FLOW PROLONGATION OF SOME TANGENT VALUED FORMS

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Abstract. We study the prolongation of semibasic projectable tangent valued k-forms on fibered manifolds with respect to a bundle functor F on the local isomorphisms that is based on the flow prolongation of vector fields and uses an auxiliary linear r-th order connection on the base manifold, where r is the base order of F. We find a general condition under which the Frölicher-Nijenhuis bracket is preserved. Special attention is paid to the curvature of connections. The first order jet functor and the tangent functor are discussed in detail. Next we clarify how this prolongation procedure can be extended to arbitrary projectable tangent valued k-forms in the case F is a fiber product preserving bundle functor on the category of fibered manifolds with m-dimensional bases and local diffeomorphisms as base maps.

Our starting point was the general procedure of prolongating a connection Γ on a fibered manifold $Y \to M$ into a connection on the fibered manifold $FY \to M$, where F is an arbitrary bundle functor on the category $\mathcal{FM}_{m,n}$ of fibered manifolds with m-dimensional bases and n-dimensional fibers and their local isomorphisms, [6]. That procedure is based on the flow prolongation of vector fields. In [6] and [8] it is clarified that if F has the base order r, then one needs an auxiliary linear r-th order connection Λ on the base manifold M to construct the induced connection $\mathcal{F}(\Gamma, \Lambda)$ on FY. In Section 2 of the present paper we deduce that every projectable morphism $\Phi : Y \times_M \wedge^k TM \to TY$ linear in the second factor, which is equivalent to a projectable semibasic tangent valued k-form on Y, induces a morphism $\mathcal{F}(\Phi, \Lambda) : FY \times_M \wedge^k TM \to TFY$ in a similar way. But first of all we clarify, in Proposition 1, that the set of such forms, or equivalently morphisms, is closed with respect to the Frölicher-Nijenhuis bracket. In Section 3 we present the coordinate expression of $\mathcal{J}^1(\Phi, \Lambda)$ in the case of the functor J^1 of the first jet prolongation. In Proposition 2 we deduce that the curvature $C\big(\mathcal{J}^1(\Gamma, \Lambda)\big)$ of the connection $\mathcal{J}^1(\Gamma, \Lambda)$ differs from the prolongation $\mathcal{J}^1(C\Gamma, \Lambda)$ of the curvature $C\Gamma$ of Γ by a term depending on the curvature of the connection $\widetilde{\Lambda}$ conjugate to the auxiliary linear connection Λ. A similar formula holds for the mixed curvature of two connections on Y.

Then we define the integrability of Λ by using the viewpoint of the theory of G-structures. Proposition 3 reads that if Λ is integrable, the operation $\Phi \mapsto \mathcal{F}(\Phi, \Lambda)$ preserves the Frölicher-Nijenhuis bracket for those morphisms, whose underlying

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base map is constant with respect to Λ. In Section 6 we interpret the construction of tangent bundles of fibered manifolds as functor transforming $Y \to M$ into $TY \to M$ and we deduce an explicit formula for the difference of $C\big(\mathcal{T}(\Gamma,\Lambda)\big)$ and $\mathcal{T}(C\Gamma,\Lambda)$. The motivation goes out from a general difference between the functors J^1 and T: the former preserves the fiber products, but the latter does not.

The final part of the paper is inspired by a classical result from the theory of tangent valued forms on manifolds. If F is a product preserving bundle functor on the category $\mathcal{M}f$ of all manifolds, the prolongation of a tangent valued k-form from a manifold M to FM can be constructed by using the canonical exchange diffeomorphism between FTM and TFM, [1], [3], [13]. For a fiber product preserving bundle functor F on the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and fibered manifold morphisms with local diffeomorphisms as base maps, we show that one can use a kind of exchange map that has been recently introduced by the second author, [4]. In this case we can construct a prolongation $\mathcal{F}(\varphi,\Lambda)$ of every projectable tangent valued k-form φ on $Y \to M$ by means of an auxiliary r-th order linear connection Λ on M. If φ is semibasic, we obtain the same result as in Section 2. As an example, in Section 8 we discuss the case $F = J^1$ in detail.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [6].

