FUNCTORIAL PROLONGATIONS OF LIE ALGEBROIDS

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Abstract. We study the prolongation of Lie algebroids and their actions with respect to product and fiber product preserving bundle functors. Our investigation of the fiber product case is based on the concept of formal action. Some explicit formulae using the Weil algebra technique are deduced.

Let $\mathcal{M}f$ be the category of smooth manifolds and $\mathcal{FM}$ be the category of smooth fibered manifolds. Our starting point are the facts that every product preserving bundle functor on $\mathcal{M}f$ coincides with a Weil functor T^A, [5], and every fiber product preserving bundle functor F on the subcategory $\mathcal{FM}_m \subset \mathcal{FM}$ of fibered manifolds with m-dimensional bases and $\mathcal{FM}$-morphisms with local diffeomorphisms as base maps can be identified with a triple $F = (A, H, t)$, [6], where A is a Weil algebra and H and t are described in Section 1 below. In the first case, every principal bundle $P(M, G)$ induces a principal bundle $T^A P(T^A M, T^A G)$, [5]. The case of $\mathcal{FM}_m$ is more sophisticated, but even in this case F induces from P a principal bundle $W^F P \to M$, [1]. Thus, in each case the Lie algebroid LP of P determines the Lie algebroid of the induced principal bundle. The main purpose of the present paper is to study these prolongation procedures within the framework of the theory of Lie algebroids.

In Section 1 we recall the concept of Lie algebroid and the basic facts concerning the functorial prolongations of principal fiber bundles. In Section 2 we construct the vertical A-prolongation $\mathcal{V}^A E \to M$ of a regular Lie algebroid $E \to M$ in a rather elementary way. Section 3 is devoted to the prolongation $T^A E \to T^A M$ and the Weil algebra technique plays an irreplaceable role here.

In the case of $F = (A, H, t)$, we meet, in an essential way, the fact that a functor can be applied to maps, while the action as well as the bracket of a Lie algebroid are certain kinds of differential operators. It is well known that a classical k-th order differential operator C between two fibered manifolds $Y \to M$ and $Z \to M$ can be characterized by its formal version $\mathcal{C} : J^k Y \to Z$. This is a smooth map with the property $C(s) = \mathcal{C} \circ j^k s$ for every section $s : M \to Y$. In Section 4 we study an action φ of a Lie algebroid $E \to M$ on a vector bundle $D \to M$ and define its formal version as a map $\Phi : E \times_{TM} TD \to D$ such that $\varphi(\varrho, \sigma) = \Phi\big(\varrho, (T\sigma) \circ (q\varrho)\big)$ for every sections $\varrho : M \to E$ and $\sigma : M \to D$, where $q : E \to TM$ is the anchor map of E. Our next step is based on a natural map $\mu^F_Y : J^r TM \times_{FTM} FTY \to TFY$

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for every fibered manifold $Y \to M$, which we introduced in [3] in connection with the study of the flow prolongation of projectable vector fields on Y with respect to the functor F. In Section 5 we clarify that μ can be used for constructing a map $\mathcal{F}\Phi$ that looks like the formal version of an action of $\mathcal{W}^F E = J^r TM \times_{FTM} FE$ on the vector bundle FD. Proposition 4 reads that $\mathcal{W}^F E$ is a Lie algebroid for every transitive Lie algebroid E. In Section 7 we prove in the case of the canonical action of the Lie algebroid LP on a vector bundle D associated to the principal bundle P that $\mathcal{F}\Phi$ is the formal version of the induced action of $\mathcal{W}^F(LP)$ on FD. We underline that our proofs are frequently based on some explicit formulae using the Weil algebra technique. These formulae seem to be of individual interest.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notations from [5].

