ON THE TORSION-FREE CONNECTIONS ON HIGHER ORDER FRAME BUNDLES

IVAN KOLÁŘ

ABSTRACT. Using the r-jets of flows of vector fields, we show that every torsion-free linear r-th order connection Γ on the tangent bundle of a manifold M determines a reduction of the $(r+1)$-st order frame bundle of M to the general linear group. We deduce that this reduction coincides with the well known reduction determined by the principal connection induced by Γ on the r-th order frame bundle of M.

Introduction. Our starting point is the fact that the principal connections on the r-th order frame bundle P^rM of a manifold M are in bijection with the linear r-th order connections on the tangent bundle TM, i.e. with the linear splittings

$$\Gamma : TM \to J^rTM \tag{1}$$

of the jet projection $J^rTM \to TM$. We shall write

$$\widetilde{\Gamma} : P^rM \to J^1P^rM \tag{2}$$

for the principal connection corresponding to (1). Its lifting map, denoted by the same symbol,

$$\widetilde{\Gamma} : P^rM \times_M TM \to TP^rM$$

is determined by

$$\widetilde{\Gamma}(u, A) = \mathcal{P}^rX(u)\,, \quad u \in P^rM,\ A \in T_xM,\ x \in M, \tag{3}$$

where $\mathcal{P}^rX$ is the flow prolongation of a vector field $X : M \to TM$ satisfying $j^r_xX = \Gamma(A)$, [3].

The structure group of P^rM is G^r_m, $m = \dim M$. There is a canonical $\mathbb{R}^m \times \mathfrak{g}^{r-1}_m$-valued one-form Θ_r on P^rM. P. C. Yuen, [5], introduced the torsion of $\widetilde{\Gamma}$ as the exterior covariant differential

$$D_{\widetilde{\Gamma}}\,\Theta_r\,, \tag{4}$$

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see also [1]. On the other hand, the $(r-1)$-jet at x of the bracket $[X,Y]$ of two vector fields X, Y on M depends on $j^r_x X$ and $j^r_x Y$ only. This defines a map

$$[\ ,\ ]_{r-1} : J^r TM \times_M J^r TM \to J^{r-1}TM\,. \tag{5}$$

The torsion of Γ can be introduced, [6], as a map

$$\tau_\Gamma : TM \times_M TM \to J^{r-1}TM$$

defined by

$$\tau_\Gamma(A,B) = [\Gamma(A), \Gamma(B)]_{r-1}\,, \qquad A,B \in T_x M\,. \tag{6}$$

In [3] we deduced that the torsions $D_{\widetilde{\Gamma}}\Theta_r$ and τ_Γ coincide in a natural way.

There is a canonical injection $i_{r+1} : P^{r+1}M \to J^1 P^r M$, see formula (13) below. If $\widetilde{\Gamma}$ is torsion-free, there is a well-known map $\varrho(\widetilde{\Gamma}) : P^1 M \to P^{r+1}M$ defined by the induction

$$i_{r+1} \circ \varrho(\widetilde{\Gamma}) = \widetilde{\Gamma} \circ \varrho(\widetilde{\Gamma}_{r-1})\,, \tag{7}$$

where $\widetilde{\Gamma}_{r-1}$ is the underlying connection on $P^{r-1}M$, so that $\varrho(\widetilde{\Gamma}_{r-1}) : P^1 M \to P^r M$ by the induction hypothesis. If we consider the canonical injection $GL(m,\mathbb{R}) \hookrightarrow G^{r+1}_m$, [4], p. 130, then $\varrho(\widetilde{\Gamma})(P^1 M)$ is a reduction of $P^{r+1}M$ to $GL(m,\mathbb{R})$. This establishes the well-known bijection between the torsion-free connections on $P^r M$ and the reductions of $P^{r+1}M$ to $GL(m,\mathbb{R})$, [2], [3].

