ON THE TORSION OF LINEAR HIGHER ORDER CONNECTIONS

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Abstract. For a linear r-th order connection on the tangent bundle we characterize geometrically its integrability in the sense of the theory of higher order G-structures. Our main tool is a bijection between these connections and the principal connections on the r-th order frame bundle and the comparison of the torsions under both approaches.

Our starting point was the problem of integrability (in the sense of the theory of higher order G-structures) of a linear r-th order connection Γ on the tangent bundle of a manifold M. This kind of integrability plays an important role in our theory (jointly with A. Cabras) of the flow prolongation of some tangent valued forms, [1]. Our solution of the problem is heavily based on the concept of the torsion of Γ. In particular, we have to compare two different approaches to the torsion in higher order. This leads us to certain new geometric results. In our opinion, the greater part of them is of individual interest.

In Section 1 we recall the definition of the torsion τ_Γ of Γ by A. Zajtz, [6], in which the jet factorization of the bracket of vector fields is used. Then we deduce that the difference of two torsion free linear r-th order connections that coincide up to order $r-1$ is an arbitrary tensor field of the type $TM \otimes S^{r+1}T^*M$. In Section 2 we clarify that Γ is equivalent to a principal connection $\widetilde{\Gamma}$ on the r-th order frame bundle P^rM. In the latter case, the torsion was introduced by P.C.Yuen, [5], as the exterior covariant differential $D_{\widetilde{\Gamma}}\Theta_r$ of the canonical $(\mathbb{R}^m \times \mathfrak{g}_m^{r-1})$-valued 1-form Θ_r on P^rM. The main result of Section 3 reads that τ_Γ and $D_{\widetilde{\Gamma}}\Theta_r$ coincide in fact. In the proof of Lemma 2 we use as an essential tool the fact that the linear natural operators commute with the Lie differentiation. Section 4 starts with a modification of the concept of the higher order exponential operator from [3] to the case of higher order linear connections on the tangent bundle. This idea enables us to characterize the integrability of Γ in a transparent way.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [4].

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1. Linear higher order connections on TM. Consider the tangent bundle $p : TM \to M$ of an m-dimensional manifold M and its r-th jet prolongation J^rTM. We recall that π^r_k denotes the canonical projection of r-jets into k-jets, $r \geq k \geq 0$.

Definition 1. A linear r-th order connection on M means a linear base preserving morphism $\Gamma : TM \to J^rTM$ satisfying $\pi^r_0 \circ \Gamma = \mathrm{id}_{TM}$.

If x^i are some local coordinates on M, $X^i = dx^i$ are the induced coordinates on TM and X^i_α are the jet coordinates on J^rTM, then the equations of Γ are

$$X^i_\alpha = \Gamma^i_{\alpha j}(x)X^j\,, \qquad 1 \leq |\alpha| \leq r\,, \tag{1}$$

where α is a multiindex of range m.

The $(r-1)$-jet at $x \in M$ of the bracket $[\xi, \eta]$ of two vector fields ξ, η on M depends on the r-jets $j^r_x\xi$ and $j^r_x\eta$. This defines a map

$$[\ ,\ ]_{r-1} : J^rTM \times_M J^rTM \to J^{r-1}TM\,.$$

A. Zajtz, [6], introduced the torsion of Γ as a map $\tau_M : TM \times_M TM \to J^{r-1}TM$ defined by

$$\tau_\Gamma(X, Y) = [\Gamma(X), \Gamma(Y)]_{r-1}\,, \qquad X, Y \in T_xM\,. \tag{2}$$

For $r = 1$, (1) is of the form $X^i_k = \Gamma^i_{kj}(x)X^j$ and one deduces easily that the coordinate expression of τ_Γ is $\Gamma^i_{jk} - \Gamma^i_{kj}$, so that we have the classical concept of torsion.