1. Semibasic forms and morphisms. Consider a fibered manifold $p: Y \to M$. A tangent valued k-form φ on Y is called semibasic, if

$$\varphi(Z_1,\dots,Z_k) = 0 \tag{1}$$

whenever at least one of the vectors $Z_1,\dots,Z_k$ is vertical. Such a form defines a morphism linear in the second factor $\Phi: Y \times_M \wedge^k TM \to TY$ by

$$\Phi(y, X_1,\dots,X_k) = \varphi(Z_1,\dots,Z_k), \quad Z_i \in T_yY,\ Tp(Z_i) = X_i. \tag{2}$$

This definition is correct: if we take another $\overline{Z}_i \in T_yY$ satisfying $Tp(\overline{Z}_i) = X_i$, $i = 1,\dots,k$, then $\overline{Z}_i = Z_i + W_i$, where W_i are vertical vectors. Then multilinearity yields

$$\varphi(Z_1 + W_1,\dots,Z_k + W_k) = \varphi(Z_1,\dots,Z_k).$$

Conversely, if $\Phi : Y \times_M \wedge^k TM \to TY$ is a morphism linear in the second factor, then (2) defines a semibasic tangent valued k-form on Y, which will be sometimes denote by $\omega(\Phi)$.

Let x^i, y^p, $i = 1,\dots,m = \dim M$, $p = 1,\dots,\dim Y - \dim M$, be local fiber coordinates on Y. If Φ is projectable, which is the same as $\omega(\Phi)$ is projectable, then the coordinate expression of both Φ and $\omega(\Phi)$ is

$$dx^j = a^j_{i_1\dots i_k}(x)\, dx^{i_1} \wedge \dots \wedge dx^{i_k}\,, \quad dy^p = a^p_{i_1\dots i_k}(x,y)\, dx^{i_1} \wedge \dots \wedge dx^{i_k}\,. \tag{3}$$

Consider another tangent valued l-form ψ on Y.

Proposition 1. *If both φ and ψ are projectable and semibasic, then the Frölicher-Nijenhuis bracket $[\varphi,\psi]$ is also projectable and semibasic.*

Proof. It is well known that $[\varphi,\psi]$ is also projectable, [6]. To prove the main assertion, we use the Lie bracket formula for $[\varphi,\psi]$, [6], p. 71. Assume that one entry is a vertical vector field. Since the Lie bracket of a projectable vector field and a vertical vector field is vertical, we have one vertical vector field in each term on the right hand side of the formula. Hence the value of $[\varphi,\psi]$ vanishes, so $[\varphi,\psi]$ is semibasic. $\square$

Thus our approach defines the Frölicher-Nijenhuis bracket of two linear projectable morphisms $\Phi : Y\times_M \wedge^k TM \to TY$ and Ψ: $Y\times_M \wedge^l TM\to TY$. Write $\underline{\Phi} : \wedge^k TM\to TM$ and $\underline{\Psi} : \Lambda^l TM \to TM$ for the underlying base maps. Then the explicit formula for $[\Phi,\Psi] : Y\times_M \wedge^{k+l} TM\to TY$ is (an analogous expression was used in another situation in [10])

$$[\Phi,\Psi](\xi_1,\dots,\xi_{k+l}) = \tag{4}$$

$$\begin{aligned}
&= \frac{1}{k!\,l!}\sum_\sigma \operatorname{sgn}\sigma\big[\Phi(\xi_{\sigma(1)},\dots,\xi_{\sigma(k)}),\Psi(\xi_{\sigma(k+1)},\dots,\xi_{\sigma(k+l)})\big]\\
&+\frac{-1}{k!(l-1)!}\sum_\sigma \operatorname{sgn}\sigma\,\Psi\big(\big[\underline{\Phi}(\xi_{\sigma(1)},\dots,\xi_{\sigma(k)}),\xi_{\sigma(k+1)}\big],\xi_{\sigma(k+2)},\dots,\xi_{\sigma(k+l)}\big)\\
&+\frac{(-1)^{kl}}{(k-1)!\,l!}\sum_\sigma \operatorname{sgn}\sigma\,\Phi\big(\big[\underline{\Psi}(\xi_{\sigma(1)},\dots,\xi_{\sigma(l)}),\xi_{\sigma(l+1)}\big],\xi_{\sigma(l+2)},\dots,\xi_{\sigma(k+l)}\big)\\
&+\frac{(-1)^{k-1}}{(k-1)!(l-1)!\,2}\sum_\sigma \operatorname{sgn}\sigma\,\Psi\big(\underline{\Phi}([\xi_{\sigma(1)},\xi_{\sigma(2)}],\dots,\xi_{\sigma(k+1)}),\xi_{\sigma(k+2)},\dots,\xi_{\sigma(k+l)}\big)\\
&+\frac{(-1)^{(k-1)l}}{(k-1)!(l-1)!\,2}\sum_\sigma \operatorname{sgn}\sigma\,\Phi\big(\underline{\Psi}([\xi_{\sigma(1)},\xi_{\sigma(2)}],\dots,\xi_{\sigma(l+1)}),\xi_{\sigma(l+2)},\dots,\xi_{\sigma(k+l)}\big)\,,
\end{aligned}$$

where $\xi_1,\dots,\xi_{k+l}$ are vector fields on M and the summations are with respect to all permutations of $k+l$ letters.

