WEIL BUNDLES AS GENERALIZED JET SPACES

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Contents

Abstract. We generalize the concept of higher order velocity and we interpret a Weil bundle as the space of A-velocities for an arbitrary Weil algebra A. Using a recent identification of every product preserving bundle functor on manifolds with a Weil functor T^A, we deduce geometric results on T^A-prolongations of various geometric objects. We characterize every fiber product preserving bundle functor F on fibered manifolds in a jet-like manner and we study F-prolongations of several geometric structures.

Preface

Having in mind the methods of algebraic geometry, A. Weil introduced an infinitely near point on a smooth manifold M as an algebra homomorphism of the algebra $C^\infty(M,\mathbb{R})$ of smooth functions on M into a local algebra A, [23]. Nowadays A is called Weil algebra and the space T^AM of the corresponding near points on M is said to be a Weil bundle. About 1985, it was deduced that the product preserving bundle functors on the category of all smooth manifolds $\mathcal{M}f$ are just the Weil functors, see [14] or Section 2.6 below. This result clarified that Weil bundles should be a good instrument for differential geometry. In this connection, the so-called covariant approach to Weil bundles was developed, [14]. Under this approach, T^AM is interpreted as a generalization of the bundle T^r_kM of (k,r)-velocities introduced by C. Ehresmann in the framework of his jet theory. So the iteration $T^s_lT^r_k$ of two classical velocities functors is the simplest new example of a Weil functor. In 1999, the fiber product preserving bundle functors on the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and fibered manifold morphisms with local diffeomorphisms as base maps were also described in terms of Weil algebras,

[15]. This clarified that even these functors can be viewed as a reasonable generalization of various types of jet bundles.

In this paper we treat systematically the covariant approach and we present the most interesting geometric results deduced in this way. We aim to differential geometry and its applications in analysis and mathematical physics. So we start with a rather detailed presentation of the algebraic properties of Weil algebras in Chapter 1. In particular, every Weil algebra of width k and order r is expressed as a factor algebra of the algebra $\mathbb{D}_k^r$ of r-jets of smooth functions on $\mathbb{R}^k$ at 0. Our definition of Weil bundle T^AM in Chapter 2 is of "practical" character: if A is a Weil (k,r)-algebra, then T^AM can be viewed as a certain factor space of the velocities bundle T_k^rM. An important theoretical result reads that the natural transformations $T^{A_1}M \to T^{A_2}M$ of two Weil bundles are in bijection with algebra homomorphisms $\mu : A_1 \to A_2$. Under our approach, each μ can be interpreted as a kind of reparametrization. Even this is suitable for applications. In 2.11 we construct a canonical exchange isomorphism $\varkappa_M^A : T^ATM \to TT^AM$ that can be used for a simple construction of flow prolongations of vector fields. In 2.16 we clarify that the product preserving bundle functors on the category $\mathcal{FM}$ of all fibered manifolds are in bijection with the Weil algebra homomorphisms. The simplest example of these functors are the fiber velocities bundles described in 2.17.

Chapter 3 is devoted to concrete geometric problems related with T^A-prolongations. First we deduce that each $a \in A$ determines a tensor field of type $(1,1)$ on T^AM. Then we discuss T^A-prolongations of Lie groups, their actions, principal and associated bundles. Next we show that T^A-prolongation preserves the Frölicher-Nijenhuis bracket of tangent valued forms. That is why we use the tangent valued forms as one of the approaches to connections in 3.11. In particular, this yields a general formula for the curvature of the T^A-induced connection and implies some interesting properties of an original concept of a-torsion, $a \in A$. We also discuss a few examples demostrating that the use of the algebra multiplication is very convenient for concrete evaluations.

In Chapter 4 we study a fiber product preserving bundle functor F on the category $\mathcal{FM}_m$. The basic examples are the r-th jet prolongation, the r-th vertical jet prolongation, the vertical Weil functor and their iterations. The iteration of jet functors leads to nonholonomic jets, whose composition is described in 4.2. The construction of product fibered manifolds is a functor $i : \mathcal{M}f_m \times \mathcal{M}f \to \mathcal{FM}_m$, where $\mathcal{M}f_m$ is the category of m-dimensional manifolds and local diffeomorphisms. If F is of base order r, then $F \circ i$ is characterized by a Weil algebra A and a group homomorphism $H : G_m^r \to \mathrm{Aut}\, A$ of the r-th jet group in dimension m into the group of all algebra automorphisms of A. Theorem 4.7 reads that the original functor F is identified with a triple (A, H, t), where $t : \mathbb{D}_m^r \to A$ is an equivariant algebra homomorphism. For every fibered manifold $Y \to M$, $m = \dim M$, FY is expressed in the form of a fiber bundle associated to the r-th order frame bundle of M. Then we describe the natural transformations and iterations of such functors. In

4.11 we introduce the general concept of r-th order jet functor as a suitable subfunctor of the nonholonomic one and we characterize it in terms of Weil algebras.

Chapter 5 is devoted to some applications of Theorem 4.7. In 5.2 we introduce an analogy ψ_Y^F of the flow natural exchange $\varkappa_M^A$ from the manifold case, but the construction of ψ_Y^F is much more sophisticated. Then we show how ψ_Y^F can be applied to F-prolongations of projectable tangent valued forms, the connections being a special case. In 5.7 we generalize the classical theory of jet prolongations of associated fiber bundles to the case of arbitrary F. We find remarkable that F-prolongation of Lie groupoids, studied in 5.8, is essentially based on the expression of FY in the form of an associated bundle from 4.7. Finally, we outline in 5.10 how our velocity-like approach to Weil bundles can be modified to the functional bundle of all smooth maps between the individual fibers of two fibered manifolds over the same base.

Except Section 5.10, we consider the classical smooth manifolds and maps, i.e. all manifolds are finite dimensional and smooth means C^∞-differentiable. All manifolds are assumed to be Hausdorff and separable, [14]. A bundle functor F on the category $\mathcal{M}f$ of all smooth manifolds and all smooth maps transforms every manifold M into a fibered manifold $\pi_M : FM \to M$ and every map $f : M_1 \to M_2$ into a fibered manifold morphism $Ff : FM_1 \to FM_2$ over f. Further, F is assumed to have the localization property: if $i_U : U \hookrightarrow M$ is the injection of an open subset, then $FU = \pi_M^{-1}(U) \subset FM$ and Fi_U is the injection of $\pi_M^{-1}(U)$, [14]. We shall consider the bundle functors also on the following categories:

$\mathcal{M}f_m \subset \mathcal{M}f$ - m-dimensional manifolds and local diffeomorphisms,

$\mathcal{FM}$ - all fibered manifolds and their morphisms,

$\mathcal{FM}_m \subset \mathcal{FM}$ - fibered manifolds with m-dimensional bases and local diffeomorphisms as base maps.

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1. Weil algebras

1.1. Algebras. An algebra is a vector space V together with a bilinear map $f : V \times V \to V$, which is called algebra multiplication. We write $f(x,y) = xy$. The bilinearity of f implies

$$ox = o\,, \quad xo = o\,, \qquad x \in V,\ o = \text{ the zero vector of } V.$$

Let $(\overline{V}, \overline{f})$ be another algebra. An algebra homomorphism $\mu : (V, f) \to (\overline{V}, \overline{f})$ is a linear map $\mu : V \to \overline{V}$ preserving the multiplications. In what follows all algebras are assumed to be both commutative and associative.

An ideal $I \subset V$ is a linear subspace such that

$$xa \in I \quad \text{for all} \quad x \in I,\ a \in V\,.$$

For every subset $S \subset V$, we denote by $\langle S \rangle$ the ideal generated by S, i.e. the smallest ideal in V containing S. The factor vector space V/I is an algebra with respect to the multiplication

$$(a+I)\,(b+I) = ab + I\,.$$

On the other hand, the kernel of every algebra homomorphism is an ideal.

An element $a \in V$ is said to be nilpotent, if $a^n = o$ for some integer n. It is easy to see that the set N of all nilpotent elements of V is an ideal.

A unit of V is an element $e \neq o$ satisfying $ex = x$ for all $x \in V$. If the unit exists, it is unique. An algebra with unit is said to be unital. The unit defines an injection $\mathbb{R} \hookrightarrow V$, $c \mapsto ce$. In this case, we write $\mathbb{R} \subset V$ and we identify e with $1 \in \mathbb{R}$ and the zero vector o with $0 \in \mathbb{R}$. The homomorphisms of unital algebras $V \to \overline{V}$ are assumed to transform the unit of V into the unit of $\overline{V}$.

1.2. Weil algebras. The following concept was called local algebra in the original paper by A. Weil, [23].

Definition. A Weil algebra A is a finite dimensional, commutative, associative and unital algebra of the form

$$A = \mathbb{R} \times N\,,$$

where N is the ideal of all nilpotent elements of A.

In particular, $\mathbb{R}$ is a trivial Weil algebra with $N = 0$. Let $\overline{A} = \mathbb{R} \times \overline{N}$ be another Weil algebra and $\mu : A \to \overline{A}$ be a homomorphism. Then the restriction and corestriction of μ to $\mathbb{R} \subset A$ and $\mathbb{R} \subset \overline{A}$ is the identity and μ transforms N into $\overline{N}$. The zero homomorphism $\mathcal{O} : A \to \overline{A}$ maps N into $0 \in \overline{A}$. We write $\mathrm{Hom}\,(A, \overline{A})$ for the set of all algebra homomorphisms of A into $\overline{A}$. An ideal $I \subset A$ is assumed to be different from A, so that $I \subset N$.

1.3. The width and the order. In general, for a linear subspace $W \subset V$ we define

$$W^n = \{a_1 \dots a_n + \dots + b_1 \dots b_n;\ a_1, \dots, a_n, \dots, b_1, \dots, b_n \in W\}\,,$$

i.e. the elements of W^n are finite sums of the products of n elements of W.

Since $A = \mathbb{R} \times N$ is finite dimensional, there exists an integer r such that $N^{r+1} = 0$. The smallest r with this property is called the order $\mathrm{ord}\, A$ of A. (A. Weil used the term "depth", [23].) On the other hand, the dimension wA of the vector space N/N^2 is said to be the width of A. A Weil algebra of width k and order r will be called Weil (k, r)-algebra.

Every algebra homomorphism $\mu : A \to \overline{A}$ induces a linear map

$$\widetilde{\mu} : N/N^2 \to \overline{N}/\overline{N}^2\,, \quad \widetilde{\mu}(a + N^2) = \mu(a) + \overline{N}^2. \tag{1}$$

Clearly, if B is another Weil algebra and $\nu : \overline{A} \to B$ is an algebra homomorphism, then $\widetilde{\nu \circ \mu} = \widetilde{\nu} \circ \widetilde{\mu}$.

1.4. The algebra $\mathbb{D}_k^r$. Let $\mathbb{R}[x_1, \dots, x_k]$ be the algebra of all polynomials in k undetermined. The simplest example of a Weil algebra is

$$\mathbb{D}_k^r = \mathbb{R}[x_1, \dots, x_k]/\langle x_1, \dots, x_k\rangle^{r+1} . \tag{2}$$

As a vector space, $\mathbb{D}_k^r$ is the set of all polynomials of degree at most r in k undetermined with the standard addition and multiplication by real scalars. The product of P, $Q \in \mathbb{D}_k^r$ is the "truncated" one: we multiply P and Q as polynomials and we neglect the terms of degree higher than r. Clearly, $\operatorname{ord}(\mathbb{D}_k^r) = r$ and $w(\mathbb{D}_k^r) = k$.

We shall write N_k^r for the nilpotent part of $\mathbb{D}_k^r$. The elements of N_k^r are the polynomials without absolute term.

In particular, $\mathbb{D}_1^1$ is the classical algebra $\mathbb{D}$ of dual (or Study) numbers. Its elements can be written as $a + be$, $a, b \in \mathbb{R}$, with e satisfying $e^2 = 0$.

1.5. A as a factor algebra. The following assertion gives a rather concrete description of Weil algebras.

Proposition. *Every Weil algebra A of order $\leq r$ and of width k is a factor algebra of $\mathbb{D}_k^r$.*

Proof. Choose $a_1, \dots, a_k \in N$ such that

$$a_1 + N^2, \dots, a_k + N^2 \tag{3}$$

is a basis of the vector space N/N^2. Define $\pi : \mathbb{D}_k^r \to A$, $\pi(P) = P(a_1, \dots, a_k)$. First we deduce that π is a homomorphism, i.e. $(PQ)(a_1, \dots, a_k)$ $= P(a_1, \dots, a_k)\, Q(a_1, \dots, a_k)$. Indeed, PQ on the left hand side is the product in $\mathbb{D}_k^r$, while $P(a_1, \dots, a_k)\, Q(a_1, \dots, a_k)$ on the right hand side is the standard product of polynomials in $a_1, \dots, a_k$. But the condition $N^{r+1} = 0$ suppresses the terms of degree $> r$.

It remains to show that π is surjective. Since (3) is a basis, for every $a \in N$ we have

$$a + N^2 = c_1(a_1 + N^2) + \dots + c_k(a_k + N^2) , \quad c_1, \dots, c_k \in \mathbb{R} ,$$

i.e. $a - c_1 a_1 - \dots - c_k a_k = n_1 \in N^2$. One verifies directly that the elements $a_i a_j + N^3$ generate linearly N^2/N^3, so that there exist $c_{ij} \in \mathbb{R}$ such that

$$n_1 = \sum_{i,j=1}^{k} c_{ij} a_i a_j + n_2 , \qquad n_2 \in N^3 .$$

In the $(l-1)$-st step of such procedure, we obtain

$$n_{l-1} = \sum_{i_1, \dots, i_l = 1}^{k} c_{i_1 \dots i_l} a_{i_1} \dots a_{i_l} + n_l , \qquad n_l \in N^{l+1} .$$

But $N^{r+1} = 0$, so that after r steps we have

$$a = \sum_{i=1}^{k} c_i a_i + \dots + \sum_{i_1, \dots, i_r = 1}^{k} c_{i_1 \dots i_r} a_{i_1} \dots a_{i_r} .$$

The c's determine a polynomial $P \in N^r_k$ satisfying $a = P(a_1, \dots, a_k)$.

This proof yields directly the following assertion. Every Weil (k,r)-algebra A is a factor algebra

$$A = \mathbb{R}[x_1, \dots, x_k]/I, \tag{4}$$

where I is an ideal satisfying $\langle x_1, \dots, x_k\rangle^2 \supset I \supset \langle x_1, \dots, x_k\rangle^{r+1}$ with minimal r. We write $\overline{\pi} : \mathbb{R}[x_1, \dots, x_k] \to A$ for the factor projection.

1.6. Reparametrizations. We shall use heavily the following interpretation of $\mathbb{D}^r_k$ in terms of jets. By (2), the elements of $\mathbb{D}^r_k$ are r-jets of functions on $\mathbb{R}^k$ at 0, i.e. $\mathbb{D}^r_k = J^r_0(\mathbb{R}^k, \mathbb{R})$. The addition in $\mathbb{D}^r_k$, the multiplication by reals and the multiplication in $\mathbb{D}^r_k$ are expressed by the formulae

$$j^r_0\gamma + j^r_0\delta = j^r_0(\gamma + \delta), \quad c j^r_0\gamma = j^r_0(c\gamma), \quad (j^r_0\gamma)(j^r_0\delta) = j^r_0(\gamma\delta),$$

$\gamma, \delta : \mathbb{R}^k \to \mathbb{R}$, $c \in \mathbb{R}$.

We denote the composition of jets by the same symbol $\circ$ as the composition of maps.

Every r-jet $X \in J^r_0(\mathbb{R}^l, \mathbb{R}^k)_0$ induces an algebra homomorphism $\mathbb{D}^r_k \to \mathbb{D}^r_l$ by

$$Y \to Y \circ X, \quad Y \in \mathbb{D}^r_k = J^r_0(\mathbb{R}^k, \mathbb{R}). \tag{5}$$

Geometrically, (5) is a reparametrization of the elements of $\mathbb{D}^r_k$.

Proposition. *We have* $\mathrm{Hom}\,(\mathbb{D}^r_k, \mathbb{D}^r_l) = J^r_0(\mathbb{R}^l, \mathbb{R}^k)_0$.

Proof. Let $\mu : \mathbb{D}^r_k \to \mathbb{D}^r_l$ be an algebra homomorphism. Then $\mu(x_i) = P_i \in N^r_l$ for every $i = 1, \dots, k$. Hence $P = (P_1, \dots, P_k)$ is a k-tuple of polynomials of degree at most r without the absolute term in l undetermined. This defines an r-jet $P \in J^r_0(\mathbb{R}^l, \mathbb{R}^k)_0$. By the definition of jet composition, $\mu(Y) = Y \circ P$ for all $Y \in \mathbb{D}^r_k$.