Let Δ be another linear r-th order connection on M satisfying $\pi^r_{r-1}\circ\Delta = \pi^r_{r-1}\circ\Gamma$. Since the kernel of $\pi^r_{r-1} : J^rTM \to J^{r-1}TM$ is $TM \otimes S^rT^*M$, the difference of Γ and Δ is a section

$$\Gamma - \Delta : M \to TM \otimes S^rT^*M \otimes T^*M\,. \tag{3}$$

Proposition 1. *If both Γ and Δ are torsion free, then the values of $\Gamma - \Delta$ lie in $TM \otimes S^{r+1}T^*M$.*

Proof. The only r-th order terms in $j^{r-1}[\xi, \eta]$ are

$$\xi^j \frac{\partial^r \eta^i}{\partial x^j\, \partial_\beta x} - \eta^j \frac{\partial^r \xi^i}{\partial x^j\, \partial_\beta x}\,, \qquad |\beta| = r - 1\,. \tag{4}$$

Let $\Delta^i_{\alpha j}$ be the Christoffel's of Δ. Then $\Gamma - \Delta$ can be interpreted as a map $TM \times_M TM \to TM \otimes S^{r-1}T^*M$ of the form

$$\big(\Gamma^i_{\beta jk} - \Delta^i_{\beta jk}\big)\xi^j\eta^k = 0\,.$$

If both Γ and Δ are torsion free, then (4) implies

$$\big(\Gamma^i_{\beta jk} - \Gamma^i_{\beta kj} - \Delta^i_{\beta jk} + \Delta^i_{\beta kj}\big)\xi^j\eta^k\,, \qquad |\beta| = r - 1\,.$$

This proves our claim. $\square$

Remark 1. One sees directly from the proof that the difference $\Gamma-\Delta$ is an arbitrary section of $TM \otimes S^{r+1}T^*M$. In a different form this result was deduced in [3].

On $\mathbb{R}^m$ there is a distinguished linear r-th order connection $I_r : T\mathbb{R}^m \to J^rT\mathbb{R}^m$ called the canonical integrable connection. It is defined by

$$I_r(X) = j^r_x \widetilde{X}\,,$$

where $\widetilde{X}$ is the constant vector field constructed from X by means of the translations on $\mathbb{R}^m$. In [1] we introduced

Definition 2. A linear r-th order connection Γ on M is said to be integrable, if for every $x \in M$ there exists a neighbourhood U and a diffeomorphism $f : U \to \mathbb{R}^m$ satisfying $I \circ Tf = J^rTf \circ (\Gamma|_U)$.

Clearly, every integrable Γ is torsion free.

2. Principal connections on P^rM. Consider the r-th order frame bundle P^rM of M. This is an open subset of the space T^r_mM of all (m,r)-velocities on M. We recall there is a natural exchange diffeomorphism $\varkappa_M : T^r_mTM \to TT^r_mM$ with the property that for every vector field $\xi : M \to TM$ the flow prolongation $\mathcal{T}^r_m\xi$ satisfies $\mathcal{T}^r_m\xi = \varkappa_M \circ T^r_m\xi$, where $T^r_m\xi : T^r_mM \to T^r_mTM$, [4]. The flow prolongation $\mathcal{P}^r\xi$ is the restriction of $\mathcal{T}^r_m\xi$ to $P^rM \subset T^r_mM$.

To establish a bijection between the linear r-th order connections on M and the principal connections on P^rM, we introduce a map

$$i : J^rTM \times_M P^rM \to TP^rM\,.$$

For $X = j^r_x\xi$, we construct the flow prolongation $\mathcal{P}^r\xi$. The value $\mathcal{P}^r\xi(u)$, $u \in P^r_xM$, depends on $j^r_x\xi$ only, so that we can set

$$i(X,u) = \mathcal{P}^r\xi(u)\,. \tag{5}$$

On the other hand, for $Y \in T_uP^rM$ we have $\varkappa_M^{-1}(Y) \in T^r_mTM = J^r_0(\mathbb{R}^m, TM)$. Since $u \in J^r_0(\mathbb{R}^m, M)$, we can construct the jet composition $\varkappa_M^{-1}(Y) \circ u^{-1} \in J^r_x(M,TM)$. We have $(T^r_mp)(\varkappa_M^{-1}Y \circ u^{-1}) = (T^r_mp)(\varkappa_M^{-1}Y) \circ u^{-1} = u \circ u^{-1} = j^r_x\,\mathrm{id}_M$, so that $(\varkappa_M^{-1}Y) \circ u \in J^r_xTM$. Clearly,

$$i\big((\varkappa_M^{-1}Y) \circ u^{-1}, u\big) = Y\,.$$

Hence i is a diffeomorphism.

