \le i \le n$, and let $M \in \mathcal{S}_i$. We have a short exact sequence

$$0 \longrightarrow K \longrightarrow R^{(\omega)} \longrightarrow M \longrightarrow 0.$$

We claim that $K \in \mathcal{A}_{i+1}$. Indeed, if $N \in \mathcal{B}_{i+1} = \mathcal{B}_{n+1} \cap \bigcap_{k \ge i+1} \mathcal{A}^{\perp_k}$ then its first cosyzygy C belongs to $\mathcal{B}_i$, so $\mathrm{Ext}_R^2(\mathcal{A}_i, N) = 0$, and in particular, $\mathrm{Ext}_R^2(M, N) = 0$, hence $\mathrm{Ext}_R^1(K, N) = 0$. This proves the claim.

By inductive premise, $\mathfrak{C}_{i+1}$ is of finite type, hence cogenerated by $\mathcal{A}_{i+1}^{<\omega}$. By Theorem 1.7, it follows that K is a direct summand in a $\mathcal{A}_{i+1}^{<\omega}$-filtered module. Using [9, Lemma 3.3] (see also [18, 5.2.20, p. 215]), we obtain the exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow M \longrightarrow 0$$

with H and G countably generated $\mathcal{A}_{i+1}^{<\omega}$-filtered modules. W.l.o.g. we can assume that H is a submodule of G. As in the proof of [9, Lemma 3.6], we show that $M \in \varinjlim \mathcal{A}_i^{<\omega}$. We state here the argument for the reader's convenience.

By [9, Corollary 3.2] (see also [18, 4.2.6]), we can write $H = \bigcup_{k < \omega} H_k$ and $G = \bigcup_{k < \omega} G_k$ where, for every $k < \omega$, H_k and G_k are finitely presented $\mathcal{A}_{i+1}^{<\omega}$-filtered modules, and H/H_k, G/G_k are $\mathcal{A}_{i+1}^{<\omega}$-filtered. Given $k < \omega$, there is j_k such that $H_k \subseteq G_{j_k}$. Moreover, we can choose the sequence $(j_k \mid k < \omega)$ to be strictly increasing.

We claim that $G_{j_k}/H_k \in \mathcal{A}_i^{<\omega}$. Clearly, G_{j_k}/H_k being finitely presented over R right coherent implies $G_{j_k}/H_k \in \mathrm{mod}\,R$, thus we have to show that $\mathrm{Ext}_R^1(G_{j_k}/H_k, B) = 0$ for each $B \in \mathcal{B}_i$. Since $G_{j_k} \in \mathcal{A}_{i+1} \subseteq \mathcal{A}_i$, we need only to check that every $f \in \mathrm{Hom}_R(H_k, B)$ can be extended to a homomorphism from G_{j_k} to B. We have $\mathrm{Ext}_R^1(H/H_k, B) = 0$ because $H/H_k \in \mathcal{A}_{i+1}$, thus we may extend f to a homomorphism f' from H to B, and then, since $G/H \cong M \in \mathcal{A}_i$, to a homomorphism g from G to B. The restriction of g to G_{j_k} obviously induces an extension of f to G_{j_k}. Our claim is proved.

Set $C_k = G_{j_k}/H_k$. Since $(j_k \mid k < \omega)$ is increasing and unbounded in ω, the inclusions $G_{j_k} \subseteq G_{j_{k+1}}$ induce maps $f_k : C_k \to C_{k+1}$, and M is a direct limit of the direct system $((C_k, f_k) \mid k < \omega)$.

But then, since $M \in \mathcal{S}_i$ was arbitrary, it follows that $\mathcal{S}_i \subseteq \varinjlim \mathcal{A}_i^{<\omega}$, and so $\mathfrak{C}_i$ is of finite type by Proposition 4.1(3). $\square$

Theorem 4.10. *Let R be a right noetherian ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary smashing cotorsion pair. If either*

(i) *$\mathcal{A}$ consists of modules of bounded projective dimension, or*

(ii) $\mathcal{B}$ *consists of modules of bounded injective dimension,*
   then $\mathfrak{C}$ *is of finite type.*[1]

Proof. By [31], (i) implies that $\mathfrak{C}$ is a tilting cotorsion pair, hence $\mathfrak{C}$ is of finite type by [9] (Indeed, this holds for an arbitrary ring R, cf. [18, 5.1.16 and 5.2.20].)

Assume (ii). Then it follows from [28, Corollary 1.10] that $\mathfrak{C}$ is cogenerated by a class of countably presented modules, so it is of finite type by Proposition 4.9. □

Corollary 4.11. *Let R be a right noetherian ring, and $(\mathcal{A}, \mathcal{B})$ an n-cotilting cotorsion pair. Then the following statements are equivalent.*

(1) $(\mathcal{A}, \mathcal{B})$ *is of finite type.*
(2) $\mathcal{B}$ *is definable.*
(3) *There is a Σ-pure-injective cotilting module C such that $\mathcal{A} = {}^{\perp}C$.*

Proof. By Proposition 3.2, condition (3) means that $(\mathcal{A}, \mathcal{B})$ is smashing. So, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). (3) $\Rightarrow$ (1) is an immediate consequence of Theorem 4.10. □

5. Extensions of small cotorsion pairs

We close the paper by pointing out some asymmetries that can occur in the behaviour of the classes involved in a cotorsion pair. Throughout this section Λ denotes an artin algebra.