1.7. Supplement to Proposition 1.5. Now we can prove

Proposition. *Let A be a Weil (k,r)-algebra and $\pi, \varrho : \mathbb{D}^r_k \to A$ be two surjective algebra homomorphisms. Then there is an algebra isomorphism $\sigma : \mathbb{D}^r_k \to \mathbb{D}^r_k$ satisfying $\pi = \varrho \circ \sigma$.*

Proof. Write $a_i = \pi(x_i)$ and choose some $P_i \in \mathbb{D}^r_k$ satisfying $\varrho(P_i) = a_i$. Consider the homomorphism $\sigma : \mathbb{D}^r_k \to \mathbb{D}^r_k$ transforming x_i into P_i. Then $\varrho\big(\sigma(x_i)\big) = \pi(x_i)$, so that $\pi = \varrho \circ \sigma$. Consider the induced maps $\widetilde{\pi}$, $\widetilde{\varrho}$: $N^r_k/(N^r_k)^2 \to N/N^2$, $\widetilde{\sigma} : N^r_k/(N^r_k)^2 \to N^r_k/(N^r_k)^2$ from 1.3. Both $\widetilde{\pi}$ and $\widetilde{\varrho}$ are linear isomorphisms by the dimension argument, so $\widetilde{\sigma}$ is too. But σ is determined by a reparametrization $Y \mapsto Y \circ X$. One verifies directly that the invertibility of $\widetilde{\sigma}$ is equivalent to the fact that the linear map corresponding to the underlying 1-jet of X is invertible. Hence X is an invertible r-jet, so that σ is an isomorphism.

1.8. Algebra homomorphisms. Let A and B be two Weil algebras with $k = wA$, $l = wB$, $r = \max(\operatorname{ord} A, \operatorname{ord} B)$ and $\mu : A \to B$ be an algebra homomorphism. Consider two surjective homomorphisms $\pi : \mathbb{D}_k^r \to A$ and $\varrho : \mathbb{D}_l^r \to B$. Analogously to 1.7, one deduces there is an algebra homomorphism $\sigma : \mathbb{D}_k^r \to \mathbb{D}_l^r$ such that the following diagram commutes

(6)
$$\begin{tikzcd} \mathbb{D}_k^r \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \mathbb{D}_l^r \arrow[d, "\varrho"] \\ A \arrow[r, "\mu"] & B \end{tikzcd}$$

An element $X \in J_0^r(\mathbb{R}^l, \mathbb{R}^k)_0$ such that (6) with $\sigma = X$ commutes will be called μ-admissible.

Let A be expressed by (4) and $P_h(x_1, \dots, x_k)$ be some generators of ideal I, $h = 1, \dots, s$. To determine all algebra homomorphisms $A \to B$, we first consider an arbitrary k-tuple b_i of elements in the nilpotent part of B to be the images of $\overline{\pi}(x_i)$. Then the rule $b_i = \mu\big(\overline{\pi}(x_i)\big)$ generates an algebra homomorphism $\mu : A \to B$, if and only if $P_h(b_1, \dots, b_k) = 0$ for all h. An example can be found in 1.12 below.

1.9. Automorphisms. The group $\operatorname{Aut} A$ of all algebra automorphisms of A is a closed subgroup in the group $GL(A)$ of all linear automorphisms of A, so a Lie group. By 1.6, the group $\operatorname{Aut}(\mathbb{D}_k^r)$ coincides with the jet group $G_k^r = \operatorname{inv} J_0^r(\mathbb{R}^k, \mathbb{R}^k)_0$ of all invertible r-jets of $\mathbb{R}^k$ into $\mathbb{R}^k$ with source and target 0.

A derivation of A is a linear map $f : A \to A$ satisfying $f(a_1 a_2) = f(a_1)a_2 + a_1 f(a_2)$ for all $a_1, a_2 \in A$. According to the general theory, the Lie algebra $\mathfrak{Aut} A$ of $\operatorname{Aut} A$ coincides with the Lie algebra $\operatorname{Der} A$ of all derivations of A.

1.10. Sums. For two Weil algebras $A = \mathbb{R} \times N_A$, $B = \mathbb{R} \times N_B$, the vector space

$$A \oplus B = \mathbb{R} \times N_A \times N_B$$

is also a Weil algebra with respect to the multiplication

$$(c_1, a_1, b_1)(c_2, a_2, b_2) = (c_1 c_2, c_1 a_2 + c_2 a_1 + a_1 a_2, c_1 b_2 + c_2 b_1 + b_1 b_2)\,,$$

$c_i \in \mathbb{R}$, $a_i \in N_A$, $b_i \in N_B$, $i = 1, 2$. One can say that $A \oplus B$ is the sum of A and B.

If A is expressed by (4) and B analogously by

(7)
$$\mathbb{R}[y_1, \dots, y_l]/J\,,$$

then $\langle I, J, x_i y_p\rangle$, $i = 1, \dots, k$, $p = 1, \dots, l$ is an ideal in $\mathbb{R}[x_1, \dots, x_k, y_1, \dots, y_l]$ and we have

$$A \oplus B = \mathbb{R}[x_1, \dots, y_l]/\langle I, J, x_i y_p\rangle\,.$$

In particular, this implies

$$w(A \oplus B) = wA + wB\,, \quad \operatorname{ord}(A \oplus B) = \max(\operatorname{ord} A, \operatorname{ord} B)\,.$$

1.11. Tensor products. In general, the tensor product $V_1 \otimes V_2$ of two algebras (V_1, f_1) and (V_2, f_2) is also an algebra, whose multiplication f is the tensor product of f_1 and f_2. Thus, for the decomposable tensors $v_1 \otimes v_2$, $\overline{v}_1 \otimes \overline{v}_2$, we have

$$f(v_1 \otimes v_2, \overline{v}_1 \otimes \overline{v}_2) = f_1(v_1, \overline{v}_1) \otimes f_2(v_2, \overline{v}_2) .$$

In the case of two Weil algebras $A = \mathbb{R} \times N_A$, $B = \mathbb{R} \times N_B$, $A \otimes B$ is also a Weil algebra with the nilpotent part $N_A \times N_B \times N_A \otimes N_B$. If A is expressed by (4) and B by (7), then $\langle I, J \rangle$ is an ideal in $\mathbb{R}[x_1, \dots, x_k, y_1, \dots, y_l]$ and we have

$$A \otimes B = \mathbb{R}[x_1, \dots, y_l]/\langle I, J \rangle .$$

In particular, this implies,

$$w(A \otimes B) = wA + wB , \quad \operatorname{ord}(A \otimes B) = \operatorname{ord} A + \operatorname{ord} B .$$

For instance, $\mathbb{D} \otimes \mathbb{D}$ is of the form

$$\mathbb{R}[x, y]/\langle x^2, y^2 \rangle .$$

1.12. Example. We determine all algebra homomorphisms $\mu : \mathbb{D} \otimes \mathbb{D} \to \mathbb{D} \otimes \mathbb{D}$. Write

$$\mu(x) = c_1 x + c_2 y + c_3 xy , \quad \mu(y) = c_4 x + c_5 y + c_6 xy .$$

The conditions $\big(\mu(x)\big)^2 = 0$ and $\big(\mu(y)\big)^2 = 0$ imply $c_1 c_2 = 0$ and $c_4 c_5 = 0$. By 1.8, all algebra homomorphisms $\mathbb{D} \otimes \mathbb{D} \to \mathbb{D} \otimes \mathbb{D}$ form the following 4 four-parameter families

$$c_1 = 0 = c_4 , \quad c_2 = 0 = c_4 , \quad c_1 = 0 = c_5 , \quad c_2 = 0 = c_5$$

with arbitrary c_3 and c_6.

2. Weil bundles

2.1. A-velocities. Having in mind the applications in concrete differential geometric problems, we introduce Weil bundles by using the concept of A-velocity. This generalizes the classical concept of (k, r)-velocity by C. Ehresmann. We recall that the construction of (k, r)-velocities is a bundle functor T^r_k on $\mathcal{M}f$ defined by

$$T^r_k M = J^r_0(\mathbb{R}^k, M) , \ T^r_k f(j^r_0 \gamma) = j^r_0(f \circ \gamma) , \quad \gamma : \mathbb{R}^k \to M$$

for every manifold M and every smooth map $f : M \to N$.

Consider a Weil (k, r)-algebra A together with an algebra homomorphism $\pi : \mathbb{D}^r_k \to A$ from 1.5. By 1.7, π is determined up to an isomorphism $\mathbb{D}^r_k \to \mathbb{D}^r_k$, so that the following definition is independent of π. Let M be a manifold.

Definition. Two maps $\gamma, \delta : \mathbb{R}^k \to M$ are said to determine the same A-velocity $j^A \gamma = j^A \delta$, if for every smooth function $\varphi : M \to \mathbb{R}$

$$\pi(j^r_0(\varphi \circ \gamma)) = \pi(j^r_0(\varphi \circ \delta)) .$$

Proposition. *Let $\gamma^i(t_1, \dots, t_k)$ or $\delta^i(t_1, \dots, t_k)$ be the coordinate expressions of γ or δ. Then $j^A\gamma = j^A\delta$ if and only if $j^A\gamma^i = j^A\delta^i$ for all $i = 1, \dots, \dim M$.*

Proof. The kernel of π is an ideal I in $\mathbb{D}^r_k$. So $j^A\gamma^i = j^A\delta^i$ means

$$j^r_0\delta^i = j^r_0\gamma^i + j^r_0\varepsilon^i\,, \qquad j^r_0\varepsilon^i \in I\,.$$

Since our assertion depends on $j^r_x\varphi$ only, we may assume φ is a polynomial of degree r. This is a sum of monomials $cx^{i_1}\dots x^{i_l}$. The value of each monomial at $j^r_0\gamma^i + j^r_0\varepsilon^i$ is

$$c(j^r_0\gamma^{i_1} + j^r_0\varepsilon^{i_1})\dots(j^r_0\gamma^{i_l} + j^r_0\varepsilon^{i_l}) = cj^r_0\gamma^{i_1}\dots j^r_0\gamma^{i_l} + \quad \text{a term in } I.$$

2.2. Weil functors. Proposition 2.1 implies

$$T^A\mathbb{R} = \pi(\mathbb{D}^r_k) = A \quad \text{and} \quad T^A\mathbb{R}^m = A^m\,.$$

So $T^AM \to M$ is a fibered manifold that is said to be a Weil bundle. For a smooth map $f : M \to N$, we define

$$T^Af : T^AM \to T^AN \quad \text{by} \quad T^Af(j^A\gamma) = j^A(f\circ\gamma)\,.$$

Using the same argument as in the proof of Proposition 2.1, one verifies that this is a correct definition. Hence we obtain a bundle functor T^A on $\mathcal{M}f$ called Weil functor. Clearly, $T^{\mathbb{D}^r_k} = T^r_k$, so that $T^{\mathbb{D}}$ is the tangent functor T. Our construction yields a surjective map $\pi_M : T^r_kM \to T^AM$ such that the following diagram commutes for every $f : M \to N$

$$\begin{array}{ccc} T^r_kM & \xrightarrow{T^r_kf} & T^r_kN \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ T^AM & \xrightarrow{T^Af} & T^AN \end{array} \tag{8}$$

A section $M \to T^AM$ is said to be an A-field on M.

2.3. Remarks. Definition 2.1 is a modification of what is called the covariant approach to Weil functors in [14]. Let $\mathcal{E}_k$ be the algebra of germs of smooth functions on $\mathbb{R}^k$ at 0. By (4), A can be viewed as a factor algebra $A = \mathcal{E}_k/\mathcal{I}$, where $\mathcal{I}$ is the ideal determined by I in $\mathcal{E}_k$. Then two maps γ, $\delta : \mathbb{R}^k \to M$ satisfy $j^A\gamma = j^A\delta$, if and only if $\varphi\circ\gamma - \varphi\circ\delta \in \mathcal{I}$ for every germ φ of smooth function on M at $x = \gamma(0) = \delta(0)$.

The original ideas by A. Weil, [23], were inspired by the algebraic geometry. So his approach is of contravariant character. All smooth functions on M form an algebra $C^\infty(M,\mathbb{R})$ with respect to the pointwise multiplication. Weil defined a so-called infinitely near point of type A on M as an algebra homomorphism

$$C^\infty(M,\mathbb{R}) \to A\,.$$

We show that the set of all algebra homomorphisms $\mathrm{Hom}\,(C^\infty(M,\mathbb{R}), A)$ is canonically isomorphic to T^AM.

For every $f \in C^\infty(M, \mathbb{R})$ and every $j^A\gamma \in T^AM$, we define

$$(j^A\gamma)(f) = j^A(f \circ \gamma) \in A\,. \tag{9}$$

This is an algebra homomorphism. Indeed, $(j^A\gamma)(f_1+f_2) = j^A(f_1\circ\gamma + f_2\circ\gamma)$ and $(j^A\gamma)(f_1 f_2) = j^A((f_1\circ\gamma)(f_2\circ\gamma))$. Since our operations behave well with respect to localization, it suffices to consider the case $M = \mathbb{R}^m$, so that $T^A\mathbb{R}^m = A^m$. Then Proposition 2.1 implies that (9) establishes a bijection between $T^A\mathbb{R}^m$ and $\mathrm{Hom}\,(C^\infty(\mathbb{R}^m, \mathbb{R}), A)$.

We also remark that some motivation for the covariant approach came from the so-called synthetic differential geometry, which was developed within the framework of the theory of categories, [8].

2.4. The coordinate expression of T^Af. Write $T^A_x f : T^A_x M \to T^A_{f(x)} N$ for the restricted and corestricted map. The map $T^A_x f$ depends on $j^r_x f$ only. Indeed, (8) is a commutative diagram with surjective columns, where the top row depends on $j^r_x f$ only.

Using (8), we deduce the coordinate expression of T^Af in the case of $f : \mathbb{R}^m \to \mathbb{R}^n$ with the components

$$y^p = f^p(x^i)\,, \qquad i = 1, \ldots, m,\ p = 1, \ldots, n\,. \tag{10}$$

Hence $T^Af : A^m \to A^n$. First consider the map $T^r_k f : T^r_k\mathbb{R}^m \to T^r_k\mathbb{R}^n$. This is determined by the Taylor expansions of order r of the components f^p

$$f^p(x^i) + \sum_{|\alpha|\le r} \frac{1}{\alpha!} D_\alpha f^p(x^i) z^\alpha\,, \quad (z^i) \in \mathbb{R}^m\,,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$ denotes a multiindex of range m and $z^\alpha = (z^1)^{\alpha_1} \ldots (z^m)^{\alpha_m}$. Write $x^i + n^i$ for the i-th component in A^m, $n^i \in N_A$, and b^p for the p-th component in A^n. If we express $T^r_k f : (\mathbb{D}^r_k)^m \to (\mathbb{D}^r_k)^n$ in the algebra form and use the homomorphism $\pi : \mathbb{D}^r_k \to A$, we obtain the following expression of T^Af

$$b^p = f^p(x^i) + \sum_{|\alpha|\le r} \frac{1}{\alpha!} D_\alpha f^p(x^i) n^\alpha\,, \tag{11}$$

where $n^\alpha = (n^1)^{\alpha_1} \ldots (n^m)^{\alpha_m}$ with the multiplication in A.

In particular, if $f : \mathbb{R}^m \to \mathbb{R}$ is a real valued function, then (11) with no superscript p expresses the A-valued function $T^Af : T^A\mathbb{R}^m \to A$. We shall need an explicit formula in the simplest case $A = \mathbb{D}$. Write x^i_1 for the additional coordinates on $T\mathbb{R}^m$. Then the coordinate form of $Tf : T\mathbb{R}^m \to \mathbb{D}$ is

$$f(x^i) + e\Big(\tfrac{\partial f}{\partial x^i}\, x^i_1\Big)\,. \tag{12}$$

2.5. Example. Consider the iterated tangent functor TT. The elements of its Weil algebra $\mathbb{D}\otimes\mathbb{D} = \mathbb{R}[t, \tau]/\langle t^2, \tau^2\rangle$ are of the form $x + ut + v\tau + wt\tau$. The corresponding coordinates on $TT\mathbb{R}^m = (\mathbb{D}\otimes\mathbb{D})^m$ are x^i, u^i, v^i, w^i. From the classical point of view, x^i are the "original" coordinates on $\mathbb{R}^m$, the role of the "first order" coordinates u^i and v^i is more or less symmetric and w^i appear clearly to be the "second order" coordinates.

Further, consider the map (10) and write $y^p + \overline{u}^p t + \overline{v}^p \tau + \overline{w}^p t\tau$ for p-th component of $(\mathbb{D} \otimes \mathbb{D})^n$. Since $t^2 = 0 = \tau^2$, (11) implies

$$y^p + \overline{u}^p t + \overline{v}^p \tau + \overline{w}^p t\tau = f^p + \tfrac{\partial f^p}{\partial x^i}(u^i t + v^i \tau + w^i t\tau) + \tfrac{\partial^2 f^p}{\partial x^i \partial x^j} u^i v^j t\tau\,.$$

Passing to the individual components, we obtain the standard coordinate expression of TTf.