2. The flow prolongation of projectable semibasic forms. Consider a bundle functor F on the category $\mathcal{FM}_{m,n}$ of fibered manifolds with m-dimensional bases and n-dimensional fibers and their local isomorphisms, [6]. The definition of the order of F is based on the concept of (q,s,r)-jet, $s\ge q\le r$, of fibered manifold morphisms, [6], p. 126. Consider two fibered manifolds $p: Y\to M$, $\overline{p}:\overline{Y}\to\overline{M}$ and two $\mathcal{FM}_{m,n}$-morphisms $f,g : Y\to\overline{Y}$. We say that F is of the order (q,s,r), if $j^{q,s,r}_y f = j^{q,s,r}_y g$ implies $Ff|F_yY = Fg|F_yY$, $y\in Y$. The number r is called the base order of F.

For a projectable morphism linear in the second factor $\Phi : Y\times_M\wedge^k TM\to TY$ and k vector fields $\xi_1,\dots,\xi_k$ on M, $\Phi(\xi_1,\dots,\xi_k)$ is a projectable vector field on Y. The restriction of the flow prolongation $\mathcal{F}\big(\Phi(\xi_1,\dots,\xi_k)\big)$ to F_xY depends on the r-jets $j^r_x\xi_1,\dots,j^r_x\xi_k$, $x\in M$, cf. [8]. This defines a map

$$\mathcal{F}\Phi : FY\times_M \wedge^k J^r TM\to TFY \tag{5}$$

linear in the second factor that will be called the flow prolongation of Φ.

Consider a linear r-th order connection Λ on M, i.e. a base preserving linear morphism $\Lambda : TM\to J^rTM$ satisfying $\beta\circ\Lambda = \mathrm{id}_{TM}$, where β is the target jet projection.

Definition 1. The composition

$$\mathcal{F}(\Phi,\Lambda) := \mathcal{F}\Phi \circ (\mathrm{id}_{FY} \times_M \wedge^k \Lambda) : FY \times_M \wedge^k TM \to TFY \tag{6}$$

is said to be the flow prolongation of Φ with respect to Λ.

Clearly, if the values of Φ lie in the vertical tangent bundle VY of Y, then the values of $\mathcal{F}(\Phi,\Lambda)$ lie in $V(FY \to M)$.

3. To supply an example, we are going to discuss the case $F = J^1$ of the first jet prolongation in detail. For a projectable vector field η on Y with the coordinate expression

$$\xi^i(x)\frac{\partial}{\partial x^i} + \eta^p(x,y)\frac{\partial}{\partial y^p}\,, \tag{7}$$

its flow prolongation $\mathcal{J}^1\eta$ is given be the well known formula

$$\eta + \left(\frac{\partial \eta^p}{\partial x^i} + \frac{\partial \eta^p}{\partial y^q}\,y^q_i - \frac{\partial \xi^j}{\partial x^i}\,y^p_j\right)\frac{\partial}{\partial y^p_i}\,, \tag{8}$$

where y^p_i are the induced coordinates on J^1Y, [6]. The coordinate form of $\Lambda : TM \to J^1TM$ is

$$\xi^i_j = \Gamma^i_{kj}(x)\xi^k\,. \tag{9}$$

Consider Φ with the coordinate expression (3). Using direct evaluations, we obtain the following additional coordinate expression of $\mathcal{J}^1(\Phi,\Lambda) : J^1Y \times_M \wedge^k TM \to TJ^1Y$

$$\begin{aligned} dy^p_j = \Bigg[&\frac{\partial a^p_{i_1\dots i_k}}{\partial x^j} + \frac{\partial a^p_{i_1\dots i_k}}{\partial y^q}\,y^q_j - \frac{\partial a^k_{i_1\dots i_k}}{\partial x^j}\,y^p_k \\ &+ (a^p_{li_2\dots i_k} - a^k_{li_2\dots i_k}y^p_k)\Gamma^l_{i_1 j} + \dots + (a^p_{i_1\dots i_{k-1}l} - a^k_{i_1\dots i_{k-1}l}y^p_k)\Gamma^l_{i_k j}\Bigg]\,dx^{i_1}\wedge\dots\wedge dx^{i_k}\,. \end{aligned} \tag{10}$$