Definition (*[29]*). Let $(\mathcal{S}, \mathcal{T})$ be a cotorsion pair in mod Λ. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in ModΛ is said to be an *extension* of $(\mathcal{S}, \mathcal{T})$ if $\mathcal{X}^{<\omega} = \mathcal{S}$ and $\mathcal{Y}^{<\omega} = \mathcal{T}$.

We have seen above three different ways of extending $(\mathcal{S}, \mathcal{T})$.

Proposition 5.1. *Let $(\mathcal{S}, \mathcal{T})$ be a cotorsion pair in* mod Λ. *The following cotorsion pairs are extensions of $(\mathcal{S}, \mathcal{T})$:*

(1) *the complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ cogenerated by $\mathcal{S}$,*
(2) *the perfect cotorsion pair $(\varinjlim \mathcal{S}, \mathcal{Y})$,*
(3) *the perfect cotorsion pair $(\mathcal{C}, \mathcal{D})$ generated by $\mathcal{T}$.*

They are related by the inclusions $\mathcal{A} \subseteq \varinjlim \mathcal{S} \subseteq \mathcal{C}$ and $\mathcal{D} \subseteq \mathcal{Y} \subseteq \mathcal{B}$.

Proof. We already know that the first cotorsion pair is complete, and the third is perfect since it is of cofinite type. Observe further that $\mathcal{S}$ has properties (R1) and (R2). By Theorem 2.1 we then have that the second cotorsion pair is perfect, generated by the pure-injective modules from $\mathcal{B}$, and moreover, $\mathcal{S} = \mathcal{A}^{<\omega} = (\varinjlim \mathcal{S})^{<\omega}$. Furthermore, $\mathcal{B}^{<\omega} = (\mathcal{S}^{\perp_1})^{<\omega} = \mathcal{T}$ since $(\mathcal{S}, \mathcal{T})$ is a cotorsion pair in mod Λ, and similarly $\mathcal{C}^{<\omega} = ({}^{\perp_1}\mathcal{T})^{<\omega} = \mathcal{S}$. In particular, $\mathcal{T}$ consists of pure-injective modules from $\mathcal{B}$. We then infer that $\mathcal{A} \subseteq \varinjlim \mathcal{S} \subseteq \mathcal{C}$, and thus $\mathcal{T} \subseteq \mathcal{D} \subseteq \mathcal{Y} \subseteq \mathcal{B}$. But this implies that $\mathcal{D}^{<\omega} = \mathcal{Y}^{<\omega} = \mathcal{B}^{<\omega} = \mathcal{T}$, and the proof is complete. □

Let us look at some examples.

Example 5.2. (1) Let Λ be a tame hereditary artin algebra (w.l.o.g. basic indecomposable), and let the notation be as in Example 3.4. We set $\mathcal{S} = \text{add} (\mathbf{p} \cup \mathbf{t})$ where $\mathbf{p}$ denotes the class of all indecomposable preprojective modules, and $\mathcal{T} = \text{add} (\mathbf{q})$. Then $(\mathcal{S}, \mathcal{T})$ is a cotorsion pair in mod Λ, and the three extensions in Proposition 5.1 coincide. Note however that $(\mathcal{S}, \mathcal{T})$ is not complete in mod Λ, and that the generic module G belongs to $\mathcal{D} \setminus \varinjlim \mathcal{T}$, so $\mathcal{D} \neq \varinjlim \mathcal{D}^{<\omega}$. In particular, we see that the validity of conjecture 0.1 for $(\mathcal{A}, \mathcal{B})$ does not imply $\mathcal{B} = \varinjlim \mathcal{B}^{<\omega}$.
(2) If $(\mathcal{S}, \mathcal{T})$ is a cotorsion pair in mod Λ with $\mathcal{S} \subseteq (\mathcal{P}_1)^{<\omega}$, then the second and the third cotorsion pair in Proposition 5.1 coincide. In fact, in this case $\mathcal{B}$ is closed under epimorphic images. Moreover, since every module

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[1] *Added in proof.* The bounds on the homological dimension can be removed in the sense that if R is any ring and $\mathfrak{C}$ satisfies the assumptions of 0.1, then $\mathfrak{C}$ is of countable type, $\mathcal{A}$ is closed under pure-epimorphic images, and $\mathcal{B}$ is definable. This has been proved in a recent manuscript by the second author and J. Šťovíček, entitled "The countable telescope conjecture for module categories".

over an artin algebra is a pure submodule of a direct product of its finitely generated factor modules, it follows that the modules from $\mathcal{B}$ are pure submodules of a product of modules from $\mathcal{T}$. Now remember that $(\varinjlim \mathcal{S}, \mathcal{Y})$ is generated by the class $\mathcal{B}'$ of all pure-injective modules from $\mathcal{B}$. But $\mathcal{B}'$ consists of direct summands of products of modules from $\mathcal{T}$, and so $\varinjlim \mathcal{S} = \mathcal{C}$.