2.6. Product preserving bundle functors. Consider the product $M \xleftarrow{p_1} M \times N \xrightarrow{p_2} N$ of two manifolds together with the product projections. A bundle functor F on $\mathcal{M}f$ is said to be product preserving, if $F(M \times N) = FM \times FN$. More precisely, this means that

$$FM \xleftarrow{Fp_1} F(M \times N) \xrightarrow{Fp_2} FN$$

is also a product. Clearly, every Weil functor T^A preserves products.

The converse assertion is a fundamental theoretical result. Let F be a product preserving bundle functor on $\mathcal{M}f$. Write pt for one point set and $i_x : pt \to M$ for the map $i_x(pt) = x$, $x \in M$. Since F preserves products, we have $F(pt) = pt$. A natural injection $\nu_M : M \to FM$ is defined by $\nu_M(x) = Fi_x(pt)$. Applying F to the addition $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and the multiplication $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of reals, we obtain

$$Fa : F\mathbb{R} \times F\mathbb{R} \to F\mathbb{R}\,, \quad Fm : F\mathbb{R} \times F\mathbb{R} \to F\mathbb{R}\,.$$

One verifies easily that $F\mathbb{R}$ with the addition Fa and the multiplication by real scalars $ca = Fm\big(\nu_{\mathbb{R}}(c), a\big)$, $c \in \mathbb{R}$, $a \in F\mathbb{R}$, is a vector space. The proof of the following assertion can be found in [14].

Theorem. *$F\mathbb{R}$ is a Weil algebra with respect to the multiplication Fm and F coincides with the Weil functor $T^{F\mathbb{R}}$.*

A simple example of a bundle functor on $\mathcal{M}f$ that does not preserve products is the second tensor power $\overset{2}{\bigotimes}T$ of the tangent functor T. Indeed, $\dim \overset{2}{\bigotimes}T(M \times N) > \dim \overset{2}{\bigotimes}TM + \dim \overset{2}{\bigotimes}TN$ provided $\dim M > 0$ and $\dim N > 0$.

2.7. Natural transformations. Let $t : T^A \to T^B$ be a natural transformation of functors T^A and T^B. If we apply t to the addition and the multiplication of reals, we obtain two commutative diagrams of the form

$$\begin{array}{ccc} A \times A & \longrightarrow & A \\ {\scriptstyle t_{\mathbb{R}}}\downarrow \quad \downarrow{\scriptstyle t_{\mathbb{R}}} & & \downarrow{\scriptstyle t_{\mathbb{R}}} \\ B \times B & \longrightarrow & B \end{array}$$

This implies easily that $t_{\mathbb{R}} : A \to B$ is an algebra homomorphism.

The converse assertion is also true. Every algebra homomorphism $\mu : A \to B$ induces a natural transformation (denoted by the same symbol) $\mu : T^A \to T^B$ as follows. Every $X \in J^r_0(\mathbb{R}^l, \mathbb{R}^k)_0 = \operatorname{Hom}(\mathbb{D}^r_k, \mathbb{D}^r_l)$ defines a natural transformation $X_M : T^r_k M \to T^r_l M$ by the reparametrization $Y \to Y \circ X$, $Y \in T^r_k M$. For a general μ, we may consider the situation from (6) with $\sigma = X$. Taking into account that π_M and ϱ_M are surjective, we deduce by (6) that there is a unique map $\mu_M : T^A M \to T^B M$ making the following diagram commutative

(13)
$$\begin{tikzcd}
T^r_k M \arrow[r, "X_M"] \arrow[d, "\pi_M"'] & T^r_l M \arrow[d, "\varrho_M"] \\
T^A M \arrow[r, "\mu_M"] & T^B M
\end{tikzcd}$$

Moreover, for every $f : M \to N$ the following diagram commutes

$$\begin{tikzcd}
T^r_k M \arrow[rrr, "T^r_k f"] \arrow[dr, "\pi_M"] \arrow[ddd, "X_M"'] & & & T^r_k N \arrow[dl, "\pi_N"'] \arrow[ddd, "X_N"] \\
& T^A M \arrow[r, "T^A f"] \arrow[d, "\mu_M"'] & T^A N \arrow[d, "\mu_N"] & \\
& T^B M \arrow[r, "T^B f"] & T^B N & \\
T^r_l M \arrow[rrr, "T^r_l f"] \arrow[ur, "\varrho_M"] & & & T^r_l N \arrow[ul, "\varrho_N"']
\end{tikzcd}$$

The inner square yields that μ_M form a natural transformation $T^A \to T^B$. Diagram (13) implies that there exists a map $\overline{\mu} : \mathbb{R}^l \to \mathbb{R}^k$ such that $\mu_M : T^A M \to T^B M$ is of the form

$$\mu_M(j^A\gamma) = j^B(\gamma \circ \overline{\mu}), \quad \gamma : \mathbb{R}^k \to M . \tag{14}$$

Thus we have proved

Proposition. *The natural transformations $T^A \to T^B$ are in bijection with the algebra homomorphisms $\mu : A \to B$. If $wA = k$ and $wB = l$, then there exists a map $\overline{\mu} : \mathbb{R}^l \to \mathbb{R}^k$ such that μ_M is of the form* (14).

In other words, even in the case of an arbitrary algebra homomorphism $\mu : A \to B$, the natural transformation μ_M is determined by a reparametrization. The admissibility of μ in the sense of 1.8 depends on $j^A\overline{\mu}$ only.

For example, the natural transformations $TT \to TT$, that correspond to the algebra homomorphisms $\mathbb{D} \otimes \mathbb{D} \to \mathbb{D} \otimes \mathbb{D}$ from 1.12, can be immediately expressed in this way. Each of them can be easily interpreted as a geometric construction on the iterated tangent bundle.

2.8. Multilinear maps. For every vector space V, the vector space structure on T^AV is defined by

$$j^A\gamma + j^A\delta = j^A(\gamma+\delta),\ cj^A\gamma = j^A(c\gamma)\,, \quad c \in \mathbb{R}\,,$$

with pointwise addition and scalar multiplication on the right-hand sides. Consider the map $\otimes : V \times A \to T^AV$,

$$\otimes(v, j^A\varphi(t_1,\dots,t_k)) = j^A(\varphi(t_1,\dots,t_k)v)\,, \quad v\in V,\ \varphi:\mathbb{R}^k \to \mathbb{R}\,. \tag{15}$$

In coordinates, we have $V=\mathbb{R}^m$, $T^AV = A^m$ and (15) is of the form

$$\big((v_1,\dots,v_n),a\big) \longmapsto (v_1a,\dots,v_na)\,, \quad v_i\in\mathbb{R},\ a\in A\,,\ i=1,\dots,n\,.$$

This implies $T^AV = V\otimes A$. In particular, if $A=\mathbb{D}$, then $V\otimes\mathbb{D} = V\times V$ and $TV = V\times V$ is the classical expression of the tangent bundle of a vector space.

Consider another vector space W and a linear map $f : V\to W$. Then

$$T^Af(v\otimes a) = T^Af(j^A(\varphi v)) = j^A(\varphi f(v)) = f(v)\otimes a\,.$$

One verifies directly that $T^Af : T^AV \to T^AW$ is also a linear map. Hence $T^Af = f\otimes \mathrm{id}_A$.

Proposition. *Let $f : V_1\times\dots\times V_l \to W$ be a multilinear map. Then $T^Af : V_1\otimes A\times\dots\times V_l\otimes A \to W\otimes A$ is also multilinear and*

$$T^Af(v_1\otimes a_1,\dots,v_l\otimes a_l) = f(v_1,\dots,v_l)\otimes a_1\dots a_l\,,$$

where the product $a_1\dots a_l$ is in A.

Proof. The multilinearity of f implies

$$T^Af(j^A(\varphi_1v_1),\dots,j^A(\varphi_lv_l)) = j^A((\varphi_1\dots\varphi_l)f(v_1,\dots,v_l))$$

with the pointwise product $\varphi_1(t_1,\dots,t_k)\dots\varphi_l(t_1,\dots,t_k)$ in $\mathbb{R}$.

Corollary. *If (V,f) is an algebra, then $T^AV = V\otimes A$ is also an algebra with the multiplication determined by*

$$T^Af(v_1\otimes a_1, v_2\otimes a_2) = f(v_1,v_2)\otimes a_1a_2\,.$$

2.9. Natural transformations on vector spaces. For every vector space V and every algebra homomorphism $\mu : A\to B$,

$$\mu_V : T^AV = V\otimes A \to V\otimes B = T^BV$$

is defined by the reparametrization (14), so that μ_V is a linear map. Applying μ_V to (15), we obtain $\mu_V(v\otimes a) = v\otimes\mu(a)$. Hence we have

$$\mu_V = \mathrm{id}_V\otimes\mu : V\otimes A\to V\otimes B\,.$$

2.10. The iteration. By 2.6, the Weil functors coincide with the product preserving bundle functors on $\mathcal{M}f$. Since the iteration T^AT^B of two Weil functors preserves products as well, this must also be a Weil functor.

Proposition. *We have $T^AT^B = T^{B\otimes A}$.*

Proof. The Weil algebra of T^AT^B is $(T^AT^B)(\mathbb{R}) = T^A(T^B\mathbb{R}) = T^AB = B\otimes A$.

We know that every algebra homomorphism induces a natural transformation. In particular, the exchange isomorphism of the tensor product ex : $B\otimes A \to A\otimes B$ induces an exchange natural equivalence

$$\mathrm{ex}_M : T^AT^BM \to T^BT^AM\,.$$

Geometrically, ex_M can be constructed as follows. Let $t\in\mathbb{R}^k$ and $\tau\in\mathbb{R}^l$. So every $Z\in T^A(T^BM)$ is of the form

$$Z = j^A\big(t\mapsto j^B(\tau\mapsto\delta(t,\tau))\big)\,,$$

where $\delta:\mathbb{R}^k\times\mathbb{R}^l\to M$. Then

$$\mathrm{ex}_M(Z) = j^B\big(\tau\mapsto j^A(t\mapsto\delta(t,\tau))\big)\,.$$

If we write $\pi^A_M : T^AM\to M$ for the bundle projection, then we have

$$T^A\pi^B_M = \pi^B_{T^AM}\circ\mathrm{ex}_M\,,\quad T^B\pi^A_M\circ\mathrm{ex}_M = \pi^A_{T^BM}\,. \tag{16}$$

In the special case $A=\mathbb{D}^r_k$, $B=\mathbb{D}^s_l$, which is important for applications, the natural exchange isomorphism $\mathrm{ex}_M : T^r_kT^s_lM\to T^s_lT^r_kM$ can be expressed in the jet form

$$\mathrm{ex}_M\big(j^r_0(t\mapsto j^s_0(\tau\mapsto\delta(t,\tau)))\big) = j^s_0\big(\tau\mapsto j^r_0(t\mapsto\delta(t,\tau))\big)\,.$$

In the case $k=l=r=s=1$, we evaluated all algebra homomorphisms $\mathbb{D}\otimes\mathbb{D}\to\mathbb{D}\otimes\mathbb{D}$ in 1.12. Clearly, the exchange isomorphism ex, that determines the well known canonical involution $TTM\to TTM$, corresponds to the values $c_2=c_4=1$, $c_1=c_3=c_5=c_6=0$.

2.11. The flow natural exchange. In the case of arbitrary A and $T^B=T$, we obtain a canonical exchange isomorphism

$$\varkappa^A_M : T^ATM\to TT^AM\,.$$

This map is said to be flow natural because of the following important property.

In general, let F be a bundle functor on the category $\mathcal{M}f_m$ of m-dimensional manifolds and local diffeomorphisms. The flow prolongation of a vector field $X:M\to TM$ is a vector field $\mathcal{F}X : FM\to T(FM)$ defined as follows. The flow Fl^X is locally a one-parameter family of diffeomorphisms $Fl^X_t : M\to M$, $t\in\mathbb{R}$. We construct $F(Fl^X_t): FM\to FM$ for every t and we set

$$\mathcal{F}X = \tfrac{\partial}{\partial t}\big|_0 F(Fl^X_t) : FM\to TFM\,.$$

In the case of a Weil functor T^A, beside $\mathcal{T}^AX : T^AM\to TT^AM$ we can consider the functorial prolongation $T^AX : T^AM\to T^ATM$ of the map X.

Proposition. *We have $\mathcal{T}^AX = \varkappa^A_M\circ T^AX$.*

Proof. Let $\overline{x} = \varphi(x,t)$ be the flow of X, $x \in M$, $t \in \mathbb{R}$. Hence $X(x) = \frac{\partial}{\partial t}\big|_0 \varphi(x,t)$. For $u = j^A\gamma(\tau) \in T^A M$, consider $\varphi(\gamma(\tau),t) : \mathbb{R}^k \times \mathbb{R} \to M$. Then

$$\frac{\partial}{\partial t}\Big|_0 j^A \varphi(\gamma(\tau),t) = \frac{\partial}{\partial t}\Big|_0 (T^A Fl_t^X)(u) = (\mathcal{T}^A X)(u)\,.$$

Of course, $\frac{\partial}{\partial t}\big|_0$ is identified with the first order jet with respect to t. If we exchange the order of j^A and $\frac{\partial}{\partial t}\big|_0$, we obtain

$$j^A\Big(\frac{\partial}{\partial t}\Big|_0 \varphi(\gamma(\tau),t)\Big) = j^A(X \circ \gamma) = (T^A X)(u)\,.$$

So the concrete evaluation of $\mathcal{T}^A X$ for a vector field $X = X^i(x^1,\dots,x^m)\partial/\partial x^i$ is also based on 2.4. For example, if $T^A = T$ is the tangent functor, then (12) implies

$$\mathcal{T}X = X^i \frac{\partial}{\partial x^i} + \Big(\frac{\partial X^i}{\partial x^j}\, x_1^j\Big) \frac{\partial}{\partial x_1^i}\,.$$

2.12. Fiber products. If F and G are two product preserving bundle functors on $\mathcal{M}f$, then the bundle functor $F \oplus G$ on $\mathcal{M}f$ defined by $(F \oplus G)(M) = FM \times_M GM$ and $(F \oplus G)(f) = Ff \times_f Gf$ also preserves products.

Proposition. *We have* $T^A \oplus T^B = T^{A \oplus B}$.

Proof. $T^A\mathbb{R}$ or $T^B\mathbb{R}$ is the product fibered manifold $\mathbb{R} \times N_A \to \mathbb{R}$ or $\mathbb{R} \times N_B \to \mathbb{R}$, respectively. Hence $(T^A \oplus T^B)(\mathbb{R}) = \mathbb{R} \times N_A \times N_B \to \mathbb{R}$. One verifies easily that the induced multiplication is that one from 1.10.

2.13. Underlying functors. An interesting feature of the theory of Weil bundles is that every r-th order Weil functor T^A induces the underlying k-th order functors for all $k \le r = \operatorname{ord} A$.

Clearly, N_A^{k+1} is an ideal in A. Write $\pi_k : A \to A/N_A^{k+1}$ for the factor projection.

Definition. The factor algebra $A_k = A/N_A^{k+1}$ is called the underlying Weil algebra of order k. The Weil functor T^{A_k} is said to be the underlying k-th order functor of T^A.

So $(\pi_k)_M : T^A M \to T^{A_k} M$ is a surjective natural transformation. The following lemma is a direct consequence of $\mu(N_A) \subset N_B$.

Lemma. *For every algebra homomorphism* $\mu : A \to B$*, we have* $\mu(N_A^k) \subset N_B^k$.

So μ factorizes through an underlying algebra homomorphism $\mu_k : A_k \to B_k$. From the geometric point of view, every natural transformation $t : T^A \to T^B$ is projectable over a natural transformation $t_k : T^{A_k} \to T^{B_k}$ for all $k \le r$.

For example, the underlying first order functor of the iterated tangent functor TT is $T \oplus T$.

In [9], the following result is deduced.

Proposition. $T^A M \to T^{A_{r-1}} M$ *is an affine bundle, whose associated vector bundle is the pullback of* $TM \otimes N^r$ *over* $T^{A_{r-1}} M$.

In the special case of T_k^r, we obtain the classical result that $T_k^r M \to T_k^{r-1} M$ is an affine bundle, whose associated vector bundle is the pullback of $TM \otimes S^r \mathbb{R}^{k*}$ over $T_k^{r-1} M$.

2.14. Regular A-velocities. Consider the vector space $V_A = N_A / N_A^2$. One finds easily that the underlying Weil algebra of the first order is $A_1 = \mathbb{R} \times V_A$ with the zero multiplication in V_A. This implies $T^{A_1} M = TM \otimes V_A$.

Definition. An A-velocity $X \in T_x^A M$ is called regular, if the linear map $V_A^* \to T_x M$ determined by $\pi_1(X) \in T_x M \otimes V_A$ is injective.

In the classical case of $X \in (T_k^r M)_x$, $\pi_1(X) \in T_k^1 M$ is identified with a k-tuple of vectors in $T_x M$ and X is regular, if and only if these vectors are linearly independent. In general, one verifies easily that an A-velocity $j^A \gamma$ is regular, if and only if γ is an immersion at $0 \in \mathbb{R}^k$.