Since P^r is a functor with values in the category of principal bundles, $\mathcal{P}^r\xi$ is a right invariant vector field on P^rM, [4]. Hence for every Γ the rule

$$\widetilde{\Gamma}(u,X) = \mathcal{P}^r\xi(u)\,, \qquad\qquad \Gamma(X) = j^r_x\xi\,, \tag{6}$$

defines the lifting map $\widetilde{\Gamma} : P^rM \times_M TM \to TP^rM$ of a principal connection $\widetilde{\Gamma}$ on P^rM. Indeed, the linearity in TM follows from the basic properties of $\varkappa_M$, [4]. Conversely, if $\Delta : TM \times_M P^rM \to TP^rM$ is the lifting map of a right invariant connection on P^rM, then

$$i^{-1} \circ \Delta : TM \times_M P^rM \to J^rTM \times_M P^rM$$

is of the form $\Gamma \times_M \mathrm{id}_{P^rM}$, where $\Gamma : TM \to J^rTM$ is a linear splitting. This proves

Proposition 2. *The rule (6) establishes a bijection between the linear r-th order connections on M and the principal connections on P^rM.*

Write $e_{r-1} = j_0^{r-1}\,\mathrm{id}_{\mathbb{R}^m}$ and $V_m^{r-1} = \mathbb{R}^m \times \mathfrak{g}_m^{r-1} = T_{e_{r-1}}P^{r-1}\mathbb{R}^m$. The canonical V_m^{r-1}-valued 1-form on P^rM is defined as follows, [4]. Every $u = j_0^r f$, $f : \mathbb{R}^m \to M$, induces $P^{r-1}f : P^{r-1}\mathbb{R}^m \to P^{r-1}M$. The tangent map $\widetilde{u} := T_{e_{r-1}}P^{r-1}f : T_{e_{r-1}}P^{r-1}\mathbb{R}^m \to T_{u_{r-1}}P^{r-1}M$, $u_{r-1} = \pi^r_{r-1}(u)$ depends on u only. Then one defines

$$\Theta_r(A) = \widetilde{u}^{-1}\big(T\pi^r_{r-1}(A)\big)\,, \qquad A \in T_uP^rM\,. \tag{7}$$

P.C.Yuen, [5], introduced the torsion of $\widetilde{\Gamma}$ as the exterior covariant differential $D_{\widetilde{\Gamma}}\Theta_r$ of Θ_r.

3. The comparison of torsions. Since $J^{r-1}TM$ is an r-th order natural bundle, it is a bundle associated to P^rM with the standard fiber $U_m^{r-1} := J_0^{r-1}T\mathbb{R}^m$, [4]. Every $u = j_0^r h \in P_x^rM$ is interpreted as a frame map $\overline{u} : U_m^{r-1} \to J_x^{r-1}TM$ as follows. We construct $Th : T\mathbb{R}^m \to TM$ and set $\overline{u} = J_0^{r-1}Th$. We have $J_0^{r-1}T\mathbb{R}^m \subset T_m^{r-1}T\mathbb{R}^m$ and the exchange $\varkappa_M : T_m^{r-1}TM \to TT_m^{r-1}M$ maps U_m^{r-1} into V_m^{r-1}. Indeed, $j_0^{r-1}\frac{\partial}{\partial t}\Big|_0 \psi(t,\tau) \in J_0^{r-1}T\mathbb{R}^m$ if and only if $\psi(0,\tau) = \mathrm{id}_{\mathbb{R}^m}$. Then $\frac{\partial}{\partial t}\Big|_0 j_0^{r-1}\psi(t,\tau) \in T_{e_{r-1}}P^{r-1}M$ and vice versa. This yields an identification

$$\varkappa_0 : U_m^{r-1} \to V_m^{r-1}\,.$$

Lemma 1. *For every vector field ξ on M and every $u \in P_x^rM$, we have*

$$\Theta_r\big(\mathcal{P}^r\xi(u)\big) = \varkappa_0\big(\overline{u}^{-1}(j_x^{r-1}\xi)\big)\,. \tag{8}$$