(3) The following example shows that the assumption "smashing" in Theorem 4.6 is essential. Let Λ be the algebra from [23]. We set $\mathcal{S} = (\mathcal{P}_1)^{<\omega}$, and $\mathcal{T} = (\mathcal{S}^{\perp_1})^{<\omega}$. As above we have $^{\perp_1}\mathcal{T} = \varinjlim \mathcal{S} = \mathcal{P}_1$, so $(^{\perp_1}\mathcal{T})^{<\omega} = \mathcal{S}$, and $(\mathcal{S}, \mathcal{T})$ is a cotorsion pair in mod Λ. Here again, $(\mathcal{S}, \mathcal{T})$ is not complete. Moreover, although $\mathcal{C} = \varinjlim \mathcal{C}^{<\omega}$, the cotorsion pair $(\mathcal{C}, \mathcal{D}) = (\varinjlim \mathcal{S}, \mathcal{Y})$ is not of finite type. This follows from 2.3, since we know from [4] that the first two cotorsion pairs in Proposition 5.1 do not coincide. In particular, $(\mathcal{C}, \mathcal{D})$ is not smashing (because it cannot be a tilting cotorsion pair, see [1,8], see also [18, Section 5.2]). However, it is of cofinite type, hence cosmashing.

As a consequence of a result of Krause and Solberg in [22], we can now describe when a cotorsion pair has the shape $(\varinjlim \mathcal{S}, \varinjlim \mathcal{T})$ for some cotorsion pair $(\mathcal{S}, \mathcal{T})$ in mod Λ.

Theorem 5.3. *The following statements are equivalent for a cotorsion pair* $(\mathcal{X}, \mathcal{Y})$ *in* ModΛ.

(1) $(\mathcal{X}, \mathcal{Y}) = (\varinjlim \mathcal{X}^{<\omega}, \varinjlim \mathcal{Y}^{<\omega})$.
(2) $(\mathcal{X}, \mathcal{Y})$ *is the unique extension of some complete cotorsion pair* $(\mathcal{S}, \mathcal{T})$ *in* mod Λ.
(3) $(\mathcal{X}, \mathcal{Y})$ *is of finite type and* $\mathcal{X}^{<\omega}$ *is contravariantly finite in* mod Λ.

If $(\mathcal{X}, \mathcal{Y})$ *is hereditary and* $\mathcal{X} \subseteq \mathcal{P}$, *then* (1)–(3) *are further equivalent to*

(4) $\mathcal{Y} = T^{\perp}$ *for a tilting module* $T \in$ mod Λ.

Proof. (1) $\Rightarrow$ (2): Set $\mathcal{T} = \mathcal{Y}^{<\omega}$. First of all, since $\mathcal{Y} = \varinjlim \mathcal{T}$ is closed under products, it follows from 1.3 and 1.4 that there is a complete cotorsion pair $(\mathcal{S}, \mathcal{T})$ in mod Λ. We then know from [22, 2.4] that the three extensions of $(\mathcal{S}, \mathcal{T})$ in Proposition 5.1 coincide with $(\mathcal{X}, \mathcal{Y})$. Suppose now that $(\mathcal{E}, \mathcal{F})$ is a further extension of $(\mathcal{S}, \mathcal{T})$. Then $\mathcal{S} \subseteq \mathcal{E}$, thus $\mathcal{F} \subseteq \mathcal{Y} = \varinjlim \mathcal{T}$, hence $\varinjlim \mathcal{S} \subseteq \mathcal{E}$. Further $\mathcal{T} \subseteq \mathcal{F}$, thus $\mathcal{E} \subseteq \mathcal{X} = \varinjlim \mathcal{S}$. This shows $(\mathcal{E}, \mathcal{F}) = (\mathcal{X}, \mathcal{Y})$, so there is a unique extension.

(2) $\Rightarrow$ (3): Consider the complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ cogenerated by $\mathcal{S}$. Then $\mathcal{B}^{<\omega} = (\mathcal{S}^{\perp_1})^{<\omega} = \mathcal{T}$, and by Theorem 2.1(1) we have $\mathcal{A}^{<\omega} = \mathcal{S}$. So $(\mathcal{A}, \mathcal{B})$ is an extension of $(\mathcal{S}, \mathcal{T})$ and therefore coincides with $(\mathcal{X}, \mathcal{Y})$.

(3) $\Rightarrow$ (1): $(\mathcal{X}, \mathcal{Y})$ is cogenerated by $\mathcal{S} = \mathcal{X}^{<\omega}$, and there is a complete cotorsion pair $(\mathcal{S}, \mathcal{T})$ in mod Λ. By [22, 2.4] it follows that $(\mathcal{X}, \mathcal{Y}) = (\varinjlim \mathcal{S}, \varinjlim \mathcal{T})$ and $\mathcal{T} = \mathcal{Y}^{<\omega}$.

The equivalence with (4) follows from a well-known result of Auslander and Reiten [6], see [3, 4.1]. $\square$

Acknowledgements

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