2.15. Contact A-elements. We recall that a contact (k, r)-element on a manifold M, as introduced by Ehresmann, is defined as a set $Z \circ G_k^r$, where Z is a regular (k, r)-velocity on M, [14]. The space of all such elements is a fiber bundle $K_k^r M$ over M.

This idea can be directly extended to A-velocities.

Definition. A contact A-element on a manifold M is the set $(\mathrm{Aut}\, A)(Z)$, where Z is a regular A-velocity on M.

All contact A-elements on M form a fiber bundle $K^A M \to M$. We remark that the contact A-elements are studied from an algebraic point view in [22].

2.16. Product preserving bundle functors on $\mathcal{FM}$. It is remarkable that the product preserving bundle functors on fibered manifolds can be also characterized in terms of Weil algebras. The following assertion was deduced by W. Mikulski, [20].

Proposition. *The product preserving bundle functors on $\mathcal{FM}$ are in bijection with the algebra homomorphisms $\mu : A \to B$.*

On one hand, μ induces two bundle functors T^A, T^B on $\mathcal{M}f$ and a natural transformation $\mu : T^A \to T^B$. For every fibered manifold $p : Y \to M$, we have $T^B p : T^B Y \to T^B M$. Taking into account the map $\mu_M : T^A M \to T^B M$, we construct the induced bundle

$$T^\mu Y = \mu_M^* T^B Y \ ,$$

which can be also denoted by $T^\mu Y = T^A M \times_{T^B M} T^B Y$. For every $\mathcal{FM}$-morphism $f : Y \to \overline{Y}$ over $\underline{f} : M \to \overline{M}$, we have $T^B f : T^B Y \to T^B \overline{Y}$, $T^A \underline{f} : T^A M \to T^A \overline{M}$ and we construct the induced map

$$T^\mu f := T^A \underline{f} \times_{T^B \underline{f}} T^B f : T^\mu Y \to T^\mu \overline{Y} \, .$$

This defines a bundle functor T^μ on $\mathcal{FM}$. Clearly, T^μ preserves products.

Conversely, let F be a product preserving bundle functor on $\mathcal{FM}$. Write $pt_M : M \to pt$ for the unique map of M into one point set. There are two canonical injections $i_1, i_2 : \mathcal{M}f \to \mathcal{FM}$ of manifolds into fibered manifolds defined by $i_1 M = (\mathrm{id}_M : M \to M)$, $i_1 f = (f, f)$, $i_2 M = (pt_M : M \to pt)$, $i_2 f = (f, \mathrm{id}_{pt})$ and a natural transformation $t : i_1 \to i_2$, $t_M = (\mathrm{id}_M, pt_M) : i_1 M \to i_2 M$. Applying F, we obtain two bundle functors $F \circ i_1$ and $F \circ i_2$ on $\mathcal{M}f$ and a natural transformation $F \circ t : F \circ i_1 \to F \circ i_2$. By 2.6 and 2.7, there is a Weil algebra homomorphism $\mu : A \to B$ such that $F \circ i_1 = T^A$, $F \circ i_2 = T^B$ and $F \circ t = \mu$. Then $F = T^\mu$. (A rather simple proof can be found in [5].)

Further, if $\nu : C \to D$ is another Weil algebra homomorphism, then the natural transformations $T^\mu \to T^\nu$ are in bijection with the pairs of Weil algebra homomorphisms $\varphi : A \to C$, $\psi : B \to D$ satisfying $\psi \circ \mu = \nu \circ \varphi$. Moreover, the iteration $T^\nu T^\mu$ corresponds to the tensor product $\mu \otimes \nu : A \otimes C \to B \otimes D$.

2.17. Fiber velocities. The simplest examples of product preserving bundle functors on $\mathcal{FM}$ are the fiber velocities bundles, which generalize the classical bundles of (k, r)-velocities. Their definition is based on the idea of (q, s, r)-jet of $\mathcal{FM}$-morphisms, $s \geq q \leq r$. Consider two fibered manifolds $p : Y \to M$ and $\overline{p} : \overline{Y} \to \overline{M}$.

Definition. We say that two $\mathcal{FM}$-morphisms $f, g : Y \to \overline{Y}$ with the base maps $\underline{f}, \underline{g} : M \to \overline{M}$ determine the same (q, s, r)-jet $j^{q,s,r}_y f = j^{q,s,r}_y g$ at $y \in Y$, $s \geq q \leq r$, if

$$j^q_y f = j^q_y g\,, \quad j^s_y(f \mid Y_x) = j^s_y(g \mid Y_x)\,, \quad j^r_x \underline{f} = j^r_x \underline{g}\,, \qquad x = p(y)\,.$$

We write $J^{q,s,r}(Y, \overline{Y})$ for the bundle of all (q, s, r)-jets of Y into $\overline{Y}$. The composition of $\mathcal{FM}$-morphisms defines the composition of (q, s, r)-jets. The base maps induce a canonical projection $J^{q,s,r}(Y, \overline{Y}) \to J^r(M, \overline{M})$, where $J^r(M, \overline{M})$ denotes the classical bundle of r-jets of M into $\overline{M}$.

Write $\mathbb{R}^{k,l}$ for the product fibered manifold $\mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$. We introduce the bundle of fiber velocities of dimension (k, l) and order (q, s, r) on a fibered manifold Y by

$$T^{q,s,r}_{k,l} Y = J^{q,s,r}_{0,0}(\mathbb{R}^{k,l}, Y)\,.$$

For every $\mathcal{FM}$-morphism $f : Y \to \overline{Y}$, $T^{q,s,r}_{k,l} f : T^{q,s,r}_{k,l} Y \to T^{q,s,r}_{k,l} \overline{Y}$ is defined by the jet composition. Clearly, $T^{q,s,r}_{k,l}$ is a product preserving bundle functor on $\mathcal{FM}$ of order (q, s, r).

Let $m = \dim M$ and $m + n = \dim Y$. Then $P^{q,s,r} Y = \operatorname{inv} J^{q,s,r}_{0,0}(\mathbb{R}^{m,n}, Y)$ is a principal bundle over Y with structure group $G^{q,s,r}_{m,n} = \operatorname{inv} J^{q,s,r}_{0,0}(\mathbb{R}^{m,n}, \mathbb{R}^{m,n})_{0,0}$, which is called (q, s, r)-th order frame bundle of Y.

2.18. Remark. Applying Theorem 2.6, the author found an interesting abstract characterization of the classical jet bundles. He modified the property of preserving products in a suitable way and deduced that under this assumption the r-jet bundles are the only finite dimensional homomorphic images of the germs of smooth maps, [14]. In the same sense, Proposition 2.16 implies that the (q,s,r)-jets are the only homomorphic images of the germs of fibered manifold morphisms, [5].

3. On the geometry of T^A-prolongations

3.1. Natural tensor fields of type $(1,1)$. Every $a \in A$ defines a natural tensor field $L(a)_M$ of type $(1,1)$ on $T^A M$ for every manifold M as follows. The multiplication of the tangent vectors of M by reals is a map $\sigma_M : \mathbb{R} \times TM \to TM$. Applying T^A, we obtain $T^A\sigma_M : A \times T^A TM \to T^A TM$. Then we construct

$$\mathcal{T}^A \sigma_M := (\varkappa_M^A)^{-1} \circ T^A \sigma_M \circ (\mathrm{id}_A \times \varkappa_M^A) : A \times TT^A M \to TT^A M$$

and define $L(a)_M = \mathcal{T}^A\sigma_M(a, -)$. Since the multiplication in A is induced by the multiplication of reals, we have

$$L(a_1)_M \circ L(a_2)_M = L(a_1 a_2)_M \,.$$

Clearly, $L(1)_M = \mathrm{id}_{TT^A M}$. The naturality of $L(a)$ means $TT^A f \circ L(a)_M = L(a)_N \circ TT^A f$ for every map $f : M \to N$.

To find the coordinate expression of $L(a)$, we take $M = \mathbb{R}^m$. Then $TT^A\mathbb{R}^m = A^m \times A^m$ and our definition implies directly

$$L(a)_{\mathbb{R}^m}(b_1, \dots, b_m, c_1, \dots, c_m) = (b_1, \dots, b_m, ac_1, \dots, ac_m), \quad a, b_i, c_i \in A.$$

In the case of the tangent functor, i.e. $A = \mathbb{D} = \{a + be\}$, the map $L(e)_M : TTM \to VTM$ is frequently used in analytical mechanics.

3.2. Natural vector fields. Every element D of the Lie algebra $\mathfrak{Aut}\, A = \operatorname{Der} A$ is of the form

$$D = \tfrac{d}{dt}\big|_0 \gamma \,, \quad \gamma : \mathbb{R} \to \operatorname{Aut} A \,.$$

The natural transformations $\gamma_M(t) : T^A M \to T^A M$ determine a vertical vector field D_M on $T^A M$,

$$D_M(y) = \tfrac{\partial}{\partial t}\big|_0 \gamma(t)_M(y) \,, \quad y \in T^A M \,.$$

For example, in the case $A = \mathbb{D}$ we obtain the classical Liouville vector field on TM and its constant multiples.

We find worth mentioning one of the oldest geometric results deduced by the technique of Weil algebras. The problem was to determine all natural operators transforming every vector field X on a manifold M into a vector field on $T^A M$. In [14] it is proved that every such operator is of the form

$$X \mapsto L(a)_M \circ \mathcal{T}^A X + D_M$$

for all $a \in A$ and $D \in \operatorname{Der} A$, $\mathcal{T}^A X$ being the flow prolongation of X. This general result was new even in the case of (k,r)-velocities.

3.3. T^A-prolongation of functions and vector fields. Every function $f : M \to \mathbb{R}$ induces a vector valued function $T^A f : T^A M \to A$. Every vector field Z on $T^A M$ determines the Lie derivative $Z(T^A f) : T^A M \to A$ of this vector valued function. Given $a \in A$, we define $aT^A f : T^A M \to A$ by multiplying in A. We shall shorten $L(a)_M \circ Z$ to $L(a)Z$.

We start with some simple formulae, [2]. For every vector field X on M, we have

$$\mathcal{T}^A X(aT^A f) = aT^A(Xf), \quad (L(a)\mathcal{T}^A X)T^A f = aT^A(Xf).$$

For the bracket of vector fields, we have

$$\big[L(a_1)\mathcal{T}^A X_1, L(a_2)\mathcal{T}^A X_2\big] = L(a_1 a_2)\mathcal{T}^A\big([X_1, X_2]\big).$$

3.4. T^A-prolongation of Lie groups. The following assertions are easy to verify.

Let G be a Lie group with composition $\varphi : G \times G \to G$ and $\mathfrak{g} = \operatorname{Lie} G$ be its Lie algebra. Then $T^A\varphi : T^A G \times T^A G \to T^A G$ is also a Lie group and $T^A\mathfrak{g} = \mathfrak{g} \otimes A$ is its Lie algebra. The bracket in $T^A\mathfrak{g}$ is the T^A-prolongation of the bracket in $\mathfrak{g}$. Using 2.8, we obtain

$$[v_1 \otimes a_1, v_2 \otimes a_2]_{\mathfrak{g}\otimes A} = [v_1, v_2]_{\mathfrak{g}} \otimes a_1 a_2,$$

$v_1, v_2 \in \mathfrak{g}$, $a_1, a_2 \in A$, the product $a_1 a_2$ being in A. This formula is powerful even in the case of (k, r)-velocities, in which $\operatorname{Lie}(T^r_k G) = \mathfrak{g} \otimes \mathbb{D}^r_k$. For example, in the case of the tangent functor T we obtain immediately the well known expression for the bracket of $\operatorname{Lie}(TG) = \mathfrak{g} \times \mathfrak{g}$

$$\big[(v_1, \overline{v}_1), (v_2, \overline{v}_2)\big] = \big([v_1, v_2], [v_1, \overline{v}_2] + [\overline{v}_1, v_2]\big).$$

If $\exp_G : \mathfrak{g} \to G$ is the exponential map of G, then the exponential map of $T^A G$ is

$$\exp_{T^A G} = T^A(\exp_G) : T^A\mathfrak{g} \to T^A G.$$

The Maurer-Cartan form $\omega_G : TG \to \mathfrak{g}$ of G induces the Maurer-Cartan form $\omega_{T^A G} : TT^A G \to T^A\mathfrak{g}$ by

$$\omega_{T^A G} = T^A\omega_G \circ (\varkappa^A_G)^{-1}.$$

For a group homomorphism $h : G_1 \to G_2$, $T^A h : T^A G_1 \to T^A G_2$ is also a group homomorphism. If $\chi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is the induced Lie algebra homomorphism, then $T^A\chi = \chi \otimes \mathrm{id}_A : T^A\mathfrak{g}_1 \to T^A\mathfrak{g}_2$ is the Lie algebra homomorphism determined by $T^A h$.

For every Weil algebra homomorphism $\mu : A \to B$, the natural transformation $\mu_G : T^A G \to T^B G$ is a group homomorphism. The induced Lie algebra homomorphism is $\mathrm{id}_{\mathfrak{g}} \otimes \mu : \mathfrak{g} \otimes A \to \mathfrak{g} \otimes B$.

3.5. T^A-prolongation of actions. Let $l : G \times M \to M$ be a left action of G on a manifold M. One verifies directly that $T^A l : T^A G \times T^A M \to T^A M$ is a left action of $T^A G$ on $T^A M$. For every algebra homomorphism $\mu : A \to B$, the natural transformations $\mu_G : T^A G \to T^B G$ and $\mu_M : T^A M \to T^B M$ form a morphism of actions.

The infinitesimal action $\lambda : \mathfrak{g} \times M \to TM$ of l is defined by

$$\lambda = Tl \circ (i_G \times 0_M),$$

where $i_G : \mathfrak{g} \to TG$ is the canonical injection and $0_M : M \to TM$ is the zero section. We write $\lambda(v) = \lambda(v, -) : M \to TM$ for the fundamental vector field on M determined by $v \in \mathfrak{g}$. One finds easily that the infinitesimal action of $T^A l$, which will be denoted by $\mathcal{T}^A\lambda : T^A\mathfrak{g} \times T^A M \to TT^A M$, is of the form

$$\mathcal{T}^A\lambda = \varkappa^A_M \circ T^A\lambda .$$

Every $v \otimes a \in \mathfrak{g} \otimes A$ defines the fundamental vector field $(\mathcal{T}^A\lambda)(v \otimes a)$ on $T^A M$. On the other hand, $\lambda(v)$ is a vector field on M and we can construct its flow prolongation $\mathcal{T}^A\big(\lambda(v)\big)$. The proof of the following assertion, which relates the previous concept in an interesting way, can be found in [10].

Proposition. *We have*

$$(\mathcal{T}^A\lambda)(v \otimes a) = L(a)_M \circ \mathcal{T}^A\big(\lambda(v)\big) .$$

3.6. The linear case. Consider the case $M = V$ is a vector space. Then $TV = V \times V$ and the first component of $\lambda : \mathfrak{g} \times V \to V \times V$ is the product projection $\mathfrak{g} \times V \to V$. The second component will be denoted by

$$\overline{\lambda} : \mathfrak{g} \times V \to V .$$

Since $T^A V = V \otimes A$ is also a vector space, we have $T^A TV = V \otimes A \times V \otimes A$ and $TT^A V = V \otimes A \times V \otimes A$. Under these identifications, $\varkappa^A_V$ is the identity of $V \otimes A \times V \otimes A$.

For the infinitesimal action $\mathcal{T}^A\lambda : T^A\mathfrak{g} \times T^A V \to TT^A V$, we have

$$\overline{\mathcal{T}^A\lambda} : T^A\mathfrak{g} \times T^A V \to T^A V .$$

Then our previous results yield

Proposition. *We have*

$$\overline{\mathcal{T}^A\lambda} = T^A\overline{\lambda} : T^A\mathfrak{g} \times T^A V \to T^A V .$$

In particular, let l be a linear action of G on V, so that $\overline{\lambda}$ is the classical representation of Lie algebra $\mathfrak{g}$ on V. Hence $\overline{\lambda}$ is a bilinear map. By 2.8, $\overline{\mathcal{T}^A\lambda}$ is of the form

$$\overline{\mathcal{T}^A\lambda}(v \otimes a_1, z \otimes a_2) = \overline{\lambda}(v, z) \otimes a_1 a_2 ,$$

$v \in \mathfrak{g}$, $z \in V$, the product $a_1 a_2$ being in A.

3.7. Vector bundles. For a vector bundle $p : E \to M$, $T^A p : T^A E \to T^A M$ is also a vector bundle. If $X_1, X_2 \in T^A E$ satisfy $T^A p(X_1) = T^A p(X_2)$, we may write $X_1 = j^A\varphi_1$, $X_2 = j^A\varphi_2$ with $p \circ \varphi_1 = p \circ \varphi_2$, so that $\varphi_1(u)$ and $\varphi_2(u)$ are in the same fiber of $E \to M$ for all $u \in \mathbb{R}^k$. Then we define $X_1 + X_2$ by $j^A\big(\varphi_1(u) + \varphi_2(u)\big)$. Similarly, $c\big(j^A\varphi(u)\big) = j^A\big(c\varphi(u)\big)$, $c \in \mathbb{R}$. Further, if $\overline{p} : \overline{E} \to \overline{M}$ is another vector bundle and $f : E \to \overline{E}$ is a linear morphism over $\underline{f} : M \to \overline{M}$, then $T^A f : T^A E \to T^A\overline{E}$ is a linear morphism over $T^A\underline{f} : T^A M \to T^A\overline{M}$.