1. Basic concepts. We recall that a Lie algebroid over a manifold M is a vector bundle $E \to M$ together with a linear base preserving morphism $q : E \to TM$, called the anchor map, and a bracket $[\![\,,]\!] : C^\infty E \times C^\infty E \to C^\infty E$, where $C^\infty E$ is the space of all sections of E, such that

(i) $C^\infty E$ is a Lie algebra with respect to $[\![\,,\,]\!]$,

(ii) q induces a homomorphism $C^\infty E \to C^\infty TM$, i.e.

$$q\big([\![\varrho_1, \varrho_2]\!]\big) = [q\varrho_1, q\varrho_2] \quad \text{for every} \quad \varrho_1, \varrho_2 \in C^\infty E\,,$$

where $[\,,]$ is the bracket of vector fields on M,

(iii) $[\![\varrho_1, f\varrho_2]\!] = f[\![\varrho_1, \varrho_2]\!] + (q\varrho_1)(f)\varrho_2$, where $(q\varrho_1)(f)$ is the derivative of $f : M \to \mathbb{R}$ with respect to vector field $q\varrho_1$.

A Lie algebroid $(E, q) \to M$ is called regular or transitive, if q is of constant rank or surjective submersion, respectively.

Every principal bundle $P(M, G)$ induces a Lie algebroid $LP \to M$ of the right-invariant vector fields on P, for which the bracket is the standard bracket of vector fields and the anchor map is the canonical projection $TP/G \to TM$. Clearly, LP is transitive.

The Lie algebroid of the product principal bundle $M \times G$ is the product Lie algebroid $(TM \times \mathfrak{g}) \to M$, where $\mathfrak{g} = \text{Lie } G$, the anchor map is the first projection and the bracket is defined by

$$[\![(X_1, g_1), (X_2, g_2)]\!] = \big([X_1, X_2], X_1 g_2 - X_2 g_1 + [g_2, g_1]_{\mathfrak{g}}\big)\,, \tag{1}$$

where X_i are vector fields on M, $g_i : M \to \mathfrak{g}$ are $\mathfrak{g}$-valued functions, $X_i g_j$ is the derivative of g_j with respect to vector field X_i and $[g_2, g_1]_{\mathfrak{g}}$ means pointwise the bracket in $\mathfrak{g}$, $i, j = 1, 2$. A fundamental theorem reads that every transitive Lie algebroid is locally isomorphic to a product one, [8].

Every product preserving bundle functor on $\mathcal{M}f$ is a Weil functor T^A, where $A = \mathbb{R} \times N$ is a Weil algebra with nilpotent ideal N, [5]. In particular, in the case of the functor T^r_k of (k, r)-velocities, the corresponding Weil algebra is $\mathbb{D}^r_k = J^r_0(\mathbb{R}^k, \mathbb{R})$. If G is a Lie group with the composition $\gamma : G \times G \to G$, then $T^A G$ is a Lie group with the composition $T^A\gamma : T^A G \times T^A G \to T^A G$. We have

$$\text{Lie}\,(T^A G) = \mathfrak{g} \otimes A$$

with the bracket bilinearly generated by the formula

$$[v_1 \otimes a_1, v_2 \otimes a_2] = [v_1, v_2]_{\mathfrak{g}} \otimes a_1 a_2\,, \tag{2}$$

$v_1, v_2 \in \mathfrak{g}$, $a_1, a_2 \in A$, the product a_1, a_2 being in A, [5].

If $p : P \to M$ is a principal bundle with structure group G, then

$$T^A p : T^A P \to T^A M \tag{3}$$

is a principal bundle with structure group $T^A G$. Moreover, the vertical subbundle $V^A P \subset T^A P$ is also a principal bundle

$$V^A P(M, T^A G)\,. \tag{4}$$

According to [6], every fiber product preserving bundle functor F on $\mathcal{FM}_m$ is identified with a triple $F = (A, H, t)$, where A is a Weil algebra, $H : G^r_m \to \operatorname{Aut} A$ is a group homomorphism, r being the base order of F, G^r_m is the r-th jet group in dimension m and $t : \mathbb{D}^r_m \to A$ is an equivariant algebra homomorphism. By the Weil theory, every algebra homomorphism $c : A_1 \to A_2$ defines a natural map $c_Q : T^{A_1} Q \to T^{A_2} Q$ for every manifold Q, [5]. In particular, H induces an action H_Q of G^r_m on $T^A Q$. For a product fibered manifold $M \times Q$, $F(M \times Q)$ is the fiber bundle associated to the r-th order frame bundle $P^r M$ of the form

$$F(M \times Q) = P^r M[T^A Q, H_Q]\,, \tag{5}$$

[6]. For an arbitrary fibered manifold $p : Y \to M$, $FY \subset P^r M[T^A Y, H_Y]$ is the subspace of all equivalence classes $\{u, Z\}$

$$u \in P^r M,\ Z \in T^A Y \quad \text{satisfying} \quad t_M(u) = T^A p(Z)\,. \tag{6}$$