In Section 1 of the present paper we use the r-jets of flows of vector fields to construct a map $\sigma(\Gamma) : P^1 M \to P^{r+1}M$ for every torsion-free linear r-th order connection Γ on TM. Our proof of the fact that $\sigma(\Gamma)(P^1 M)$ is a reduction of $P^{r+1}M$ to $GL(m,\mathbb{R})$ is based on two interesting lemmas concerning the r-jets of the bracket of vector fields, the proofs of which we postpone to Section 3. In Section 2 we deduce $\sigma(\Gamma) = \varrho(\widetilde{\Gamma})$, i.e. both constructions of a reduction of $P^{r+1}M$ to $GL(m,\mathbb{R})$ coincide.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [4].

1. The reduction $\sigma(\Gamma)$. Consider $\Gamma : TM \to J^r TM$. For a linear frame $u \in P^1_x M$, $u = (A_1, \dots, A_m)$, $A_i \in T_x M$, we take vector fields X_i satisfying $j^r_x X_i = \Gamma(A_i)$, $i = 1, \dots, m$. Then

$$\big(Fl^{X_1}_{t^1} \circ \cdots \circ Fl^{X_m}_{t^m}\big)(x)$$

is a local map $\mathbb{R}^m \to M$ and we define

$$\sigma(\Gamma)(u) = j^{r+1}_0 \big(Fl^{X_1}_{t^1} \circ \cdots \circ Fl^{X_m}_{t^m}\big)(x) \in P^{r+1}_x M\,, \tag{8}$$

where $0 \in \mathbb{R}^m$. One verifies easily that $\sigma(\Gamma)(u)$ depends on u and Γ only.

Proposition 1. *If Γ is torsion-free, then $\sigma(\Gamma)(P^1M)$ is a reduction of $P^{r+1}M$ to $GL(m,\mathbb{R})$.*

The proof will be based on the following two lemmas, the proofs of which we postpone to the last section.

Consider two vector fields X and Y on M. Then $\big(Fl^X_t \circ Fl^Y_\tau\big)(x)$ is a local map $\mathbb{R}^2 \to M$, so that $j^{r+1}_{0,0}\big(Fl^X_t \circ Fl^Y_\tau\big)(x) \in (T^{r+1}_2 M)_x$ is a $(2, r+1)$-velocity on M.

Lemma 1. *If $j^{r-1}_x[X,Y] = 0$, then*

$$j^{r+1}_{0,0}\big(Fl^X_t \circ Fl^Y_\tau\big)(x) = j^{r+1}_{0,0}\big(Fl^Y_\tau \circ Fl^X_t\big)(x)\,. \tag{9}$$

Further, $\big(Fl^X_t \circ Fl^Y_t\big)(x)$ is a local map $\mathbb{R} \to M$, so that $j^{r+1}_0\big(Fl^X_t \circ Fl^Y_t\big)(x) \in (T^{r+1}_1 M)_x$ is a $(1, r+1)$-velocity on M.

Lemma 2. *If $j^{r-1}_x[X,Y] = 0$, then*

$$j^{r+1}_0\big(Fl^X_t \circ Fl^Y_t\big)(x) = j^{r+1}_0\big(Fl^{X+Y}_t\big)(x)\,. \tag{10}$$

Now we prove Proposition 1 by using Lemmas 1 and 2. We shall also use the well known formula

$$Fl^X_{at} = Fl^{aX}_t\,, \qquad a \in \mathbb{R}\,. \tag{11}$$

Take $g = (a^i_j) \in GL(m,\mathbb{R})$, $u = (A_i) \in P^1_x M$ and consider $ug = (a^j_i A_j)$. Write $\Gamma(A_i) = j^r_x X_i$. Since Γ is torsion-free, by (10), (11) and (9) we obtain gradually