3.8. Principal and associated bundles. Let $P(M,G)$ be a principal bundle with structure group G and projection $p : P \to M$. Write $\varrho_P : P \times G \to P$ for the right action of G on P. Then $T^A p : T^A P \to T^A M$ is a principal bundle with structure group $T^A G$ and $\varrho_{T^A P} = T^A \varrho_P : T^A P \times T^A G \to T^A P$.

Consider a fiber bundle $E = P[S,l]$ associated to P with respect to an action $l : G \times S \to S$ of G on the standard fiber S and write $q : E \to M$ for the bundle projection. Then $T^A q : T^A E \to T^A M$ is an associated bundle $T^A E = T^A P[T^A S, T^A l]$.

For every fibered manifold $Y \to M$, the vertical Weil bundle $V^A Y$ is the union

$$V^A Y = \bigcup_{x \in M} T^A(Y_x)\,, \qquad V^A Y \subset T^A Y$$

of the Weil bundles of the individual fibers of Y. Clearly, $V^A P$ is a principal bundle $V^A P(M, T^A G)$ and $V^A E$ is an associated bundle $V^A E = V^A P[T^A S, T^A l]$.

3.9. Tensor fields of type $(1,k)$. A tensor field C of type $(1,k)$ on a manifold M can be considered as a map

$$C : TM \underbrace{\times_M \cdots \times_M}_{k\text{-times}} TM \to TM\,.$$

Applying functor T^A, we obtain

$$T^A C : T^A TM \times_{T^A M} \cdots \times_{T^A M} T^A TM \to T^A TM\,.$$

Using the canonical exchange $\varkappa^A_M$, we construct

$$\mathcal{T}^A C = \varkappa^A_M \circ T^A C \circ \big((\varkappa^A_M)^{-1} \times \cdots \times (\varkappa^A_M)^{-1}\big)\,.$$

This is a tensor field of type $(1,k)$ on $T^A M$, which is called the complete lift of C to $T^A M$. In the special case $k = 0$, C is a vector field on M and $\mathcal{T}^A C$ is its flow prolongation, see 2.11.

For every k vector fields $X_1, \dots, X_k$ on M, $C(X_1, \dots, X_k)$ is also a vector field. The following assertion is deduced in [7].

Proposition. *For every $a_1, \dots, a_k \in A$, we have*

$$\mathcal{T}^A C\big(L(a_1)\mathcal{T}^A X_1, \dots, L(a_k)\mathcal{T}^A X_k\big) = L(a_1 \dots a_k)\mathcal{T}^A\big(C(X_1, \dots, X_k)\big)\,.$$

3.10. The Frölicher-Nijenhuis bracket. An antisymmetric tensor field P of type $(1,k)$ on M is said to be a tangent valued k-form on M. The Frölicher-Nijenhuis bracket is an important geometric operation on the tangent valued forms, see e.g. [14]. If Q is another tangent valued l-form on M, then the Frölicher-Nijenhuis bracket $[P,Q]$ is a tangent valued $(k+l)$-form on M. A tangent valued 0-form is a vector field. If both P and Q are tangent valued 0-forms, then $[P,Q]$ coincides with the classical bracket of vector fields.

The identity of TM can be interpreted as a tangent valued 1-form on M. For every P, we have

$$\big[\mathrm{id}_{TM}, P\big] = 0\,. \tag{17}$$

Given a tangent valued form S on $T^A M$ and $a \in A$, $L(a)_M \circ S =: L(a)S$ is a tangent valued form on $T^A M$, too.

Theorem. *For every tangent valued k-form P, every tangent valued l-form Q on M and every $a_1, a_2 \in A$, we have*

$$\big[L(a_1)\mathcal{T}^A P, L(a_2)\mathcal{T}^A Q\big] = L(a_1 a_2)\mathcal{T}^A\left([P,Q]\right).$$

The proof is based on a formula by M. Modugno and P. W. Michor that expresses $[P,Q]$ in terms of the bracket of vector fields, [2].

For $a_1 = a_2 = 1$ we obtain

Corollary. *The T^A-prolongation of tangent valued forms commutes with the Frölicher-Nijenhuis bracket.*

3.11. Connections. The first jet prolongation J^1Y of a fibered manifold $p : Y \to M$ is the bundle of 1-jets of local sections of Y. The elements of J^1Y are identified with the m-dimensional linear subspaces in TY complementary to the vertical tangent space, $m = \dim M$. Hence a connection on an arbitrary fibered manifold Y can be interpreted as a section $\Gamma : Y \to J^1Y$.

The projection $T_yY \to V_yY$ in the direction of $\Gamma(y)$, $y \in Y$, defines the connection form $\omega_\Gamma : TY \to VY \subset TY$. Clearly, ω_Γ is a tangent valued 1-form on TY satisfying $\omega_\Gamma(X) = X$ for all $X \in VY$. If we denote by $i_{VY} : VY \to TY$ the injection, this can be expressed by

$$\omega_\Gamma \circ i_{VY} = i_{VY}\,. \tag{18}$$

Conversely, if ω is a tangent valued 1-form on Y satisfying $\omega(TY) \subset VY$ and (18), then the kernels of ω determine a unique connection $\Gamma = \mathrm{Ker}\ \omega$ such that $\omega_\Gamma = \omega$.

The curvature C_Γ of Γ can be identified with the Frölicher-Nijenhuis bracket, see [14],

$$C_\Gamma = \frac{1}{2}\big[\omega_\Gamma, \omega_\Gamma\big]\,.$$

The T^A-prolongation $\mathcal{T}^A\omega_\Gamma$ of ω_Γ is a tangent valued 1-form on T^AY. Taking into account $\varkappa^A_Y(T^AVY) = V(T^AY \to T^AM)$, one verifies easily

$$\mathcal{T}^A\omega_\Gamma \circ i_{V(T^AY\to T^AM)} = i_{V(T^AY\to T^AM)}\,.$$

Hence there is a unique connection $\mathcal{T}^A\Gamma$ on $T^AY \to T^AM$ such that $\mathcal{T}^A(\omega_\Gamma) = \omega_{\mathcal{T}^A\Gamma}$. It will be called the T^A-prolongation of Γ.

Corollary 3.10 yields a formula for the curvature $C_{\mathcal{T}^A\Gamma} = \frac{1}{2}\big[\omega_{\mathcal{T}^A\Gamma}, \omega_{\mathcal{T}^A\Gamma}\big]$.

Proposition. *We have $C_{\mathcal{T}^A\Gamma} = \mathcal{T}^A(C_\Gamma)$.*

3.12. Vector valued forms. Let V be a vector space. A V-valued k-form ω on M can be interpreted as a map

$$\omega : TM \times_M \cdots \times_M TM \to V\,.$$

Applying T^A and using $\varkappa^A_M$, we obtain

$$\mathcal{T}^A\omega : TT^AM \times_{T^AM} \cdots \times_{T^AM} TT^AM \to T^AV\,,$$

which is a $V \otimes A$-valued k-form on T^AM. Taking into account 3.3, one finds easily that this operation commutes with the exterior differentiation, i.e.

$$\mathcal{T}^A(d\omega) = d(\mathcal{T}^A\omega)\,.$$

We describe the coordinate form of $\mathcal{T}^A\omega$. Let $M = \mathbb{R}^m$, $V = \mathbb{R}^n$ and

$$y^p = f^p_{i_1\dots i_k}(x^1, \dots, x^m)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

be the coordinate expression of ω. We have $TT^A\mathbb{R}^m = A^m \times A^m$ and we write a^i, da^i for the corresponding algebra coordinates. The coordinate formula for $T^Af^p_{i_1\dots i_k}$ is described in 2.4. Then 2.8 implies that $\mathcal{T}^A\omega$ is of the form

$$\big(T^Af^p_{i_1\dots i_k}\big)da^{i_1} \wedge \cdots \wedge da^{i_k}$$

with all products in A.

For example, consider the tangent functor T and $n = 1$, $k = 2$, so that

$$\omega = f_{i_1i_2}(x^1, \dots, x^m)\, dx^{i_1} \wedge dx^{i_2}\,.$$

Using (12), we obtain

$$\mathcal{T}\omega = \Big(f_{i_1i_2} + e\tfrac{\partial f_{i_1i_2}}{\partial x^i}\, x^i_1\Big)(dx^{i_1} + e dx^{i_1}_1) \wedge (dx^{i_2} + e dx^{i_2}_1)$$

$$= f_{i_1i_2}dx^{i_1} \wedge dx^{i_2} + e\left(\Big(\tfrac{\partial f_{i_1i_2}}{\partial x^i}\, x^i_1\Big)dx^{i_1} \wedge dx^{i_2} + f_{i_1i_2}(dx^{i_1} \wedge dx^{i_2}_1 + dx^{i_1}_1 \wedge dx^{i_2})\right)$$

In coordinates, a tangent valued k-form P looks like a vector valued k-form. So the procedure of finding the coordinate expression of $\mathcal{T}^AP$ is the same.

3.13. Connections in the lifting form. Taking into account the projection $T_yY \to \Gamma(y)$ in the direction of V_yY, we can interpret Γ as the lifting map (denoted by the same symbol)

$$\Gamma : Y \times_M TM \to TY\,.$$

Conversely, let $\Phi : Y \times_M TM \to TY$ be a map linear in TM and such that $\pi_Y \circ \Phi = pr_1$, $Tp \circ \Phi = pr_2$, where $\pi_Y : TY \to Y$ is the bundle projection. Then there is a unique connection Γ on Y such that Φ is its lifting map.

The lifting map of $\mathcal{T}^A\Gamma$ is

$$\mathcal{T}^A\Gamma := \varkappa^A_Y \circ \big(T^A\Gamma \circ (\mathrm{id}_{T^AY} \times_{T^AM} (\varkappa^A_M)^{-1})\big)\,,$$

where $T^A\Gamma : T^AY \times_{T^AM} T^ATM \to T^ATY$.

The algebra multiplication enables us to evaluate the equations of $\mathcal{T}^A\Gamma$ in the following simple way, [3]. If we have the product bundle $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ with the canonical coordinates x^i, y^p, then the equations of Γ are

$$dy^p = F^p_i(x,y)\,dx^i\,. \tag{19}$$

The Weil bundle $T^A(\mathbb{R}^m \times \mathbb{R}^n) \to T^A\mathbb{R}^m$ is $A^m \times A^n \to A^m$ with the algebra coordinates a^i, b^p. Then $\mathcal{T}^A(dx^i) = da^i$ and $\mathcal{T}^A(dy^p) = db^p$ are the additional coordinates on $TT^A(\mathbb{R}^m \times \mathbb{R}^m)$. Applying T^A to (19), we obtain the equations of $\mathcal{T}^A\Gamma$ in the form

$$db^p = (T^A F^p_i)\,da^i\,,$$

where the A-valued functions $T^A F^p_i$ can be evaluated by (11) and the products on the right hand side are in A.

For example, consider the tangent functor T. By (12),

$$TF^p_i = F^p_i + e\Big(\tfrac{\partial F^p_i}{\partial x^j}\,x^j_1 + \tfrac{\partial F^p_i}{\partial y^q}\,y^q_1\Big)$$

and we have $da^i = dx^i + e dx^i_1$, $db^p = dy^p + e dy^p_1$. Then

$$(dy^p + e dy^p_1) = \Big(F^p_i + e\Big(\tfrac{\partial F^p_i}{\partial x^j}\,x^j_1 + \tfrac{\partial F^p_i}{\partial y^q}\,y^q_1\Big)\Big)\,(dx^i + e dx^i_1)$$

implies that the equations of $\mathcal{T}\Gamma$ are (19) and

$$dy^p_1 = \Big(\tfrac{\partial F^p_i}{\partial x^j}\,x^j_1 + \tfrac{\partial F^p_i}{\partial y^q}\,y^q_1\Big)\,dx^i + F^p_i\,dx^i_1\,.$$

Of course, the aim of this simplest example is to illustrate the procedure. The power of the Weil algebra technique appears properly if we consider a more complicated algebra A.

3.14. Curvature in the lifting form. Consider Γ in the lifting form. The Γ-lift of a vector field X on M is a projectable vector field ΓX on Y defined by $\Gamma X = \Gamma(-, X)$. Then the curvature C_Γ is identified with a morphism

$$C_\Gamma : Y \times_M TM \times_M TM \to VY \tag{20}$$

defined as follows. Two vectors $\xi_1, \xi_2 \in T_{p(y)}M$ are extended into some vector fields X_1, X_2 on M and

$$C_\Gamma(y,\xi_1,\xi_2) = [\Gamma X_1, \Gamma X_2](y) - \Gamma\big([X_1,X_2]\big)(y)\,.$$

Using the lifting form of Γ, we construct

$$T^A C_\Gamma : T^A Y \times_{T^A M} T^A TM \times_{T^A M} T^A TM \to T^A VY\,.$$

Then we have the following characterization of the curvature of $\mathcal{T}^A\Gamma$.

Proposition. *If C_Γ is in the form (20), then*

$$C_{\mathcal{T}^A\Gamma} = \varkappa^A_Y \circ T^A C_\Gamma \circ \big(\mathrm{id}_{T^A Y} \times_{T^A M} (\varkappa^A_M)^{-1} \times_{T^A M} (\varkappa^A_M)^{-1}\big)\,.$$

3.15. Principal and linear connections. Connections in the sense of 3.11 represent a generalization of the classical concept of connection. In the classical theories, connections are studied on principal bundles and are assumed to be right-invariant. Nowadays, a right-invariant connection on P is said to be principal. Similarly, if $E \to M$ is a vector bundle, so that $J^1E \to M$ is also a vector bundle, a connection $\Gamma : E \to J^1E$ is said to be linear, if Γ is a linear morphism.

Consider a connection Γ on a principal bundle $P(M, G)$. Each fiber of $VP \to P$ is identified with $\mathfrak{g}$ in the standard way. Hence the connection form ω_Γ of Γ can be interpreted as a 1-form on P with values in the vector space $\mathfrak{g}$. If Γ is principal, then $\omega_\Gamma : TP \to \mathfrak{g}$ is the classical connection form of Γ. One deduces easily

Proposition. *If Γ is a principal connection on P with connection form $\omega_\Gamma : TP \to \mathfrak{g}$, then $\mathcal{T}^A\Gamma$ is a principal connection on $T^AP \to T^AM$ with connection form $\omega_{\mathcal{T}^A\Gamma} = \mathcal{T}^A(\omega_\Gamma)$.*

Analogously, let Γ be a linear connection on a vector bundle $E \to M$. Then the induced connection $\mathcal{T}^A\Gamma$ on $T^AE \to T^AM$ is also linear.

3.16. a-torsions. The Frölicher-Nijenhuis bracket can be applied in the theory of torsions of connections on a Weil bundle T^AM. This was pointed out in [17]. We present the basic ideas in a slightly more general setting.

For every $a \in A$, $L(a)_M$ is a natural tangent valued 1-form on T^AM. Consider a natural fibered manifold $T^AM \to Q$. (For example, Q is M or one of the underlying Weil bundles from 2.13. In the case of a fibered manifold $Y \to M$, we can consider $T^AY \to T^AM$.) Then every connection Γ on $T^AM \to Q$ can be viewed as a tangent valued 1-form ω_Γ on T^AM.

Definition. The Frölicher-Nijenhuis bracket $\big[L(a)_M, \omega_\Gamma\big]$ is called a-torsion of Γ.

Since $L(1)_M = \mathrm{id}_{TT^AM}$ and $[\mathrm{id}, \omega_\Gamma] = 0$ by (17), only the elements from the nilpotent part of A are interesting.

We have to explain how our definition generalizes the classical notion of torsion. The Weil algebra of the tangent functor T is $\mathbb{D}$ with one-dimensional nilpotent part generated by e, see 1.4. Let Γ be a classical linear connection on $TM \to M$. Using simple evaluation, one deduced that $\big[L(e)_M, \omega_\Gamma\big]$ is the classical torsion of Γ.

Consider the T^A-prolongation $\mathcal{T}^A\Gamma$ of a connection Γ on $Y \to M$. Then $\mathcal{T}^A\Gamma$ is torsion free in the following sense.

Proposition. *We have $\big[L(a)_Y, \omega_{\mathcal{T}^A\Gamma}\big] = 0$ for all $a \in A$.*

Proof. We have $L(a)_Y = L(a) \circ \mathrm{id}_{TT^AY}$. Hence Theorem 3.10 and (17) imply $\big[L(a)_Y, \omega_{\mathcal{T}^A\Gamma}\big] = L(a)\mathcal{T}^A\big([\mathrm{id}_{TY}, \omega_\Gamma]\big) = 0$.