4. The curvature of $\mathcal{J}^1(\Gamma,\Lambda)$. The lifting map of a connection on Y is a morphism $\Gamma : Y \times_M TM \to TY$ linear in the second factor. Its coordinate expression is

$$dx^i = dx^i\,, \quad dy^p = F^p_i(x,y)\,dx^i\,. \tag{11}$$

Using (10), we obtain the well known additional coordinate expression of the connection $\mathcal{J}^1(\Gamma,\Lambda)$ on $J^1Y \to M$, [6], p. 366, [9],

$$dy^p_j = \left[\frac{\partial F^p_i}{\partial x^j} + \frac{\partial F^p_i}{\partial y^q}\,y^q_j + (F^p_k - y^p_k)\Gamma^k_{ij}\right]dx^i\,. \tag{12}$$

The curvature $C\Gamma$ of Γ is a morphism $Y \times \wedge^2 TM \to VY$ with the coordinate expression

$$dx^i = 0,\ dy^p = \left(\frac{\partial F^p_j}{dx^i} + \frac{\partial F^p_j}{\partial y^q} F^q_i \right) dx^i \wedge dx^j \,. \tag{13}$$

Hence we can construct

$$\mathcal{J}^1(C\Gamma, \Lambda) : J^1Y \times_M \wedge^2 TM \to VJ^1Y \,. \tag{14}$$

To compare (14) with the curvature of $\mathcal{J}^1(C, \Gamma)$

$$C(\mathcal{J}^1(\Gamma, \Lambda)) : J^1Y \times_M \wedge^2 TM \to VJ^1Y \,, \tag{15}$$

we need the following concept.

Since $\beta : J^1Y \to Y$ is an affine bundle associated to the vector bundle $VY \otimes T^*M$, Γ defines a map $\tau\Gamma : J^1Y \to VY \otimes T^*M$,

$$\tau\Gamma(z) = \Gamma(\beta(z)) - z \,, \qquad z \in J^1Y \,.$$

The curvature of the classical linear connection Λ can be interpreted as a map

$$C\Lambda : M \to TM \otimes T^*M \otimes \wedge^2 T^*M \,.$$

The well known exact sequence

$$0 \to \beta^*(VY \otimes T^*M) \to VJ^1Y \overset{V\beta}{\to} VY \to 0$$

induces an injection $\beta^*(VY \otimes T^*M \otimes \wedge^2 T^*M) \to VJ^1Y \otimes \wedge^2 T^*M$. So we have the evaluation map

$$\langle \tau\Gamma, C\Lambda \rangle : J^1Y \to VJ^1Y \otimes \wedge^2 T^*M \,.$$

Write $\widetilde{\Lambda}$ for the classical conjugate connection of Λ. The following assertion can be proved by direct evaluation.

Proposition 2. *We have* $C(\mathcal{J}^1(\Gamma, \Lambda)) = \mathcal{J}^1(C\Gamma, \Lambda) + \langle \tau\Gamma, C\widetilde{\Lambda} \rangle$. □

Remark. A more general result will be interesting from the viewpoint of our theory. If Δ is another connection on Y, then the mixed curvature of Γ and Δ is defined to be the Frölicher-Nijenhuis bracket $[\Gamma, \Delta] : Y \times_M \wedge^2 TM \to VY$, [6], p. 232. Even in this case we deduced by direct evaluations

$$\big[\mathcal{J}^1(\Gamma, \Lambda), \mathcal{J}^1(\Delta, \Lambda)\big] = \mathcal{J}^1([\Gamma, \Delta], \Lambda) + \langle \tau\Gamma, C\widetilde{\Lambda} \rangle + \langle \tau\Delta, C\widetilde{\Lambda} \rangle \,. \tag{16}$$

Since $C\Gamma = \frac{1}{2}[\Gamma, \Gamma]$, Proposition 2 is a special case of (16).

5. We define the integrability of $\Lambda : TM \to J^rTM$ by using the viewpoint of the theory of G-structures. Using the translations on $\mathbb{R}^m$, we extend each $X \in T_x\mathbb{R}^m$ into a constant vector field $\widetilde{X}$ on $\mathbb{R}^m$. The canonical integrable connection $I : T\mathbb{R}^m \to J^rT\mathbb{R}^m$ maps every $X \in T_x\mathbb{R}^m$ into $j^r_x\widetilde{X}$.

Definition 2. A linear r-th order connection Λ on M is said to be integrable, if for every $x \in M$ there exists a neighbourhood U and a diffeomorphism $f : U \to \mathbb{R}^m$ transforming Λ into I, i.e. $I \circ Tf = J^r Tf \circ (\Lambda|_U)$.