$$\begin{aligned}
\sigma(\Gamma)(ug) &= j^{r+1}_0\big(Fl^{a^1_1 X_1 + \dots + a^m_1 X_m}_{t^1} \circ \dots \circ Fl^{a^1_m X_1 + \dots + a^m_m X_m}_{t^m}\big) \\
&= j^{r+1}_0\big(Fl^{a^1_1 X_1}_{t^1} \circ \dots \circ Fl^{a^m_1 X_m}_{t^1} \circ \dots \circ Fl^{a^1_m X_1}_{t^m} \circ \dots \circ Fl^{a^m_m X_m}_{t^m}\big) \\
&= j^{r+1}_0\big(Fl^{X_1}_{a^1_1 t^1} \circ \dots \circ Fl^{X_m}_{a^m_1 t^1} \circ \dots \circ Fl^{X_1}_{a^1_m t^m} \circ \dots \circ Fl^{X_m}_{a^m_m t^m}\big) \\
&= j^{r+1}_0\big(Fl^{X_1}_{a^1_1 t^1 + \dots + a^1_m t^m} \circ \dots \circ Fl^{X_m}_{a^m_1 t^1 + \dots + a^m_m t^m}\big)\,.
\end{aligned}$$

This proves Proposition 1.

2. The main result. We shall use the following form of the canonical injection $i_{r+1} : P^{r+1}M \to J^1 P^r M$. We have $P^r M \subset T^r_m M$, where T^r_m is the functor of (m,r)-velocities. Clearly, $j^r_0 f \in T^r_m M$, $f : \mathbb{R}^m \to M$, can be expressed in the form

$$j^r_0 f = \big(T^r_m f\big)(e)\,, \qquad e = j^r_0\, \mathrm{id}_{\mathbb{R}^m}\ . \tag{12}$$

Write $E_i = \frac{\partial}{\partial t}\big|_0 j^r_0 \tau^i_t \in T_e T^r_m \mathbb{R}^m$, where $\tau^i_t : \mathbb{R}^m \to \mathbb{R}^m$ is the translation $\bar t^1 = t^1, \dots, \bar t^i = t^i + t, \dots, \bar t^m = t^m$. If we consider $j^{r+1}_0 \psi \in P^{r+1}M$, then

$$\big(TT^r_m \psi\big)(E_i) \tag{13}$$

is an m-tuple of tangent vectors at $j^r_0\psi \in P^r M$. The linear span of these vectors defines $i_{r+1}(j^{r+1}_0\psi) \in J^1 P^r M$.

Proposition 2. *If Γ is a torsion-free linear r-th order connection on TM and $\widetilde{\Gamma}$ is the corresponding principal connection on P^rM, then $\sigma(\Gamma) = \varrho(\widetilde{\Gamma})$.*

Proof. We proceed by induction. If Γ_{r-1} and $\widetilde{\Gamma}_{r-1}$ are the underlying connections in the order $r-1$, then

$$\sigma(\Gamma_{r-1}) = \varrho(\widetilde{\Gamma}_{r-1}) \tag{14}$$

by the induction hypothesis. Consider $u = (A_1, \dots, A_m) \in P^1_x M$ and write

$$v = \sigma\big(\Gamma_{r-1}\big)(u) = \varrho\big(\widetilde{\Gamma}_{r-1}\big)(u)\,.$$

By (13), $i_{r+1}\big(j_0^{r+1}\big(Fl^{X_1}_{t^1} \circ \dots \circ Fl^{X_m}_{t^m}\big)(x)\big)$ is the linear span of the vectors

$$TT^r_m\big(Fl^{X_1}_{t^1} \circ \dots \circ Fl^{X_m}_{t^m}\big)(E_i)\,. \tag{15}$$

Using the basic properties of flows, Lemma 1 and (12), we deduce that (15) is equal to

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big|_0 T^r_m\big(Fl^{X_1}_{t^1} \circ \dots \circ Fl^{X_i}_{t+t^i} \circ \dots \circ Fl^{X_m}_{t^m}\big)(e)\\
=&\frac{\partial}{\partial t}\Big|_0 \big(Fl^{\mathcal{T}^r_m X_i}_{t} \circ Fl^{\mathcal{T}^r_m X_1}_{t^1} \circ \dots \circ Fl^{\mathcal{T}^r_m X_m}_{t^m}\big)(e)\\
=&\mathcal{T}^r_m X_i\big(T^r_m(Fl^{X_1}_{t^1} \circ \dots \circ Fl^{X_m}_{t_m})(e)\big) = \mathcal{T}^r_m X_i(v)\,,
\end{aligned}$$

where $\mathcal{T}^r_m X_i$ denotes the flow prolongation of X_i. By (3) and by the induction hypothesis, this m-tuple spans $\varrho(\widetilde{\Gamma})(v)$. $\square$