3.17. Some further structures. The first paper dealing systematically with T^A-prolongation of various geometric structures is by A. Morimoto, [21]. Beside some subjects we already mentioned, he studied T^A-prolongation of almost complex structures and the prolongation of a classical linear connection on M into a classical linear connection on $T^A M$.

Even the paper [7] by J. Gancarzewicz, W. Mikulski and Z. Pogoda is devoted, beside the general theory, to T^A-prolongation of further geometric structures. Main attention is paid to Riemannian and pseudo-Riemannian metrics, symplectic and almost symplectic structures, almost tangent structures and Kählerian structures.

3.18. Remark. All results of this paper are valid for an arbitrary Weil algebra. However, we have to point out that there are further geometric problems, in which certain special kinds of Weil algebras are specified. We mention solely a paper by M. Kureš and W. Mikulski, [19], on the natural operators transforming vector fields from a manifold M into vector fields on the bundle $K^A M$ of contact A-elements introduced in 2.15. In [19], the best results are deduced for homogeneous Weil algebras. We recall that an ideal $I \subset \mathbb{R}[x_1, \dots, x_k]$ is said to be homogeneous, if $P \in I$ implies that all homogeneous components of polynomial P are also in I. A Weil algebra is called homogeneous, if it can be expressed in the form $\mathbb{R}[x_1, \dots, x_k]/I$ with a homogeneous ideal I. Clearly, all algebras $\mathbb{D}^r_k$ and their tensor products are homogeneous. Examples of nonhomogeneous Weil algebras are constructed in [19].

4. Fiber product preserving bundle functors

4.1. Jet functors. There are 3 bundle functors, whose construction is based on the concept of r-jet only.

For every two manifolds M and N, $J^r(M, N)$ is the bundle of all r-jets of M into N. Let $f : M \to \overline{M}$ be a local diffeomorphism and $g : N \to \overline{N}$ be a map. Then the induced map $J^r(f, g) : J^r(M, N) \to J^r(\overline{M}, \overline{N})$ is defined by

$$J^r(f, g)(X) = (j^r_y g) \circ X \circ (j^r_x f)^{-1} ,$$

where x or y is the source or the target of $X \in J^r(M, N)$. Hence J^r is a bundle functor defined on the product category $\mathcal{M}f_m \times \mathcal{M}f$, $m = \dim M$.

For every fibered manifold $p : Y \to M$, $J^r Y$ is the bundle of r-jets of local sections of Y. If $\overline{p} : \overline{Y} \to \overline{M}$ is another fibered manifold and $f : Y \to \overline{Y}$ is an $\mathcal{FM}$-morphism such that the base map $\underline{f} : M \to \overline{M}$ is a local diffeomorphism, then the map $J^r(\underline{f}, f) : J^r(M, Y) \to J^r(\overline{M}, \overline{Y})$ transforms $J^r Y$ into $J^r \overline{Y}$. The restricted and corestricted map $J^r f : J^r Y \to J^r \overline{Y}$ is called the r-th jet prolongation of f. Hence J^r is a functor on the category $\mathcal{FM}_m$. In concrete problems, it is always clear which functor J^r is under consideration and we have no intention to change this convention. But in this theoretical chapter we have to distinguish. So, in this chapter we shall write J^r_h in the case of $\mathcal{FM}_m$.

The r-th vertical jet prolongation $J^r_v Y$, which is used e.g. in the theory of higher order absolute differentiation, is defined by

$$J^r_v Y = \bigcup_{x\in M} J^r_x(M, Y_x)\,.$$

The restriction and corestriction of $J^r(\underline{f}, f)$ defines $J^r_v f : J^r_v Y \to J^r_v \overline{Y}$. Hence J^r_v is a bundle functor on $\mathcal{FM}_m$. In this context, we also say that $J^r_h Y$ is the r-th horizontal jet prolongation of Y.

The construction of product fibered manifolds is a functor $i : \mathcal{M}f_m \times \mathcal{M}f \to \mathcal{FM}_m$, $i(M\times N) = (M\times N \to M)$ and $i(f\times g)$ is $f\times g$ with the base map f. We have

$$J^r_h(M\times N) = J^r(M,N) = J^r_v(M\times N)$$

and both functors $J^r_h \circ i$ and $J^r_v \circ i$ coincide with J^r.

4.2. Nonholonomic jets. The r-th nonholonomic prolongation $\widetilde{J}^r_h Y$ of a fibered manifold Y is defined by the iteration

$$\widetilde{J}^r_h Y = J^1_h(\widetilde{J}^{r-1}_h Y \to M)\,,$$

$\widetilde{J}^1_h Y = J^1_h Y$. For every $f : Y \to \overline{Y}$ in $\mathcal{FM}_m$, the iteration determines $\widetilde{J}^r_h f = J^1_h(\widetilde{J}^{r-1}_h f) : \widetilde{J}^r_h Y \to \widetilde{J}^r_h \overline{Y}$. Hence $\widetilde{J}^r_h$ is a bundle functor on $\mathcal{FM}_m$. The canonical inclusion $J^r_h Y \hookrightarrow \widetilde{J}^r_h Y$ is defined by the iteration $j^r_x s \mapsto j^1_x(u \mapsto j^{r-1}_u s)$ for every local section s of Y, $u\in M$.

The restriction $\widetilde{J}^r_h \circ i$ yields a functor $\widetilde{J}^r$ on $\mathcal{M}f_m \times \mathcal{M}f$. The space

$$\widetilde{J}^r(M,N) = \widetilde{J}^r_h(M\times N \to M)$$

is the bundle of nonholonomic r-jets of manifold M into manifold N defined by Ehresmann. The elements of $J^r(M,N) \subset \widetilde{J}^r(M,N)$ are also said to be holonomic r-jets. Let Q be a third manifold. Ehresmann introduced the composition of nonholonomic r-jets by the following induction. For $r=1$, we have the composition of 1-jets. Write $\beta : \widetilde{J}^{r-1}(M,N) \to N$ for the canonical projection. Let $X = j^1_x s(u) \in \widetilde{J}^r_x(M,N)_y$, $u\in M$, and $Z = j^1_y \sigma \in \widetilde{J}^r_y(N,Q)_z$, $y = \beta\big(s(x)\big)$. Then

$$Z\circ X := j^1_x\big(\sigma\big(\beta(s(u))\big)\circ s(u)\big) \in \widetilde{J}^r_x(M,Q)_z \tag{21}$$

with the composition of nonholonomic $(r-1)$-jets on the right hand side. If X and Z are holonomic r-jets, then (21) coincides with the classical composition. The composition of nonholonomic r-jets is associative.

The r-th vertical nonholonomic prolongation of Y is defined by

$$\widetilde{J}^r_v Y = \bigcup_{x\in M} \widetilde{J}^r_x(M, Y_x)\,.$$

Even this is a bundle functor on $\mathcal{FM}_m$. Similarly to the holonomic case, we have $\widetilde{J}^r_h \circ i = \widetilde{J}^r = \widetilde{J}^r_v \circ i$.

4.3. Bundle functors in the product case. We start with some properties of bundle functors on $\mathcal{M}f \times \mathcal{M}f$ that are needed for the main subject of this chapter. First we introduce some notation.

Let F be a bundle functor on $\mathcal{M}f \times \mathcal{M}f$. We write $F_x(M,N)$ or $F_x(M,N)_y$ for the submanifold of all elements of $F(M,N)$ over $x \in M$ or $(x,y) \in M \times N$, respectively. For $g : M \to \overline{M}$ and $f : N \to \overline{N}$, we write $F_x(g,f) : F_x(M,N) \to F_{g(x)}(\overline{M},\overline{N})$ and $F_x(g,f)_y : F_x(M,N)_y \to F_{g(x)}(\overline{M},\overline{N})_{f(y)}$ for the restricted and corestricted maps.

Definition. We say that F preserves products in the second factor, if $F(M, N_1 \times N_2) = F(M,N_1) \times_M F(M,N_2)$. We say that F has order r in the first factor, if for every $g, \overline{g} : M \to \overline{M}$ and $f : N \to \overline{N}$, $j^r_x g = j^r_x \overline{g}$ implies

$$F_x(g,f) = F_x(\overline{g},f) : F_x(M,N) \to F_{g(x)}(\overline{M},\overline{N}).$$

Clearly, if F has order r in the standard sense, i.e. $j^r_{x,y}(g,f) = j^r_{x,y}(\overline{g},\overline{f})$ implies $F_x(g,f)_y = F_x(\overline{g},\overline{f})_y$, then F has order r in the first factor.

4.4. The case of $\mathcal{M}f_m \times \mathcal{M}f$. Let F be a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ that preserves products in the second factor. We define an associated bundle functor G^F on $\mathcal{M}f$ by $G^F(N) = F_0(\mathbb{R}^m, N)$ and $G^F(f) = F_0(\mathrm{id}_{\mathbb{R}^m}, f) : G^F N \to G^F \overline{N}$. Clearly, G^F preserves products, so that $G^F = T^A$ for a Weil algebra A.

Assume further that F has order r in the first factor. For $X \in G^r_m$, $X = j^r_0 \gamma$ and every manifold N, we set

$$H^F(X)_N = F_0(\gamma, \mathrm{id}_N) : G^F N \to G^F N.$$

Since the following diagram commutes

$$\begin{tikzcd}
(\mathbb{R}^m, N) \arrow[r, "(\mathrm{id}_{\mathbb{R}^m}, f)"] \arrow[d, "(\gamma, \mathrm{id}_N)"'] & (\mathbb{R}^m, \overline{N}) \arrow[d, "(\gamma, \mathrm{id}_{\overline{N}})"] \\
(\mathbb{R}^m, N) \arrow[r, "(\mathrm{id}_{\mathbb{R}^m}, f)"] & (\mathbb{R}^m, \overline{N})
\end{tikzcd}$$

each $H^F(X)$ is a natural equivalence $T^A \to T^A$. By 2.7, $H^F(X)$ corresponds to an element of Aut A. Clearly, $H^F : G^r_m \to \mathrm{Aut}\, A$ is a group homomorphism.

Conversely, consider a Weil algebra A and a group homomorphism $H : G^r_m \to \mathrm{Aut}\, A$. For every manifold N, we have the induced left action H_N of G^r_m on $T^A N$, so that we can construct the associated bundle $P^r M[T^A N, H_N] =: (A,H)(M,N)$ for every m-dimensional manifold M. We underline that the elements of $P^r M[T^A N, H_N]$ are the equivalence classes

$$\{u, Z\}, \qquad u \in P^r M,\ Z \in T^A N.$$

For every local diffeomorphism $g : M \to \overline{M}$, we have the induced morphism $P^r g : P^r M \to P^r \overline{M}$ of principal bundles and every map $f : N \to \overline{N}$ induces a G_m^r-equivariant map $T^A f : T^A N \to T^A \overline{N}$. So we can construct the morphism of associated bundles $(A, H)(g, f) := P^r g[T^A f] : P^r M[T^A N, H_N] \to P^r \overline{M}[T^A \overline{N}, H_{\overline{N}}]$. Clearly, (A, H) is a functor. Thus, we have proved

Proposition. *The above construction establishes a bijection between the bundle functors on $\mathcal{M}f_m \times \mathcal{M}f$ that preserve products in the second factor and have order r in the first factor and the pairs (A, H) of a Weil algebra A and a group homomorphism $H : G_m^r \to \mathrm{Aut}\, A$.*

In particular, if $M = \mathbb{R}^m$, then $P^r \mathbb{R}^m = \mathbb{R}^m \times G_m^r$ and the associated bundle $(A, H)(\mathbb{R}^m, N)$ is identified with $\mathbb{R}^m \times T^A N$. Given $f : N \to \overline{N}$, we have

$$(A, H)(\mathrm{id}_{\mathbb{R}^m}, f) = \mathrm{id}_{\mathbb{R}^m} \times T^A f : \mathbb{R}^m \times T^A N \to \mathbb{R}^m \times T^A \overline{N} .$$

If $\overline{F} = (\overline{A}, \overline{H})$ is another bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ of order r in the first factor, then the natural transformations $\tau : F \to \overline{F}$ are in bijection with the equivariant algebra homomorphisms $\mu : A \to \overline{A}$, i.e. $\mu\big(H(X)(a)\big) = \overline{H}(X)\big(\mu(a)\big)$ for all $a \in A$ and $X \in G_m^r$. We have

$$\tau_{M,N} = (\mathrm{id}_{P^r M}, \mu_N) : P^r M[T^A N, H_N] \to P^r M[T^{\overline{A}} N, \overline{H}_N] .$$

In the case $M = \mathbb{R}^m$, $\tau_{\mathbb{R}^m, N} = \mathrm{id}_{\mathbb{R}^m} \times \mu_N : \mathbb{R}^m \times T^A N \to \mathbb{R}^m \times T^{\overline{A}} N$.

4.5. The algebra $\widetilde{\mathbb{D}}_m^r$. We describe the functor $\widetilde{J}^r$ in this way. The space $\widetilde{T}_m^r N := \widetilde{J}_0^r(\mathbb{R}^m, N)$ is called bundle of nonholonomic (m, r)-velocities on N. Using the translations on $\mathbb{R}^m$, one identifies $\widetilde{T}_m^r N$ with the iteration $T_m^1 \underbrace{(\dots}_{r\text{-times}} (T_m^1 N) \dots)$. Since the Weil algebra of T_m^1 is $\mathbb{D}_m^1$, 2.10 implies that the Weil algebra $\widetilde{\mathbb{D}}_m^r$ of $\widetilde{J}^r$ is $\mathbb{D}_m^1 \underbrace{\otimes \dots \otimes}_{r\text{-times}} \mathbb{D}_m^1$. The action of $G_m^r = \mathrm{inv}\, J_0^r(\mathbb{R}^m, \mathbb{R}^m)_0$ on $\widetilde{\mathbb{D}}_m^r$ is given by the composition of nonholonomic jets.

4.6. The base order of F. Let F be a bundle functor on $\mathcal{FM}$. The definition of the order of F is based on the concept of (q, s, r)-jet, see 2.17. We say that F is of order (q, s, r), $s \geq q \leq r$, if for every $\mathcal{FM}$-morphism $f : Y \to \overline{Y}$, the restriction $Ff \mid F_y Y$ depends on $j_y^{q,s,r} f$ only, $y \in Y$.

The integer r is called the base order of F.

4.7. The main result. Let F be a fiber product preserving bundle (in short: f.p.p.b.) functor on $\mathcal{FM}_m$, i.e. $F(Y_1 \times_M Y_2) = FY_1 \times_M FY_2$ for every two fibered manifolds Y_1 and Y_2 over the same base M, $\dim M = m$. Its restriction $F \circ i$ to $\mathcal{M}f_m \times \mathcal{M}f$ preserves products in the second factor. In [15], it is deduced that F has finite order. The base order r of F coincides with the order of $F \circ i$ in the second factor. By 4.4, $F \circ i = (A, H)$, so that $F(M \times N) = P^r M[T^A N, H_N]$.

Further, F determines a natural transformation $\widetilde{t}_Y : J^r_h Y \to FY$. Every element $X \in J^r_h Y$ is of the form $j^r_x s$. We interpret the local section s of Y as a local $\mathcal{FM}_m$-morphism $\widetilde{s}$ of the trivial fibered morphism $\mathrm{id}_M : M \to M$ (denoted by $i_1 M$ in 2.16) into Y and we set

$$\widetilde{t}_Y(X) = (F\widetilde{s})(x) \in FY\,. \tag{22}$$

In the product case $Y = \mathbb{R}^m \times N$, we have $J^r_h(\mathbb{R}^m \times N) = \mathbb{R}^m \times T^r_m N$ and $F(\mathbb{R}^m \times N) = \mathbb{R}^m \times T^A N$. Write

$$t_N : T^r_m N \to T^A N$$

for the restricted and corestricted map over $0 \in \mathbb{R}^m$. This is a natural transformation, so that it corresponds to an algebra homomorphism $t : \mathbb{D}^r_m \to A$. By naturality of $\widetilde{t}$, t is a G^r_m-equivariant algebra homomorphism, i.e.

$$t(X \circ g) = H(g)\big(t(X)\big)\,, \qquad X \in \mathbb{D}^r_m\,,\ g \in G^r_m\,.$$

The inclusion $Y \hookrightarrow M \times Y$, $y \mapsto \big(p(x), y\big)$ is a base preserving morphism. Applying F, we obtain an inclusion

$$FY \hookrightarrow F(M \times Y) = P^r M[T^A Y, H_Y]\,.$$

Since $P^r M \subset T^r_m M$, t defines a natural map (denoted by the same symbol) $t_M : P^r M \to T^A M$. According to [15], FY is the space of all equivalence classes $\{u, Z\}$ satisfying $t_M(u) = T^A p(Z)$, $u \in P^r M$, $Z \in T^A Y$.