We say that f is a normal coordinate system of the integrable connection Λ. One verifies easily that two normal coordinate systems of Λ differ by an affine transformation $\mathbb{R}^m \to \mathbb{R}^m$.

In the case $r = 1$, a classical result reads that a classical linear connection is integrable if and only if it is both torsion-free and curvature-free.

Assume Λ is integrable and consider a tangent valued k-form φ on M. We say that φ is constant with respect to Λ, if in every normal coordinate system of Λ all coefficients of φ are constant. In particular, the identity one-form id_{TM} is constant with respect to every integrable Λ. Clearly, if ψ is another tangent valued l-form on M constant with respect Λ, then $[\varphi, \psi] = 0$.

Now we can describe the most important case in which the operation $\Phi \mapsto \mathcal{F}(\Phi, \Lambda)$ preserves the Frölicher-Nijenhuis bracket. Consider two projectable morphisms $\Phi : Y \times_M \wedge^k TM \to TY$ and $\Psi : Y \times_M \wedge^l TM \to TY$ linear in the second factor.

Proposition 3. *If Λ is integrable and the underlying tangent valued forms $\underline{\Phi}$ and $\underline{\Psi}$ are constant with respect to Λ, then*

$$[\mathcal{F}(\Phi, \Lambda), \mathcal{F}(\Psi, \Lambda)] = \mathcal{F}([\Phi, \Psi], \Lambda)\,. \tag{17}$$

Proof. In a normal coordinate system of Λ, consider constant vector fields $\widetilde{X}_1, \dots, \widetilde{X}_{k+l}$. In particular, it holds $[\widetilde{X}_i, \widetilde{X}_j] = 0$. By Definitions 1 and 2, we have

$$\mathcal{F}(\Phi, \Lambda)(\widetilde{X}_1, \dots, \widetilde{X}_k) = \mathcal{F}(\Phi(\widetilde{X}_1, \dots, \widetilde{X}_k)) \tag{18}$$

and analogously for $\mathcal{F}(\Psi, \Lambda)$ and $\mathcal{F}([\Phi, \Psi], \Lambda)$. Consider (4) with ξ's replaced by $\widetilde{X}$'s. Since $\underline{\Phi}$ and $\underline{\Psi}$ are constant with respect to Λ, $\underline{\Phi}(\widetilde{X}_1, \dots, \widetilde{X}_k)$ and $\underline{\Psi}(\widetilde{X}_{k+1}, \dots, \widetilde{X}_{k+l})$ are also constant vector fields and all terms on the right hand side except the first ones vanish. Now we apply $\mathcal{F}$ to both sides and use the fact that the flow prolongation preserves the bracket of vector fields. □

In particular, for every connection Γ and every integrable r-th order linear connection Λ on the base manifold we have

$$C\big(\mathcal{F}(\Gamma, \Lambda)\big) = \mathcal{F}(C\Gamma, \Lambda)\,. \tag{19}$$

The same is true for the mixed curvature of two connections.

6. We find it useful to discuss the case of non-integrable Λ in another concrete situation. If we interpret the construction of the tangent bundle of a fibered manifold as a functor transforming $Y \to M$ into $TY \to M$, every connection Γ on Y and a linear connection $\Lambda : TM \to J^1 TM$ induces a connection $\mathcal{T}(\Gamma, \Lambda)$ on $TY \to M$.

Write X^i, Y^p for the induces coordinates on TY. For a projectable vector field η on Y, the coordinate form of its flow prolongation $\mathcal{T}\eta$ is

$$\eta + \frac{\partial \xi^i}{\partial x^j} X^j \frac{\partial}{\partial X^i} + \left(\frac{\partial \eta^p}{\partial x^i} X^i + \frac{\partial \eta^p}{\partial y^q} Y^q \right) \frac{\partial}{\partial Y^p}\,. \tag{20}$$

Thus, if Γ and Λ are of the form (11) and (9). the equations of $\mathcal{T}(\Gamma,\Lambda)$ are

$$
\begin{aligned}
dy^p &= F_i^p\,dx^i\,,\\
dX^j &= \Gamma^j_{ik}X^k\,dx^i\,,\\
dY^p &= \left(\frac{\partial F_i^p}{\partial x^j}X^j + \frac{\partial F_i^p}{\partial y^q}Y^q + F_j^p\Gamma^j_{ik}X^k\right)dx^i\,.
\end{aligned}
\tag{21}
$$

Both $C\big(\mathcal{T}(\Gamma,\Lambda)\big)$ and $\mathcal{T}(C\Gamma,\Lambda)$ are sections $TY\to V(TY\to M)\otimes\wedge^2T^*M$.