For example, in the most important case of the functor J^r of r-jet prolongation of fibered manifolds, we have $A = \mathbb{D}^r_m$, H is the identity of $G^r_m = \operatorname{Aut} \mathbb{D}^r_m$ and $t = \mathrm{id}_{\mathbb{D}^r_m}$.

A classical result reads, [5], that the r-jet prolongation $J^r D$ of an associated bundle $D = P[S, \ell]$ is an associated bundle $J^r D = W^r P[T^r_m S, W^r_m \ell]$, where

$$W^r P = P^r M \times_M J^r P$$

is a principal bundle over M, the structure group of which is the semidirect product

$$W^r_m G = G^r_m \rtimes T^r_m G\,,$$

and the action $W^r_m \ell$ of $W^r_m G$ on $T^r_m S$ is

$$W^r_m \ell\big((X, v), Z\big) = T^r_m \ell(v, Z) \circ X^{-1}\,,$$

$X \in G^r_m$, $v \in T^r_m G$, $Z \in T^r_m S$.

In the general case $F = (A, H, t)$, it is deduced in [1] that FD is a bundle $W^F P[T^A S, W^A_H \ell]$ associated to the principal bundle

$$W^F P = P^r M \times_M FP \tag{7}$$

over M with structure group

$$W^A_H G = G^r_m \rtimes_H T^A G\,.$$

The composition in this semidirect product is of the form

$$(X_1, Z_1)(X_2, Z_2) = \big(X_1 \circ X_2, T^A\gamma(H_G(X_2)(Z_1), Z_2)\big)\,, \tag{8}$$

where $H_G(X_2)$ denotes the natural diffeomorphism of $T^A G$ into itself determined by $H(X_2)$, $X_2 \in G^r_m$. The action $W^A_H \ell$ of $W^A_H G$ on $T^A S$ is of the form

$$W^A_H \ell\big((X, v), Z\big) = H_S(X)\big(T^A \ell(v, Z)\big)\,, \tag{9}$$

$X \in G^r_m$, $v \in T^A G$, $Z \in T^A S$.

The group homomorphism $H : G^r_m \to \mathrm{Aut}\, A$ induces a Lie algebra homomorphism $h : \mathfrak{g}^r_m \to \mathrm{Lie}\,(\mathrm{Aut}\, A) = \mathrm{Der}\, A$. The bracket in $\mathrm{Lie}\,(W^A_H G) = \mathfrak{g}^r_m \rtimes (\mathfrak{g} \otimes A)$ is of the form, [3],

$$\begin{aligned} &\big[(X_1, v_1 \otimes a_1), (X_2, v_2 \otimes a_2)\big] \\ &\qquad = \big([X_1, X_2], v_2 \otimes h(X_1)(a_2) - v_1 \otimes h(X_2)(a_1) + [v_1, v_2]_{\mathfrak{g}} \otimes a_1 a_2\big)\,, \end{aligned} \tag{10}$$

where the bracket $[X_1, X_2]$ is in $\mathfrak{g}^r_m$.

Consider an action $\psi : \mathfrak{g} \times S \to S$ of the Lie algebra $\mathfrak{g}$ on a vector space S. According to [5], the induced action of $T^A \mathfrak{g} = \mathfrak{g} \otimes A$ on $T^A S = S \otimes A$ is $T^A \psi$, so that it is bilinearly generated by

$$T^A \psi(v \otimes a, z \otimes b) = \psi(v, z) \otimes ab\,, \tag{11}$$

$v \in \mathfrak{g}$, $z \in S$, $a, b \in A$, the product ab being in A. If S is a vector space, the action H_S of G^r_m on $T^A S$ is linear and induces an action h_S of $\mathfrak{g}^r_m$ on $T^A S$ of the form

$$h_S(X, z \otimes b) = z \otimes h(X)(b)\,, \qquad X \in \mathfrak{g}^r_m, z \in S, b \in A\,. \tag{12}$$