Conversely, such a triple (A, H, t) defines a bundle functor $F = (A, H, t)$ on $\mathcal{FM}_m$ by

$$FY = \big\{\{u, Z\}; u \in P^r M, Z \in T^A Y, t_M(u) = T^A p(Z)\big\}\,. \tag{23}$$

For an $\mathcal{FM}_m$-morphism $f : Y \to \overline{Y}$ over $\underline{f} : M \to \overline{M}, (A, H)(\underline{f}, f) = P^r \underline{f}[T^A f]$ maps FY into $F\overline{Y}$ and Ff is its restriction and corestriction. This implies

Theorem. *The f.p.p.b. functors of base order r on $\mathcal{FM}_m$ are in bijection with the triples (A, H, t), where A is a Weil algebra, $H : G^r_m \to \mathrm{Aut}\ A$ is a group homomorphism and $t : \mathbb{D}^r_m \to A$ is an equivariant algebra homomorphism. The natural transformations $(A, H, t) \to (\overline{A}, \overline{H}, \overline{t})$ are in bijection with the equivariant algebra homomorphisms $\mu : A \to \overline{A}$ satisfying $\overline{t} = \mu \circ t$.*

The second assertion follows from the fact that $\overline{t} = \mu \circ t$ implies that the natural transformation $(A, H) \to (\overline{A}, \overline{H})$ maps $(A, H, t)(Y)$ into $(\overline{A}, \overline{H}, \overline{t})(Y)$.

If we use the inclusions $J^r_h Y \subset P^r M[T^r_m Y]$ and $FY \subset P^r M[T^A Y]$, then the map $\widetilde{t}_Y : J^r_h Y \to FY$ is of the form

$$\widetilde{t}_Y\big(\{u, X\}\big) = \big\{u, t_Y(X)\big\}\,. \tag{24}$$

4.8. The basic examples. The simplest examples of f.p.p.b. functors on $\mathcal{FM}_m$ are J^r_h, J^r_v and V^A. So every iteration of these functors is also a f.p.p.b. functor on $\mathcal{FM}_m$.

By 1.9, Aut $\mathbb{D}^r_m = G^r_m$. Write C for the corresponding action of G^r_m on $\mathbb{D}^r_m$. One finds easily $J^r_h = (\mathbb{D}^r_m, C, \mathrm{id}_{\mathbb{D}^r_m})$ and $J^r_v = (\mathbb{D}^r_m, C, \mathcal{O})$, where $\mathcal{O} : \mathbb{D}^r_m \to \mathbb{D}^r_m$ is the zero homomorphism. Functor V^A has base order 0. In this case, $G^0_m = \{e\}$ is the one element group, $\mathbb{D}^0_m = \mathbb{R}$ and H maps e into $\mathrm{id}_{\mathbb{R}}$.

4.9. The iteration. Let $E = (B, K, u)$ be another f.p.p.b. functor on $\mathcal{FM}_m$ of base order s, so that $K : G^s_m \to \mathrm{Aut}\, B$ and $u : \mathbb{D}^s_m \to B$. The composition $F \circ E$ preserves fiber products as well. In [6], the following expression $F \circ E = (C, L, v)$ is deduced. Clearly, the Weil algebra is $C = B \otimes A$.

Then we construct the group homomorphism $L : G^{r+s}_m \to \mathrm{Aut}\,(B \otimes A)$. Write $\iota^{r,s}_m : G^{r+s}_m \to T^r_m G^s_m$ for the map

$$j^{r+s}_0 \gamma \mapsto j^s_0\big(y \mapsto j^r_0(\gamma \circ \tau_y)\big)\,, \quad y \in \mathbb{R}^m\,,$$

where τ_y is the translation on $\mathbb{R}^m$ transforming 0 into y. For every $g \in G^{r+s}_m$, we denote by $\beta_r(g) \in G^r_m$ the underlying r-jet. We interpret K or L as a map $K : G^s_m \times B \to B$ or $L : G^{r+s}_m \times T^A B \to T^A B$, respectively. Then $T^A K : T^A G^s_m \times T^A B \to T^A B$. By 3.4, t induces a group homomorphism $t_{G^s_m} : T^r_m G^s_m \to T^A G^s_m$. According to [6], we have

$$L(g, Z) = H\big(\beta_r(g)\big)_B\big(T^A K(t_{G^s_m}(\iota^{r+s}_m(g)), Z)\big)\,, \quad Z \in T^A B\,.$$

To find the algebra homomorphism $v : \mathbb{D}^{r+s}_m \to B \otimes A$, we consider the injection $i^{r,s}_m : \mathbb{D}^{r+s}_m \to T^r_m \mathbb{D}^s_m$,

$$j^{r+s}_0 \gamma \mapsto j^r_0\big(y \mapsto j^s_0(\gamma \circ \tau_{\gamma(y)})\big)\,, \quad y \in \mathbb{R}^m\,.$$

Then we construct $T^r_m u : T^r_m \mathbb{D}^s_m \to T^r_m B$ and $t_B : T^r_m B \to T^A B$. By [6], we have

$$v = t_B \circ T^r_m u \circ i^{r,s}_m\,.$$

4.10. Applications. We present two simple geometric applications of 4.9. Consider the vertical Weil functor V^B in the role of E. The base order of both iterations $V^B F$ and $F V^B$ is r. In the first or the second case, the action of G^r_m on $A \otimes B$ or $B \otimes A$ is $H \otimes \mathrm{id}_B$ or $\mathrm{id}_B \otimes H$ and the algebra homomorphism $\mathbb{D}^r_m \to A \otimes B$ or $\mathbb{D}^r_m \to B \otimes A$ is $t \otimes \mathrm{id}_B$ or $\mathrm{id}_B \otimes t$, respectively. Hence the exchange algebra homomorphism $\mathrm{ex} : A \otimes B \to B \otimes A$ is equivariant and satisfies $\mathrm{ex} \circ (t \otimes \mathrm{id}_B) = \mathrm{id}_B \otimes t$. Thus ex determines a canonical natural equivalence

$$\varkappa^{B,F}_Y : V^B(FY) \to F(V^B Y)\,. \tag{25}$$

The special case $F = J^r_h$ and $V^B = V$ is heavily used e.g. in the variational calculus on fibered manifolds.

Another result, which is proved in [6], clarifies certain "rigidity" properties of jet functors.

Proposition. *The only natural transformation $J^r_h J^s_h \to J^r_h J^s_h$ is the identity.*

In particular, for $r = s = 1$ we obtain that there is no natural exchange map $\widetilde{J}^2_h Y \to \widetilde{J}^2_h Y$.

4.11. The general concept of jet functor. Several special kinds of nonholonomic r-jets are known. Ehresmann himself defined the r-th semiholonomic jet prolongation $\overline{J}^r_h Y$ of a fibered manifold $Y \to M$ by induction starting with $\overline{J}^1_h Y = J^1_h Y$. Then $\overline{J}^r_h Y$ is the space of 1-jets $j^1_x s$, where s is a local section of $\overline{J}^{r-1}_h Y \to M$ satisfying $s(x) = j^1_x(\beta_{r-1} \circ s)$, $\beta_{r-1} : \overline{J}^{r-1}_h Y \to \overline{J}^{r-2}_h Y$ being the canonical projection. Clearly, we have $J^r_h Y \subset \overline{J}^r_h Y$. In the product case, $\overline{J}^r_h(M \times N) =: \overline{J}^r(M, N)$ is the bundle of semiholonomic r-jets of M into N. The composition of two semiholonomic r-jets in the sense of 4.2 is a semiholonomic r-jet as well. We can also construct the r-th vertical semiholonomic jet prolongation of Y by

$$\overline{J}^r_v Y = \bigcup_{x \in M} \overline{J}^r_x(M, Y_x)\,.$$

Using the viewpoint of Weil bundles, we introduced the general concept of r-th order jet functor, [11]. This is a bundle functor G on $\mathcal{M}f_m \times \mathcal{M}f$ satisfying

$$J^r \subset G \subset \widetilde{J}^r$$

and preserving products in the second factor. By 4.4, for the corresponding Weil algebra A we have $\mathbb{D}^r_m \subset A \subset \widetilde{\mathbb{D}}^r_m$, so that there is a canonical action C_A of G^r_m on A. Conversely, every Weil algebra A with this property defines an r-th order jet functor G. Using 4.7, we can construct its horizontal version $G_h = (A, C_A, i)$, where $i : \mathbb{D}_m \to A$ is the above injection, and the vertical version $G_v = (A, C_A, \mathcal{O})$, where $\mathcal{O} : \mathbb{D}^r_m \to A$ is the zero homomorphism. We have $J^r_h \subset G_h \subset \widetilde{J}^r_h$ and $J^r_v \subset G_v \subset \widetilde{J}^r_h$.

4.12. Remark. The f.p.p.b. functors were studied on the category $\mathcal{FM}_m$ because of the relations to jet bundles. Of course, one can be also interested in the f.p.p.b. functors on the whole category $\mathcal{FM}$. Even these functors can be characterized in terms of Weil algebras. The first results on this subject are presented in a recent paper [16].

5. Some applications

5.1. F-prolongation of vector bundles. Consider a fibered manifold $p : C \to M$ and a Lie group K. We say that C is a group bundle of type K, if each fiber is a Lie group and for every $x \in M$ there exists a neighbourhood U such that $p^{-1}(U) = U \times K$. The group compositions form a base preserving morphism $\nu : C \times_M C \to C$. For every f.p.p.b. functor $F = (A, H, t)$, $F\nu : FC \times_M FC \to FC$ endows $FC \to M$ with the structure of group bundle of type $T^A K$. If $\overline{C} \to \overline{M}$ is another group bundle, an $\mathcal{FM}$-morphism $f : C \to \overline{C}$ is called group bundle morphism, if its restriction to each fiber is a group homomorphism. If $f : C \to \overline{C}$ is a group bundle morphism with local diffeomorphism as base map, then $Ff : FC \to F\overline{C}$ is also a group bundle morphism.

In particular, if $E \to M$ is a vector bundle, then FE is a bundle of Abelian groups. The multiplication of the elements of E by reals can be interpreted as a base preserving morphism $\sigma : (M \times \mathbb{R}) \times_M E \to E$. By 4.7, $F(M \times \mathbb{R}) = P^r M[A, H]$. Hence

$$F\sigma : P^r M[A, H] \times_M FE \to FE\,.$$

But $\mathbb{R} \subset A$ is an H-invariant subspace, so that $M \times \mathbb{R}$ is a subbundle of $P^r M[A, H]$. This defines the multiplication by reals on FE and one verifies easily that $FE \to M$ is a vector bundle, too. Clearly, the bundle projection $FE \to E$ is a linear morphism. Further, if $\overline{E} \to \overline{M}$ is another vector bundle and $f : E \to \overline{E}$ is a linear morphism with local diffeomorphism as base map, then $Ff : FE \to F\overline{E}$ is also a linear morphism.

5.2. The flow natural map. In the case of a f.p.p.b. functor $F = (A, H, t)$ on $\mathcal{FM}_m$, we have the following analogy of the flow natural exchange map from 2.11. First consider a vector field ξ on M. Its flow prolongation $\mathcal{P}^r\xi$ is a right-invariant vector field on r-th order frame bundle $P^r M$, whose value at every $u \in P^r_x M$ depends on $j^r_x\xi$ only. This defines a map $i : P^r M \times_M J^r TM \to TP^r M$.

For a fibered manifold $p : Y \to M$, we shall consider TY as a fibered manifold $TY \to M$. Then $Tp : TY \to TM$ is a base preserving morphism that induces $FTp : FTY \to FTM$. Taking into account the natural transformation $\widetilde{t}_{TM} : J^r TM \to FTM$ from 4.7, we construct the fiber product

$$J^r TM \times_{FTM} FTY\,. \tag{26}$$

By 4.7, we have $FTY \subset P^r M[T^A TY]$. Consider $(X, \{u, Z\})$ from (26), $X \in J^r_x TM$, $u \in P^r_x M$, $Z \in T^A TY$. Write $i(u, X) = (\partial/\partial t)_0\gamma(t)$, $\gamma : \mathbb{R} \to P^r M$. By (26), $\varkappa^A_Y(Z) \in TT^A Y$ can be expressed as $(\partial/\partial t)_0\zeta(t)$, where $\zeta : \mathbb{R} \to T^A Y$ satisfies $t_M\big(\gamma(t)\big) = T^A p\big(\zeta(t)\big)$ for all t. So $\{\gamma(t), \zeta(t)\}$ is a curve on FY and we define

$$\psi^F_Y\big(X, \{u, Z\}\big) = \tfrac{\partial}{\partial t}\big|_0\big\{\gamma(t), \zeta(t)\big\}\,.$$

By right-invariancy, this is independent of the choice of u. Hence we obtain a map

$$\psi^F_Y : J^r TM \times_{FTM} FTY \to TFY\,.$$

By 5.1, FTY is a vector bundle over $FY \times_M FTM$ and one finds easily that ψ^F_Y is linear in both $J^r TM$ and FTY.

A projectable vector field η on Y over ξ on M can be interpreted as a base preserving morphism $\eta : Y \to TY$. Then we construct the functorial prolongation $F\eta : FY \to FTY$ as well as the r-th jet prolongation $j^r\xi : M \to J^r TM$. The values of $j^r\xi \times_{\mathrm{id}_M} F\eta$ are in (26). The proof of the following assertion can be found in [10].

Proposition. *The flow prolongation $\mathcal{F}\eta$ of η satisfies*

$$\mathcal{F}\eta = \psi^F_Y \circ (j^r\xi \times_{\mathrm{id}_M} F\eta)\,.$$

It is useful to introduce also a modified map

$$\widetilde{\psi}^F_Y : J^rTM \times_{FTM} FTY \to J^rTM \times_{TM} TFY\,,$$
$$\widetilde{\psi}^F_Y(X, \{u, Z\}) = \big(X, \psi^F_Y(X, \{u, Z\})\big)\,.$$

One finds easily that $\widetilde{\psi}^F_Y$ is a diffeomorphism.

In the case $F = J^r$, we have $FTM = J^rTM$, so that

$$\psi^{J^r}_Y : J^rTY \to TJ^rY\,.$$

This map was constructed in another way by L. Mangiarotti and M. Modugno.

We present a local expression of ψ^F_Y. We take $Y = \mathbb{R}^m \times V$, where we may assume V is a vector space. The group homomorphism $H : G^r_m \to \operatorname{Aut} A$ induces a Lie algebra homomorphism $h : \mathfrak{g}^r_m \to \operatorname{Der} A$. We have $J^rT\mathbb{R}^m = T\mathbb{R}^m \times \mathfrak{g}^r_m$. Since $\widetilde{t}_{T\mathbb{R}^m}$ maps $\mathfrak{g}^r_m$ into $(N_A)^m$, it is

$$J^rT\mathbb{R}^m \times_{FT\mathbb{R}^m} FT(\mathbb{R}^m \times V) = T\mathbb{R}^m \times \mathfrak{g}^r_m \times V \otimes A \times V \otimes A\,.$$

On the other hand, $TF(\mathbb{R}^m \times V) = T\mathbb{R}^m \times V \otimes A \times V \otimes A$. For $z \in T\mathbb{R}^m$, $u \in \mathfrak{g}^r_m$, $v \otimes a \in V \otimes A$ and $w \in V \otimes A$, one finds

$$\psi^F_{\mathbb{R}^m \times V}(z, u, v \otimes a, w) = \big(z, v \otimes a, w + v \otimes h(u)(a)\big)\,. \tag{27}$$

Even in the classical case $F = J^r$, this formula is convenient for evaluating $\mathcal{J}^r\eta$.

5.3. F-prolongation of connections. The flow natural map ψ^F_Y can be used for constructing the F-prolongation of projectable tangent valued forms on Y. However, in this situation we need an auxiliary linear r-th order connection on the base M, i.e. a linear base preserving morphism

$$\Lambda : TM \to J^rTM$$

satisfying $\beta \circ \Lambda = \mathrm{id}_{TM}$. This fact is well known from the theory of connections, [14]. Let F be an arbitrary bundle functor on the category of local isomorphisms of fibered manifolds of base order r and Γ be a connection on Y. The flow prolongation of the lifted vector field ΓX depends on the r-jets of vector field $X : M \to TM$. This defines a map

$$\mathcal{F}\Gamma : FY \times_M J^rTM \to TFY\,.$$

If we add Λ to the second factor, we obtain the lifting map $FY \times_M TM \to TFY$ of a connection $\mathcal{F}(\Gamma, \Lambda)$ on FY, which is called the F-prolongation of Γ with respect to Λ.