3. The proofs of Lemmas 1 and 2. In general, if we have two maps $f, g : \mathbb{R}^m \to \mathbb{R}^m$, it suffices to verify the condition $j^r_0 f = j^r_0 g$ on all curves of the form $x^i = a^i t$, $i = 1, \dots, m$, [4]. By the flow property (11), Lemma 1 follows from the fact that $j^{r-1}_x [X, Y] = 0$ implies

$$j_0^{r+1}\big(Fl^X_t \circ Fl^Y_t\big)(x) = j_0^{r+1}\big(Fl^Y_t \circ Fl^X_t\big)(x) \in T^{r+1}_1 M\,. \tag{16}$$

But this is a direct consequence of Lemma 2. So it suffices to prove Lemma 2. We have the following 3 cases.

I. If $X(x) = Y(x) = 0$, then the $(r+1)$-jets of the flows of X and Y are in the group of all invertible $(r+1)$-jets of M into M with source x and target x and we have a well known result concerning Lie groups.

II. If $X(x) \neq 0$, we can consider such local coordinates on M that $X = \frac{\partial}{\partial x^1}$. Then $j^{r-1}_x[\frac{\partial}{\partial x^1}, Y] = 0$ means

$$D_\alpha \frac{\partial Y^i(x)}{\partial x^1} = 0\,, \qquad 0 \le \|\alpha\| \le r-1\,, \tag{17}$$

where Y^i are the coordinate components of Y and D_α denotes the partial derivative with respect to a multiindex α of the range m.

The flow $\psi^i(t,x)$ of the vector field $\frac{\partial}{\partial x^1} + Y$ satisfies

$$\frac{\partial \psi^i(t,x)}{\partial t} = \delta^i_1 + Y^i\big(\psi(t,x)\big)\,. \tag{18}$$

If $\eta^i(t,x)$ denotes the flow of Y, then the coordinate expression of $Fl^X_t \circ Fl^Y_t$ is

$$\mu^i(t,x) = \delta^i_1 t + \eta^i(t,x)\,. \tag{19}$$

Hence

$$\frac{\partial \mu^i(t,x)}{\partial t} = \delta^i_1 + \frac{\partial \eta^i(t,x)}{\partial t} = \delta^i_1 + Y^i\big(\eta(t,x)\big)\,. \tag{20}$$

From (19) we obtain

$$\frac{\partial^k \mu^i(t,x)}{\partial t^k} = \frac{\partial^k \eta^i(t,x)}{\partial t^k}\,, \qquad k \geq 2\,. \tag{21}$$

For $t = 0$, (18) and (20) yield directly $\frac{\partial \psi^i(0,x)}{\partial t} = \frac{\partial \mu^i(0,x)}{\partial t}$. Then we find by direct evaluation

$$\frac{\partial^2 \psi^i(t,x)}{\partial t^2} = \frac{\partial Y^i\big(\psi(t,x)\big)}{\partial x^j}\,\frac{\partial \psi^j(t,x)}{\partial t}\,, \tag{22}$$

$$\frac{\partial^2 \mu^i(t,x)}{\partial t^2} = \frac{\partial Y^i\big(\eta(t,x)\big)}{\partial x^j}\,\frac{\partial \eta^j(t,x)}{\partial t}\,. \tag{23}$$

Hence (17) implies

$$\frac{\partial^2 \psi^i(0,x)}{\partial t^2} = \frac{\partial^2 \mu^i(0,x)}{\partial t^2}\,. \tag{24}$$

By iteration we deduce (10) for every r.

III. The case $Y(x) \neq 0$ can be reduced to II by using $Fl^X_t \circ Fl^Y_t = \big(Fl^Y_{-t} \circ Fl^X_{-t}\big)^{-1}$.

This proves Lemmas 1 and 2.

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Ivan Kolář
Department of Algebra and Geometry
Faculty of Science, Masaryk University
Janáčkovo nám. 2a
662 95 Brno, CZECH REPUBLIC
E-mail: kolar@math.muni.cz