Further, (9), (11) and (12) implies that the induced action $\mathcal{W}^A_H \psi$ of $\mathrm{Lie}\,(W^A_H G)$ on $T^A S$ is bilinearly generated by

$$\mathcal{W}^A_H \psi\big((X, v \otimes a), z \otimes b\big) = \psi(v, z) \otimes ab + z \otimes h(X)(b)\,. \tag{13}$$

2. The Lie algebroid $\mathcal{V}^A E$. We start with the simpliest case (4). For every vector space S, we have $T^A S = S \otimes A$. In this case, the general injection $i^A_M : M \to T^A M$ has the form $s \mapsto s \otimes 1$, $s \in S$. If $D \to M$ is a vector bundle, we obtain fiberwise $V^A D = D \otimes A$ and $i^A_D : D \to V^A D$.

Consider the vertical A-prolongation of the anchor map

$$V^A q : V^A E \to V^A TM = TM \otimes A\,.$$

Write $\mathcal{V}^A E = (V^A q)^{-1}\big(i^A_{TM}(TM)\big)$, so that $V^A q$ yields a map

$$\mathcal{V}^A q : \mathcal{V}^A E \to TM\,.$$

For every section $\varrho : M \to E$, we write $i(\varrho) = i^A_E \circ \varrho : M \to \mathcal{V}^A E$. If ϱ is over the zero vector field 0_M on M, i.e. $q \circ \varrho = 0_M$, then also $i(\varrho) \otimes a : M \to \mathcal{V}^A E$ for every $a \in A$.

Proposition 1. *If E is a regular Lie algebroid, there is a unique Lie algebroid structure on $(\mathcal{V}^A E, \mathcal{V}^A q) \to M$ with the bracket $\mathbf{[}\,,\mathbf{]}$ (bold-faced type) such that for every sections ϱ and τ of E, $q \circ \tau = 0_M$ and every $a \in A$, we have*

(i) $\mathbf{[}i(\varrho_1), i(\varrho_2)\mathbf{]} = i\big([\![\varrho_1, \varrho_2]\!]\big)$,
(ii) $\mathbf{[}i(\varrho), i(\tau) \otimes a\mathbf{]} = i\big([\![\varrho, \tau]\!]\big) \otimes a$,
(iii) $\mathbf{[}i(\tau_1) \otimes a_1, i(\tau_2) \otimes a_2\mathbf{]} = i\big([\![\tau_1, \tau_2]\!]\big) \otimes a_1 a_2\,.$

In the case of the Lie algebroid LP,

$$\mathcal{V}^A(LP) = L(V^A P)\,. \tag{14}$$

Proof. Since E is regular, its adjoint bundle $\mathcal{E} \to M$, i.e. the kernel of q, is a vector bundle of fiber dimension k. Then $V^A \mathcal{E}$ is the kernel of $\mathcal{V}^A q$. If $\tau_1, \dots, \tau_k$ is a basis of local sections of $\mathcal{E}$, $i(\tau_1), \dots, i(\tau_k)$ generate $V^A \mathcal{E}$ over A. Then we compare our requirements (i)–(iii) with (2). The fact $\mathfrak{g} \otimes A$ is a Lie algebra implies directly that $\mathbf{[}\,,\mathbf{]}$ defines a Lie algebroid structure on $(\mathcal{V}^A E, \mathcal{V}^A q) \to M$. In the same way we deduce (14). $\square$

3. The Lie algebroid $\mathcal{T}^A E$. We are going to the case (3). We construct $T^A q : T^A E \to T^A TM$ and add the canonical exchange map $\varkappa^A_M : T^A TM \to TT^A M$. Then

$$\mathcal{T}^A q := \varkappa^A_M \circ T^A q : T^A E \to TT^A M$$

is a linear morphism over $\mathrm{id}_{T^A M}$, [5].