5.4. Tangent valued forms. For the sake of simplicity, we consider a projectable tangent valued 1-form $Q : TY \to TY$ over $\underline{Q} : TM \to TM$, that means $Tp \circ Q = \underline{Q} \circ Tp$. We have to interpret both Q and $\underline{Q}$ as base preserving morphisms over M. Then we construct the induced map

$$J^r\underline{Q} \times_{F\underline{Q}} FQ : J^rTM \times_{FTM} FTY \to J^rTM \times_{FTM} FTY\,.$$

Consider the following diagram

$$\begin{array}{ccc} J^rTM \times_{FTM} FTY & \xrightarrow{J^r\underline{Q}\times_{F\underline{Q}}FQ} & J^rTM \times_{FTM} FTY \\ \Big\downarrow{\scriptstyle \widetilde{\psi}^F_Y} & & \Big\downarrow{\scriptstyle \psi^F_Y} \\ J^rTM \times_{TM} TFY & \xrightarrow{\qquad\qquad} & TFY \end{array}$$

Since $\widetilde{\psi}^F_Y$ is invertible, the bottom arrow defines

$$\mathcal{F}Q = \psi^F_Y \circ (J^r\underline{Q} \times_{F\underline{Q}} FQ) \circ (\widetilde{\psi}^F_Y)^{-1} : J^rTM \times_{TM} TFY \to TFY\,.$$

Then 5.2 implies the following property of $\mathcal{F}Q$.

Proposition. *Let η be a projectable vector field on Y over ξ on M. Then*

$$\mathcal{F}\big(Q(\eta)\big) = \mathcal{F}Q \circ (j^r\xi \times_{\mathrm{id}_{TM}} \mathcal{F}\eta)\,.$$

Definition. For every linear r-th order connection $\Lambda : TM \to J^rTM$, the tangent valued 1-form on FY

$$\mathcal{F}(Q,\Lambda) := \mathcal{F}Q \circ (\Lambda \times_{\mathrm{id}_{TM}} \mathrm{id}_{TFY}) : TFY \to TFY$$

is called the F-prolongation of Q with respect to Λ.

For a projectable tangent valued k-form Q on Y, we construct a projectable tangent valued k-form $\mathcal{F}(Q,\Lambda)$ on FY in the same way.

We remark that ψ^F can be applied in an interesting way for constructing the F-prolongations of Lie algebroids and their actions, [12].

5.5. Remarks. If ω_Γ is the connection form of a connection Γ on Y, see 3.11, then $\mathcal{F}(\omega_\Gamma,\Lambda)$ is the connection form of the connection $\mathcal{F}(\Gamma,\Lambda)$.

However, finding the curvature of $\mathcal{F}(\Gamma,\Lambda)$ is a much more complicated problem than in 3.11. This can be illustrated on some special cases discussed in detail in [4].

In general, the F-prolongation of tangent valued forms with respect to Λ does not preserve the Frölicher-Nijenhuis bracket. Some very special cases, in which this bracket is preserved, are characterized in [4].

5.6. Weak principal bundles. First we recall the basic properties of r-th jet prolongations of associated bundles, which were clarified already by Ehresmann. Consider a fiber bundle $E = P[S,l]$ associated to a principal bundle $P(M,G)$. By 3.4, G^r_m acts on T^r_mG by group isomorphisms. Hence we can construct the semidirect group product

$$W^r_mG = G^r_m \rtimes T^r_mG\,.$$

Then $W^rP := (P^rM \times_M J^rP) \to M$ is a principal bundle with structure group W^r_mG. (More details will be given in 5.7 in a more general setting.) W^rP is called the r-th principal prolongation of principal bundle P, [14].

An action $l : G \times S \to S$ induces an action $W^r_m l : W^r_m G \times T^r_m S \to T^r_m S$,

$$W^r_m l\big((g, X), Z\big) = \big(T^r_m l(X, Z)\big) \circ g^{-1} ,$$

$g \in G^r_m$, $X \in T^r_m G$, $Z \in T^r_m S$. Then $J^r E$ has a canonical structure of associated bundle

$$J^r E = W^r P[T^r_m S, W^r_m l] .$$

Some geometric properties of $J^r P$ can be described by using the following concept. Consider a group bundle $C \to M$.

Definition. A fibered manifold $Q \to M$ is called a weak principal bundle with structure group bundle $C \to M$, if we are given a base preserving morphism $\varrho_Q : Q \times_M C \to Q$ such that each group C_x acts simply transitively on the right on Q_x.

The principal bundle is a weak principal bundle, the group bundle of which is a product $M \times K$.

By 4.7, we have $J^r(M \times G) = P^r M[T^r_m G]$. This is a group bundle of type $T^r_m G$. Applying J^r to $\varrho_P : P \times_M (M \times G) \to P$, we obtain

$$J^r \varrho_P : J^r P \times_M P^r M[T^r_m G] \to J^r P .$$

This defines a weak principal bundle structure on $J^r P$.

5.7. Principal F-prolongations. Consider the general case $F = (A, H, t)$. Applying F to ϱ_Q, we obtain

$$\varrho_{FQ} := F\varrho_Q : FQ \times_M FC \to FQ .$$

This endows $FQ \to M$ with the structure of a weak principal bundle with structure group bundle $FC \to M$.

The construction of $W^r P$ from 5.6 can be extended to the general case. By 3.4, the action H_G of G^r_m on $T^A G$ is by group homomorphisms. Hence we can construct the semidirect group product $W^A_H G = G^r_m \rtimes T^A G$ with the composition

$$(g_1, X_1)(g_2, X_2) = \big(g_1 \circ g_2, T^A\varphi\big(H_G(g_2^{-1})(X_1), X_2\big)\big) ,$$

where φ is the group composition of G. For an action $l : G \times S \to S$, we define

$$W^A_H l : W^A_H G \times T^A S \to T^A S , \quad W^A_H l\big((g, X), Z\big) = H_S(g)\big(T^A l(X, Z)\big) ,$$

$g \in G^r_m$, $X \in T^A G$, $Z \in T^A S$. This is an action, too. Clearly,

$$F(M \times G) = P^r M[T^A G, H_G]$$

is a group bundle of type $T^A G$. We have

$$F\varrho_P : FP \times_M P^r M[T^A G, H_G] \to FP .$$

We introduce $W^F P = P^r M \times_M FP$ and we define an action of $W^A_H G$ on $W^F P$ by

$$(u, Z)(g, X) = \big(u \circ g, F\varrho_P\big(Z, \{u \circ g, X\}\big)\big) \tag{28}$$

with $u \in P^r_x M$, $Z \in F_x P$, $g \in G^r_m$, $X \in T^A G$, so that $\{u \circ g, X\} \in P^r M[T^A G, H_G]$. One verifies directly

Proposition. *$W^F P(M, W^A_H G)$ is a principal bundle. For an associated bundle $E = P[S, l]$, FE is an associated bundle $W^F P[T^A S, W^A_H l]$.*

We say that $W^F P$ is the principal F-prolongation of principal bundle P. In the case $F = J^r$, we have $W^{J^r} P = W^r P$.

Remark. In [13], a more general construction of a principal bundle from a weak principal bundle and a suitable group bundle is discussed. Then a construction of the Lie algebroid of the principal bundle in question in terms of an action of a Lie algebroid on a Lie algebra bundle by derivations is described. In particular, the Lie algebroid of $W^F P$ can be determined in this way.

5.8. Lie groupoids. It is interesting that the description of FY as a subset of $P^r M[T^A Y]$, deduced in 4.7, is unavoidable for constructing the F-prolongations of Lie groupoids.

In the algebraic sense, a groupoid is a category in which all elements are invertible. We write a or b for the right or left unit map (also called source or target), respectively. A smooth groupoid $\Phi \underset{b}{\overset{a}{\rightrightarrows}} M$ is a groupoid such that Φ and M are manifolds, both $a, b : \Phi \to M$ are surjective submersions and the partial composition law

$$\varphi : \Phi^a \times_M \Phi^b \to \Phi\,, \quad \Phi^a := (\Phi \xrightarrow{a} M)\,, \quad \Phi^b := (\Phi \xrightarrow{b} M)$$

as well as the unit injection $e : M \to \Phi$ are smooth maps. The product groupoid is of the form

$$M \times G \times M\,,$$

where M is a manifold, G is a Lie group, $a = pr_3$, $b = pr_1$, $e(x) = (x, e_G, x)$, where e_G is the unit of G, and

$$\varphi\big((x_3, g_2, x_2)(x_2, g_1, x_1)\big) = (x_3, g_2 g_1, x_1)\,,$$

the product $g_2 g_1$ being in G. A smooth groupoid is called a Lie groupoid, if it is locally isomorphic to the product one.

For every Lie groupoid Φ and every $x \in M$,

$$\Phi_x := \{\theta \in \Phi, a\theta = x\}$$

is a principal bundle, whose structure group G_x is the isotropy group of Φ over x. Conversely, if $P(M, G)$ is a principal bundle, then the space of all equivalence classes

$$PP^{-1} = P \times P/\sim\,, \quad (v, u) \sim (vg, ug)\,, \quad u, v \in P,\ g \in G\,,$$

is a Lie groupoid over M with respect to the composition $\{w, v\}\{v, u\} = \{w, u\}$. We say that PP^{-1} is the gauge groupoid of P.

One verifies easily that for every Lie groupoid $\Phi \overset{a}{\underset{b}{\rightrightarrows}} M$, $T^A\Phi \overset{T^A a}{\underset{T^A b}{\rightrightarrows}} T^A M$ is also a Lie groupoid with partial composition law $T^A\varphi$ and unit injection $T^A e : T^A M \to T^A\Phi$. If $\Phi = PP^{-1}$, then $T^A\Phi = T^A P(T^A P)^{-1}$, where $T^A P \to T^A M$ is the principal bundle from 3.8.

However, in the case $F = (A, H, t)$ we cannot apply F to φ, for φ is not an $\mathcal{FM}_m$-morphism. But we can consider the F-prolongation of $\Phi \xrightarrow{a} M$. We write $\pi : F(\Phi^a) \to \Phi^a$ for the bundle projection and $\overline{a} = a \circ \pi$, $\overline{b} = b \circ \pi : F\Phi^a \to M$. Further consider the groupoid $\Pi^r M \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M$ of all invertible r-jets of M into M. Hence $(\overline{a}, \overline{b}) : F\Phi^a \to M \times M$ and $(\alpha, \beta) : \Pi^r M \to M \times M$, so that we can construct the fiber product $\Pi^r M \times_{M\times M} F\Phi^a$.

Definition. The F-prolongation of a Lie groupoid Φ is the subset

$$\mathcal{F}\Phi \subset \Pi^r M \times_{M\times M} F\Phi^a \tag{29}$$

of all pairs $(v \circ u^{-1}, \{u, Z\})$ satisfying $t_M v = T^A b(Z)$.

Hence the elements of $\mathcal{F}\Phi$ are the equivalence classes, with respect to the action of G^r_m,

$$\{v, Z, u\} \quad \text{satisfying} \quad T^A a(Z) = t_M u\,,\ T^A b(Z) = t_M v\,,$$

$u, v \in P^r M$, $Z \in T^A\Phi$. Given another $\{w, \overline{Z}, v\} \in \mathcal{F}\Phi$, we define the composition $*$ by

$$\{w, \overline{Z}, v\} * \{v, Z, u\} = \{w, T^A\varphi(\overline{Z}, Z), u\}\,. \tag{30}$$

Write $\widetilde{a}, \widetilde{b} : \mathcal{F}\Phi \to M$ for the projections determined by (29). Define $\widetilde{e} : M \to \mathcal{F}\Phi$ by $\widetilde{e}(x) = \{u, j^A\widehat{e(x)}, u\}$, where $\widehat{\ }$ denotes the constant map of $\mathbb{R}^k$ into $e(x)$. Then one verifies directly

Proposition. *$\mathcal{F}\Phi \overset{\widetilde{a}}{\underset{\widetilde{b}}{\rightrightarrows}} M$ with the partial composition law* (30) *and the unit injection $\widetilde{e}$ is a Lie groupoid over M.*

For a principal bundle P, we have $\mathcal{F}(PP^{-1}) = (W^F P)(W^F P)^{-1}$, where $W^F P$ is the principal bundle from 5.7.

5.9. F-prolongation of actions. An action of a Lie groupoid Φ on a fibered manifold $p : Y \to M$ is a map $\psi : \Phi^a \times_M Y \to Y$ such that

$$p\big(\psi(\theta, y)\big) = b(\theta)\,,\ \psi\big(\varphi(\theta_2, \theta_1), y\big) = \psi\big(\theta_2, \psi(\theta_1, y)\big)\,,\ \psi\big(e(x), y\big) = y\,.$$

Every left group action $l : G \times S \to S$ induces an action of $M \times G \times M$ on the product fibered manifold $M \times S \to M$ by

$$\psi\big((x_2, g, x_1), (x_1, s)\big) = \big(x_2, l(g, s)\big)\,.$$

In the principal bundle form, $M \times S$ is a fiber bundle associated to $M \times G$.

If $\psi : \Phi^a \times_M Y \to Y$ is an action of Φ on Y, then $T^A\psi : (T^A\Phi)^{T^A a} \times_{T^A M} T^A Y \to T^A Y$ is an action of $T^A\Phi$ on $T^A Y \to T^A M$. In the case $F = (A, H, t)$, we have $FY \subset P^r M[T^A Y]$ and we define $\mathcal{F}\psi : (F\Phi)^{\tilde{a}} \times_M FY \to FY$ by

$$\mathcal{F}\psi(\{v, Z, u\}, \{u, Q\}) = \{v, T^A\psi(Z, Q)\},$$

$u, v \in P^r M$, $Z \in T^A\Phi$, $Q \in T^A Y$.

Analogously to 5.8, one verifies directly

Proposition. *$\mathcal{F}\psi$ is an action of $\mathcal{F}\Phi$ on $FY \to M$.*

In the case $\Phi = PP^{-1}$, we obtain the associated bundle structure on FY described in 5.7.

5.10. Remark. The theory of Weil bundles can be extended to some infinite dimensional spaces. The basic information can be found in [18]. We describe the case of a rather simple functional bundle and we present one more advanced result.

Two fibered manifolds $p_1 : Y_1 \to M$, $p_2 : Y_2 \to M$ over the same base define an infinite dimensional bundle

$$\bigcup_{x \in M} C^\infty(Y_{1x}, Y_{2x}) =: \mathcal{F}(Y_1, Y_2) \xrightarrow{\pi} M$$

of all smooth maps between the individual fibers over the same base point. This is a smooth space in the sense of Frölicher. We describe the basic ideas of this theory only. Given a manifold Q, a map $f : Q \to \mathcal{F}(Y_1, Y_2)$ is said to be smooth in the sense of Frölicher, if $\underline{f} = \pi \circ f : Q \to M$ is smooth, so that we can construct the pullback

$$\underline{f}^* Y_1 = \{(q, y) \in Q \times Y_1, \pi(f(q)) = p_1(y)\},$$

and the associated map

$$\tilde{f} : \underline{f}^* Y_1 \to Y_2, \quad \tilde{f}(q, y) = f(q)(y)$$

is also smooth.

Write $T^A_X Y_i = (T^A p_i)^{-1}(X)$, $X \in T^A M$, $i = 1, 2$. Consider two maps $f, g : \mathbb{R}^k \to \mathcal{F}(Y_1, Y_2)$ that are smooth in the sense of Frölicher and satisfy $j^A(\pi \circ f) = j^A(\pi \circ g) = X$. Then we can construct a map $\mathbf{j}^A f : T^A_X Y_1 \to T^A_X Y_2$,

$$(\mathbf{j}^A f)(j^A(\gamma(u))) = j^A(f(u)(\gamma(u))), \quad u \in \mathbb{R}^k,$$

and the same for g. If the maps $\mathbf{j}^A f$ and $\mathbf{j}^A g$ coincide, we say that f and g determine the same A-velocity on $\mathcal{F}(Y_1, Y_2)$. This defines an infinite dimensional bundle $T^A\mathcal{F}(Y_1, Y_2) \to T^A M$. Since each algebra homomorphism $\mu : A \to B$ can be interpreted as a reparametrization, we have an induced map

$$\mu_{\mathcal{F}(Y_1, Y_2)} : T^A\mathcal{F}(Y_1, Y_2) \to T^B\mathcal{F}(Y_1, Y_2).$$

There are no flows in this situation. However, if $X : \mathcal{F}(Y_1, Y_2) \to T\mathcal{F}(Y_1, Y_2)$ is a vector field that is differentiable in the sense of Frölicher, then the formula

$$\mathcal{T}^A X := \varkappa^A_{\mathcal{F}(Y_1, Y_2)} \circ T^A X$$

defines a vector field on $T^A\mathcal{F}(Y_1, Y_2)$. The proof of the fact that this operation preserves the bracket of vector fields is heavily based on the Weil algebra technique, [1].

Further, if F is a f.p.p.b. functor on $\mathcal{FM}_m$, then the decomposition $F = (A, H, t)$ enables us to construct an infinite dimensional bundle $F\mathcal{F}(Y_1, Y_2) \to M$ analogously to 4.7, [1]. The r-jet prolongation $J^r\mathcal{F}(Y_1, Y_2)$ can be also interpreted in a direct geometric way. A section $s : M \to \mathcal{F}(Y_1, Y_2)$ smooth in the sense of Frölicher is identified with a base-preserving morphism $Y_1 \to Y_2$. Then $J^r\mathcal{F}(Y_1, Y_2)$ is the space of all fiber r-jets $\mathbf{j}^r_x s$, $x \in M$, introduced in [14].

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