Since $TY\to Y$ is a vector bundle, the identification $TY\times_Y TY = V(TY\to Y)$ and the inclusion $V(TY\to Y)\hookrightarrow V(TY\to M)$ define a map $TY\times_Y TY\to V(TY\to M)$. If we add $\Gamma : Y\times_M TM\to TY$ to the second factor, we construct a map $TY\times_M TM\to V(TY\to M)$. Dualizing in TM, we obtain a section

$$
\begin{aligned}
\nu\Gamma &: TY\to V(TY\to M)\otimes T^*M\,,\\
\nu\Gamma(x^i,y^p,X^i,Y^p) &= (x^i,y^p,X^i,Y^p,0,dx^i,F_i^p dx^i)\,.
\end{aligned}
\tag{22}
$$

The clasical curvature $C\widetilde{\Lambda}$ of the conjugate connection of Λ can be interpreted as a map $TM\to TM\otimes\wedge^2T^*M$. Hence $C\widetilde{\Lambda}\circ Tp : TY\to TM\otimes\wedge^2T^*M$ and we can construct the evaluation map

$$\langle\nu\Gamma, C\widetilde{\Lambda}\circ Tp\rangle : TY\to V(TY\to M)\otimes\wedge^2T^*M\,.$$

Using direct evaluations, one deduces

Proposition 4. *We have* $C\big(\mathcal{T}(\Gamma,\Lambda)\big) = \mathcal{T}(C\Gamma,\Lambda) + \langle\nu\Gamma, C\widetilde{\Lambda}\circ Tp\rangle$. □

7. The case of a fiber product preserving bundle functor. If F is a fiber product preserving bundle functor of the base order r on the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and fibered manifold morphisms with local diffeomorphisms as base maps, then a natural transformation $t_Y : J^rY\to FY$ is defined as follows. Every section $s: M\to Y$ can be interpreted as a morphism $\widetilde{s}$ of the trivial fibered manifold $\mathrm{id}_M : M\to M$ into Y and we set

$$t_Y(j^r_x s) = (F\widetilde{s})(x)\,,\qquad x\in M\,. \tag{23}$$

Taking into account $t_{TM} : J^rTM\to FTM$ and the surjective submersion $FTp : FTY\to FTM$, we can construct the fiber product $FTY\times_{FTM} J^rTM$.

In [4] the second author introduced a map

$$\mu^F_Y : FTY\times_{FTM} J^rTM\to TFY$$

with the following property. If η is a projectable vector field on Y over a vector field ξ on M and $F\eta : FY\to F(TY\to M)$ is its functorial prolongation, then the flow prolongation $\mathcal{F}\eta : FY\to TFY$ satisfies

$$\mathcal{F}\eta = \mu^F_Y\circ(F\eta\times_M j^r\xi)\,. \tag{24}$$

Moreover, the map

$$(25)\quad \widetilde{\mu}^F_Y : FTY \times_{FTM} J^rTM \to TFY \times_{TM} J^rTM\,, \quad \widetilde{\mu}^F_Y(A,B) = (\mu^F_Y(A), B)$$

is a diffeomorphism.

To clarify the basic ideas, we start with the case of a projectable tangent valued one-form $\varphi : TY \to TY$ over $\underline{\varphi} : TM \to TM$. We have the induced map

$$F\varphi \times_{F\underline{\varphi}} J^r\underline{\varphi} : FTY \times_{FTM} J^rTM \to FTY \times_{FTM} J^rTM\,.$$

Consider the following diagram

$$(26)\qquad \begin{array}{ccc} FTY \times_{FTM} J^rTM & \xrightarrow{F\varphi \times_{F\underline{\varphi}} J^r\underline{\varphi}} & FTY \times_{FTM} J^rTM \\ \Big\downarrow{\scriptstyle \widetilde{\mu}^F_Y} & & \Big\downarrow{\scriptstyle \mu^F_Y} \\ TFY \times_{TM} J^rTM & \longrightarrow & TFY \end{array}$$

Since $\widetilde{\mu}^F_Y$ is invertible, the bottom arrow defines

$$(27)\qquad \mathcal{F}\varphi = \mu^F_Y \circ (F\varphi \times_{F\underline{\varphi}} J^r\underline{\varphi}) \circ (\widetilde{\mu}^F_Y)^{-1} : TFY \times_{TM} J^rTM \to TFY\,.$$

If η is a projectable vector field on Y, then $\varphi(\eta)$ is also a projectable vector field on Y.