For every vector bundle $D \to M$, the multiplication by reals is a map $m : \mathbb{R} \times D \to D$. Applying T^A and using $T^A \mathbb{R} = A$, we obtain an action

$$T^A m : A \times T^A D \to T^A D$$

of A on $T^A D$. We write $T^A m(a, Z) = aZ$, $a \in A$, $Z \in T^A D$.

For every section $\varrho : M \to E$, we have $T^A \varrho : T^A M \to T^A E$.

Proposition 2. *If E is a regular Lie algebroid, there is a unique Lie algebroid structure on $(T^A E, \mathcal{T}^A q) \to T^A M$ with the bracket $[\,,]$ such that*

(i) $[T^A \varrho_1, T^A \varrho_2] = T^A(\llbracket \varrho_1, \varrho_2 \rrbracket)$ *for every* $\varrho_1, \varrho_2 \in C^\infty E$,

(ii) *for every sections* $R_1, R_2 : T^A M \to T^A E$ *and every A-valued function* $f : T^A M \to A$, *we have*

$$[R_1, f R_2] = f[R_1, R_2] + (\mathcal{T}^A q)(R_1)(f) R_2 \,,$$

where $(\mathcal{T}^A q)(R_1)$ is a vector field on $T^A M$. In the case of the Lie algebroid LP,

$$T^A(LP) = L(T^A P)\,. \tag{15}$$

Proof. The adjoint bundle $\mathcal{T}^A \mathcal{E}$ of $(T^A E, \mathcal{T}^A q) \to T^A M$ coincides with the T^A-prolongation $T^A \mathcal{E} \to T^A M$ of the adjoint bundle $\mathcal{E} \to M$ of E. If $\tau_1, \dots, \tau_k$ is a basis of local sections of $\mathcal{E}$, then $T^A \tau_1, \dots, T^A \tau_k$ generate $T^A \mathcal{E}$ over A. In this situation, the standard manipulations demonstrate that the requirements (i) and (ii) define correctly a Lie algebroid structure. For the product Lie algebroid $TM \times \mathfrak{g}$, (2) implies $T^A(TM \times \mathfrak{g}) = TT^A M \times (\mathfrak{g} \otimes A)$. □

4. The formal version of an action. We recall, [2], that an action of a Lie algebroid $(E, q) \to M$ on a vector bundle $D \to M$ is an $\mathbb{R}$-bilinear map $\varphi : C^\infty E \times C^\infty D \to C^\infty D$ such that, for every $\varrho \in C^\infty E$, $\sigma \in C^\infty D$ and $f : M \to \mathbb{R}$,

(i) $\varphi(f\varrho, \sigma) = f\varphi(\varrho, \sigma)$,

(ii) $\varphi(\varrho, f\sigma) = f\varphi(\varrho, \sigma) + (q\varrho)(f)\sigma$,

(iii) $\varphi(\llbracket \varrho_1, \varrho_2 \rrbracket, \sigma) = \varphi(\varrho_1, \varphi(\varrho_2, \sigma)) - \varphi(\varrho_2, \varphi(\varrho_1, \sigma))$.

Example 1. If $D = P[S, \ell]$ is a vector bundle associated to a principal bundle $P(M, G)$, we have the following canonical action of LP on D. Let ϱ be a right-invariant vector field on P, $\varrho(u) = \frac{\partial}{\partial t}\big|_0 \gamma(t)$, $\gamma : \mathbb{R} \to P$, $\gamma(0) = u \in P$. This induces a vector field ϱ_D on D

$$\varrho_D(\{u, s\}) = \frac{\partial}{\partial t}\Big|_0 \{\gamma(t), s\}\,, \qquad s \in S\,. \tag{16}$$

Since ϱ is right-invariant, (16) is independent of the choice of $u \in P$. Obviously, ϱ_D is a linear vector field on D. Further, every section $\sigma : M \to D$ induces, by translations, a fiberwise constant vector field σ_D on M. By direct evaluation we deduce that the bracket $[\varrho_D, \sigma_D]$ is also a fiberwise constant vector field. Hence the rule

$$\big(\varphi(\varrho, \sigma)\big)_D = [\varrho_D, \sigma_D] \tag{17}$$

defines a map $C^\infty E \times C^\infty D \to C^\infty D$. One verifies easily that this is an action. In the product case $P = M \times G$, we have $D \approx M \times S$. Let $\psi : \mathfrak{g} \times S \to S$ be the action of the Lie algebra $\mathfrak{g}$ on S corresponding to the linear action ℓ of G on S. We have $\varrho = (X, g)$ and σ is identified with a map $\sigma : M \to S$. Then (16) yields the following product expression of φ

$$\varphi\big((X, g), \sigma\big) = X\sigma - \psi(g, \sigma)\,. \tag{18}$$

The motivation of the following definition was explained in the introduction.