Proof. The naturality of $\varkappa$ on $h : \mathbb{R}^m \to M$ yields a commutative diagram

$$\begin{tikzcd} T_m^{r-1}T\mathbb{R}^m \arrow[r, "T_m^{r-1}Th"] \arrow[d, "\varkappa_{\mathbb{R}^m}"'] & T_m^{r-1}TM \arrow[d, "\varkappa_M"] \\ TT^{r-1}\mathbb{R} \arrow[r, "TT_m^{r-1}h"] & TT_m^{r-1}M \end{tikzcd} \tag{9}$$

The right column of (9) maps $j_x^{r-1}\xi$ into $\mathcal{P}_x^{r-1}\xi(u_{r-1})$, $u = j_0^r h$. Then (8) follows directly from the definitions of $\widetilde{u}$, $\overline{u}$ and $\varkappa_0$. □

Since $D_{\widetilde{\Gamma}}\Theta_r$ is a horizontal V_m^{r-1}-valued 2-form on P^rM, it can be interpreted as a map $P^rM \to V_m^{r-1} \otimes \wedge^2T^*M$, see e.g. [4], p. 112. Taking into account the identification $\widetilde{u}_1 : \mathbb{R}^m \to T_xM$, $u_1 = \pi_1^r(u)$, we construct

$$\overline{D}_{\widetilde{\Gamma}}\Theta_r : P^rM \to V_m^{r-1} \otimes \wedge^2\mathbb{R}^m\,. \tag{10}$$

On the other hand, τ_Γ can be interpreted as a section of $J^{r-1}TM \otimes \wedge^2T^*M$. So its frame form, see [4], is a map

$$\overline{\tau}_\Gamma : P^rM \to U_m^{r-1} \otimes \wedge^2\mathbb{R}^{m*}\,.$$

Proposition 3. *Under the identification $\varkappa_0$, we have $\overline{D}_{\widetilde{\Gamma}}\Theta_r = \frac{1}{2}\overline{\tau}_\Gamma$.*

Proof. The most important part of the proof is the following lemma. Let ξ, η be two vector fields on M. Then $\Theta_r(\mathcal{P}^r\eta)$ is a V_m^{r-1}-valued function on P^rM, so that we can construct its derivation $(\mathcal{P}^r\xi)\Theta_r(\mathcal{P}^r\eta)$ in the direction of $\mathcal{P}^r\xi$, which is another V_m^{r-1}-valued function on P^rM.

Lemma 2. *We have $(\mathcal{P}^r\xi)\Theta_r(\mathcal{P}^r\eta) = \Theta_r(\mathcal{P}^r([\xi,\eta]))$.*

Proof. As a direct consequence of the definition, Θ_r is natural, i.e. for every local diffeomorphism $f : M \to \overline{M}$ we have

$$\Theta_{r,M} = \Theta_{r,\overline{M}} \circ TP^rf\,.$$

Consider the operator $\widetilde{\Theta}_r$ transforming each vector field on M into a V_m^{r-1}-valued function on P^rM

$$\widetilde{\Theta}_r(\eta) = \Theta_r(\mathcal{P}^r\eta)\,.$$

Clearly, $\widetilde{\Theta}_r$ is linear and natural. By Proposition 49.5 of [4], $\widetilde{\Theta}_r$ commutes with the Lie differentiation, i.e.

$$\mathscr{L}_\xi\widetilde{\Theta}_r(\eta) = \widetilde{\Theta}_r(\mathscr{L}_\xi\eta)\,.$$

The geometrical meaning of the Lie derivative on the left hand side is $(\mathcal{P}^r\xi)\Theta_r(\mathcal{P}^r\eta)$. On the right hand side, we use the well known fact $\mathscr{L}_\xi\eta = [\xi,\eta]$. □