Proposition 5. *We have $\mathcal{F}(\varphi(\eta)) = \mathcal{F}\varphi \circ (\mathcal{F}\eta \times_{TM} j^r\xi)$.*

Proof. If we interpret ξ as the morphism $\widetilde{\xi}$ of $M \to M$ into TM, we have $j^r_x\xi = J^r\widetilde{\xi}(x)$. Hence $j^r_x(\underline{\varphi} \circ \xi) = (J^r\underline{\varphi} \circ J^r\widetilde{\xi})(x)$. By functoriality,

$$\begin{aligned} &\mathcal{F}(\omega(\eta)) = \mu^F_Y \circ (F\varphi \circ F\eta \times_M J^r\underline{\varphi} \circ J^r\widetilde{\xi}) = \mu^F_Y \circ (F\varphi \times_{F\underline{\varphi}} J^r\underline{\varphi})\circ \\ &(F\eta \times_M j^r\xi) = \mu^F_Y \circ (F\varphi \times_{F\underline{\varphi}} J^r\underline{\varphi}) \circ (\widetilde{\mu}^F_Y)^{-1} \circ \widetilde{\mu}^F_Y \circ (F\eta \times_M J^r\xi) \\ &\;= \mathcal{F}\varphi \circ (\mathcal{F}\eta \times_{TM} j^r\xi)\,. \end{aligned}$$

□

It can be deduced that $\mathcal{F}\varphi$ is a bilinear morphism (this purely technical assertion will be proved in a more general framework in a next paper [5]).

Definiton 3. For every linear r-th order connection $\Lambda : TM \to J^rTM$, the tangent valued one-form on FY

$$\mathcal{F}(\varphi, \Lambda) := \mathcal{F}\varphi \circ (\mathrm{id}_{TFY} \times_{TM} \Lambda) : TFY \to TFY$$

will be called the F-prolongation of φ with respect to Λ.

Consider a projectable morphism $\Phi : Y \times_M TM \to TY$ linear in the second factor and the corresponding semibasic one-form $\omega(\Phi)$ on Y. Write $q : FY \to M$. For $\mathcal{F}\Phi : FY \times_M J^rTM \to TFY$ we define analogously

$$(28)\qquad \omega(\mathcal{F}\Phi) : TFY \times_{TM} J^rTM \to TFY\,, \quad \omega(\mathcal{F}\Phi)(Z, X) = \Phi(z, X)\,,$$

where $Z \in T_zFY$, $X \in J^r_xTM$, $Tq(Z) = \beta X \in T_xM$.

Lemma. *We have* $\omega(\mathcal{F}\Phi)=\mathcal{F}\big(\omega(\Phi)\big)$.

Proof. By definition, $\omega(\Phi)(\eta)=\Phi(\xi)$ is the same projectable vector field on Y. Hence

$$\mathcal{F}(\omega(\Phi)(\eta))=\mathcal{F}(\Phi(\xi))\,. \tag{29}$$

By Proposition 3, $\mathcal{F}\big(\omega(\Phi)(\eta)\big)=\mathcal{F}\varphi\circ(\mathcal{F}\eta\times_{TM}j^r\xi)$. Then (29) implies $\mathcal{F}\big(\omega(\Phi)(\eta)\big)$ depends on $j^r\xi$ only and the induced map $FY\times_M J^rTM\to TFY$ coincides with $\mathcal{F}\Phi$. □

If we add $\Lambda : TM\to J^rTM$, the above Lemma implies

Proposition 6. *We have* $\mathcal{F}\big(\omega(\Phi),\Lambda\big)=\omega\big(\mathcal{F}(\Phi,\Lambda)\big)$. □

8. As an example, we discuss the case $F=J^1$ in detail. If $F=J^r$, then $J^rTY\times_{J^rTM}J^rTM=J^rTY$ and the map $\mu_Y^{J^r}:J^rTY\to TJ^rY$ coincides with that one introduced by L. Mangiarotti and M. Modugno, [11], see [4]. Thus, we can use a result from [6] to find the coordinate expression of $\widetilde{\mu}_Y^{J^1}:J^1TY\to TJ^1Y\times_{TM}J^1TM$. Let x^i,y^p,X^i,Y^p be the above coordinates on TY and y^p_i,X^i_j,Y^p_i be the induced jet coordinates on $J^1(TY\to M)$. Further, let x^i,y^p,y^p_i be the standard coordinates on J^1Y and dx^i,dy^p,dy^p_i be the induced coordinates on TJ^1Y. Moreover, write x^i,dx^i,ξ^i_j for the corresponding coordinates on J^1TM. By [6], p. 340, the equations of $\widetilde{\mu}_Y^{J^1}$ are $x^i=x^i$, $y^p=y^p$, $y^p_i=y^p_i$ and