Definition 1. The formal version of an action φ of $(E,q)\to M$ on $D\to M$ is a map $\Phi : E\times_{TM} TD\to D$ satisfying

$$\varphi(\varrho,\sigma)=\Phi\big(\varrho,(T\sigma)\circ(q\varrho)\big) \tag{19}$$

for every $\varrho\in C^\infty E$ and $\sigma\in C^\infty D$.

If such a map Φ determines an action of E on D by (19), we say Φ is a formal action.

Proposition 3. *If E is transitive, there exists the formal version of every action of E.*

Proof. If E is transitive, we have locally $E=TM\times\mathfrak{g}$. Consider some local coordinates x^i, dx^i, y^p on E and x^i, z^a on D. The induced coordinates on $E\times_{TM} TD$ are x^i, dx^i, y^p, z^a, dz^a. Consider the corresponding local sections $\sigma_a(x)$ of D and $\left(\frac{\partial}{\partial x^i},0\right)$, $\big(0,e_p(x)\big)$ of E. Then we have

$$\varphi\Big(\frac{\partial}{\partial x^i},\sigma_a\Big)=\varphi^b_{ia}(x)\sigma_b\,,\quad \varphi(e_p,\sigma_a)=\varphi^b_{pa}(x)\sigma_b\,. \tag{20}$$

The axioms of action yield

$$\varphi\Big(X^i(x)\frac{\partial}{\partial x^i}+g^p(x)e_p,f^a(x)\sigma_a\Big)=\Big(X^i\frac{\partial f^a}{\partial x^i}\Big)\sigma_a+(X^if^a\varphi^b_{ia}+g^pf^a\varphi^b_{pa})\sigma_b\,.$$

This implies that the coordinate expression of Φ is $x^i=x^i$ and

$$z^a=dz^a+\varphi^a_{ib}z^bdx^i+\varphi^a_{pb}y^pz^b\,. \tag{21}$$

□

In the product case of the canonical action (18),

$$\Phi : TM\times\mathfrak{g}\times S\times S\to M\times S$$

is of the form $x=x$ and

$$z=Z-\psi(y,z)\,, \tag{22}$$

where $(x,dx)\in TM$, $(z,Z)\in S\times S$, $y\in\mathfrak{g}$.

5. F-prolongation of a formal action. Consider a bundle functor $F=(A,H,t)$ on $\mathcal{FM}_m$ and a vector bundle $D\to M$. The vector addition is a map $c : D\times_M D\to D$ and the multiplication by reals can be interpreted as a map $m : (M\times\mathbb{R})\times_M D\to D$. Applying F and taking into account $F(M\times\mathbb{R})=P^rM[A,H]$, we obtain $Fc : FD\times_M FD\to FD$ and $Fm : P^rM[A,H]\times_M FD\to FD$. Obviously, $FD\to M$ is a bundle of Abelian groups with respect to Fc. Further, $\mathbb{R}\subset A$ is an H-invariant subspace, so that Fm can be restricted to $M\times\mathbb{R}\subset P^rM[A,H]$ in the first factor. This defines the multiplication by reals on FD and $FD\to M$ is also a vector bundle.

For every fibered manifold $p : Y \to M$, F induces a natural transformation

$$\widetilde{t}_Y : J^r Y \to FY\,, \qquad \widetilde{t}_Y(j^r_x s) = (F\widetilde{s})(x)\,, \qquad x \in M,$$

where a section $s : M \to Y$ is interpreted as a morphism $\widetilde{s}$ of the trivial fibered manifold $\mathrm{id}_M : M \to M$ into Y. In particular, we have $\widetilde{t}_{TM} : J^r TM \to FTM$.