Consider now $u \in P_x^rM$ and $X, Y \in T_xM$, $\Gamma(X) = j_x^r\xi$, $\Gamma(Y) = j_x^r\eta$. If we interpret $\overline{D}_{\widetilde{\Gamma}}\Theta_r$ as a map $P^rM \times \mathbb{R}^m \times \mathbb{R}^m \to V_m^{r-1}$, we have

$$\overline{D}_{\widetilde{\Gamma}}\Theta_r\big(u, \widetilde{u}_1^{-1}(X), \widetilde{u}_1^{-1}(Y)\big) = d\Theta_r\big(\mathcal{P}^r\xi(u), \mathcal{P}^r\eta(u)\big)\,.$$

Applying Lemma 2, we obtain

$$\begin{aligned} 2d\Theta_r(\mathcal{P}^r\xi, \mathcal{P}^r\eta) &= (\mathcal{P}^r\xi)\Theta_r(\mathcal{P}^r\eta) - (\mathcal{P}^r\eta)\Theta_r(\mathcal{P}^r\xi) - \Theta_r\big([\mathcal{P}^r\xi, \mathcal{P}^r\eta]\big) \\ &= \Theta_r\big([\mathcal{P}^r\xi, \mathcal{P}^r\eta]\big)\,. \end{aligned}$$

By Lemma 1, the value of the last expression at u corresponds to $[\Gamma(X), \Gamma(Y)]_{r-1}$. This proves Proposition 3.

4. The integrability of Γ. Consider a classical torsion free connection $\Gamma : TM \to J^1TM$ and denote by $\exp_x : T_xM \to M$ its exponential mapping at $x \in M$. We define the r-th exponential prolongation of Γ

$$\varepsilon_r(\Gamma) : TM \to J^rTM$$

as follows. Every $X \in T_xM$ is extended into a constant vector field $\widetilde{X}$ on T_xM. This vector field is transformed by $\exp_x$ into a vector field on M and we set

$$\varepsilon_r(\Gamma)(X) = j_x^r(T\exp_x \circ \widetilde{X} \circ \exp_x^{-1})\,. \tag{11}$$

The linearity is obvious, so that $\varepsilon_r(\Gamma)$ is a linear r-th order connection on M. The fact $[\widetilde{X}, \widetilde{Y}] = 0$ for another $Y \in T_xM$ implies that $\varepsilon_r(\Gamma)$ is torsion free.

Clearly, if Γ is integrable, then $\varepsilon_r(\Gamma)$ is so.

Remark 2. We remark that the definition of ε_r is based on a similar idea as the concept of the exponential operator E_r from [3]. In that paper we used the fact that the torsion free connections on P^rM are in bijection with some reductions of $P^{r+1}M$. But a complete comparison of both approaches is beyond the scope of the present paper.

Since every integrable linear r-th order connection Γ on M is torsion free, the torsion of Γ is the first obstruction to the integrability of Γ. Consider the underlying connections $\Gamma_k = \pi^r_k \circ \Gamma$, $k = 1, \dots, r$. Clearly, if Γ is torsion free or integrable, then each Γ_k is so. Assume Γ is torsion free. By a classical result, the torsion free connection Γ_1 is integrable, if and only if its curvature vanishes. This is the second obstruction to the integrability of Γ. If the curvature vanishes, each connection $\varepsilon_k(\Gamma_1)$ is integrable. The difference

$$\Gamma_2 - \varepsilon_2(\Gamma_1)$$

is a tensor field of the type $TM \otimes S^3T^*M$ that is the third obstruction to the integrability of Γ. Assume by induction that the first up to $(k+1)$-st obstructions to the integrability of Γ vanish. Then $\Gamma_k = \varepsilon_k(\Gamma_1)$ and the tensor field of type $TM \otimes S^{k+2}T^*M$

$$\Gamma_{k+1} - \varepsilon_{k+1}(\Gamma_1)$$

is the next obstruction to the integrability of Γ. If all the $r+1$ obstructions vanish, then $\Gamma = \varepsilon_r(\Gamma_1)$ is integrable. Thus, we have proved

Proposition 4. *Γ is integrable, if and only if all the following conditions are satisfied*

a) Γ is torsion free,

b) Γ_1 is curvature free,

c) all the gradually defined tensor fields $\Gamma_k - \varepsilon_k(\Gamma_1)$, $k = 2, \dots, r$ vanish. □

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