$$dx^i=X^i\,,\ dy^p=Y^p\,,\ dy^p_i=Y^p_i-X^j_iy^p_j\,,\ \xi^i_j=X^i_j\,, \tag{30}$$

while in the coordinate form of $\mu_Y^{J^1}$ the last equation of (30) is missing. Hence the only nontrivial equation for $(\widetilde{\mu}_Y^{J^1})^{-1}$ is

$$Y^p_i=dy^p_i+\xi^j_iy^p_j\,. \tag{31}$$

Thus, if $x^i=x^i$, $y^p=y^p$ and

$$X^i=a^i_j(x)X^j\,,\ Y^p=a^p_i(x,y)X^i+a^p_q(x,y)Y^q \tag{32}$$

is the coordinate expression of φ, then the additional equations of $J^1\varphi$ are

$$X^i_j=\frac{\partial a^i_k}{\partial x^j}X^k+a^i_kX^k_j\,, \tag{33}$$

$$Y^p_i=\left(\frac{\partial a^p_j}{\partial x^i}+\frac{\partial a^p_j}{dy^q}y^q_i\right)X^j+a^p_jX^j_i+\left(\frac{\partial a^p_q}{\partial x^i}+\frac{\partial a^p_q}{\partial y^r}y^r_i\right)Y^q+a^p_qY^q_i\,. \tag{34}$$

In the case $F=J^1$, we have $J^1\varphi\times_{J^1\underline{\varphi}}J^1\underline{\varphi}=J^1\varphi$. Using (30) and (31), we deduce that the coordinate expression of $\mathcal{J}^1\varphi:\mu_Y^{J_1}\circ J^1\varphi\circ(\widetilde{\mu}_Y^{J^1})^{-1}:TJ^1Y\times_{TM}J^1TM\to TJ^1Y$ is

$$dx^i=a^i_j\,dx^j\,,\ dy^p=a^p_i\,dx^i+a^p_q\,dy^q\,, \tag{35}$$

$$(36) \qquad dy_i^p = \left(\frac{\partial a_j^p}{\partial x^i} + \frac{\partial a_j^p}{\partial y^q}\, y_i^q\right) dx^j + a_j^p \xi_i^j + \left(\frac{\partial a_q^p}{\partial x^i} + \frac{\partial a_q^p}{\partial y^r}\, y_i^r\right) dy^q$$
$$+ a_q^p (dy_i^q + \xi_i^j y_j^q) - y_j^p \frac{\partial a_k^j}{\partial x^i}\, dx^k - y_j^p a_k^j \xi_i^k \,.$$

Now one can observe even on the coordinate expressions that if φ is semibasic, i.e. $a_q^p = 0$, and we substitute $\xi_j^i = \Gamma_{kj}^i\, dx^k$ into (36), then the result is equal to (10) with $k = 1$.

9. The case of a projectable k-form. Such a k-form φ on Y can be interpreted as a map

$$\varphi : TY \times_Y \cdots \times_Y TY \to TY \quad \text{over} \quad \underline{\varphi} : TM \times_M \cdots \times_M TM \to TM \,.$$

The induced maps are

$$F\varphi : FTY \times_{FY} \cdots \times_{FY} FTY \to FTY \,, \quad F\underline{\varphi} : FTM \times_M \cdots \times_M FTM \to FTM \,,$$
$$J^r\underline{\varphi} : J^rTM \times_M \cdots \times_M J^rTM \to J^rTM \,.$$

Then we can construct

$$F\varphi \times_{F\underline{\varphi}} J^r\underline{\varphi} : (FTY \times_{FTM} J^rTM) \times_{FY} \cdots \times_{FY} (FTY \times_{FTM} J^rTM)$$
$$\to FTY \times_{FTM} J^rTM$$

and a diagram analogous to (26) defines a map $\mathcal{F}\varphi$. The antisymmetry of φ induces the antisymmetry of $\mathcal{F}\varphi$, so that it can be interpreted as a map

$$(37) \qquad \mathcal{F}\varphi : \wedge^k TFY \times_{\wedge^k TM} \wedge^k J^rTM \to TFY \,.$$

If we add the k-th exterior power of $\Lambda : TM \to J^rTM$ to the second factor, we obtain a tangent valued k-form $\mathcal{F}(\varphi, \Lambda)$ on FY, which will be called the F-prolongation of φ with respect to Λ.

If $\varphi = \omega(\Phi)$, we deduce analogously to Proposition 6

$$(38) \qquad \omega\big(\mathcal{F}(\Phi, \Lambda)\big) = \mathcal{F}\big(\omega(\Phi), \Lambda\big) \,.$$

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