In [3] and [4] we introduced a natural diffeomorphism

$$\widetilde{\mu}^F_Y : J^r TM \times_{FTM} FTY \to J^r TM \times_{TM} TFY \tag{23}$$

over $\mathrm{id}_{J^r TM}$. We write $\mu^F_Y = pr_2 \circ \widetilde{\mu}^F_Y : J^r TM \times_{FTM} FTY \to TFY$. The most important property of $\widetilde{\mu}^F_Y$ is that for every projectable vector field η on Y over ξ on M the flow prolongation $\mathcal{F}\eta : FY \to TFY$ satisfies

$$\mathcal{F}\eta = \mu^F_Y \circ (j^r \xi \times_M F\eta)\,, \tag{24}$$

where $F\eta : FY \to FTY$ is the functorial prolongation of η interpreted as an $\mathcal{FM}$-morphism $\eta : Y \to TY$ over id_M. In the case $Y = \mathbb{R}^m \times S$, where S is a vector space, we have

$$\begin{aligned} J^r T\mathbb{R}^m \times_{FT\mathbb{R}^m} F\big(T(\mathbb{R}^m \times S)\big) &= T\mathbb{R}^m \times \mathfrak{g}^r_m \times S \otimes A \times S \otimes A\,, \\ J^r T\mathbb{R}^m \times_{TM} TF\big(\mathbb{R}^m \times S\big) &= T\mathbb{R}^m \times \mathfrak{g}^r_m \times S \otimes A \times S \otimes A \end{aligned} \tag{25}$$

and the explicit formula for $\widetilde{\mu}^F_Y$ is, [4],

$$\widetilde{\mu}^F_{\mathbb{R}^m \times S}(z, X, v \otimes a, w) = \big(z, X, v \otimes a, w + v \otimes h(X)(a)\big)\,, \tag{26}$$

$z \in T\mathbb{R}^m$, $X \in \mathfrak{g}^r_m$, $v \in S$, $a \in A$, $w \in S \otimes A$.

Applying F to the formal action $\Phi : E \times_{TM} TD \to D$, we obtain

$$F\Phi : FE \times_{FTM} FTD \to FD\,.$$

If we add the product projection

$$\pi : J^r TM \times_{FTM} FE \times_{FTM} FTD \to FE \times_{FTM} FTD\,,$$

we obtain

$$F'\Phi = F\Phi \circ \pi : J^r TM \times_{FTM} FE \times_{FTM} FTD \to FD\,.$$

Write $\overline{\mu}^F_D = pr_2 \circ (\widetilde{\mu}^F_D)^{-1} : J^r TM \times_{TM} TFD \to FTD$. Then we define

$$\begin{aligned} \mathcal{F}\Phi &: (J^r TM \times_{FTM} FE) \times_{TM} TFD \to FD\,, \\ \mathcal{F}\Phi(c_1, c_2, c_3) &= F'\Phi\big(c_1, c_2, \overline{\mu}^F_D(c_1, c_3)\big)\,, \end{aligned} \tag{27}$$

$c_1 \in J^r TM$, $c_2 \in FE$, $c_3 \in TFD$.

In the case of the canonical action of $E = T\mathbb{R}^m \times \mathfrak{g}$ on $D = \mathbb{R}^m \times S$, (22), (11) and (26) imply that the explicit formula for

$$\mathcal{F}\Phi : T\mathbb{R}^m \times \mathfrak{g}^r_m \times \mathfrak{g} \otimes A \times S \otimes A \times S \otimes A \to \mathbb{R}^m \times S \otimes A$$

is $x = x$ and

$$w = W - \psi(y, z) \otimes ab - z \otimes h(X)(b)\,, \tag{28}$$

$(x, dx) \in T\mathbb{R}^m$, $X \in \mathfrak{g}^r_m$, $y \otimes a \in \mathfrak{g} \otimes A$, $(z \otimes b, W) \in S \otimes A \times S \otimes A$, $w \in S \otimes A$.

Remark 1. We remark that one can also introduce the concept of formal bracket of a Lie algebroid $(E, q) \to M$. This is based on the fact that the bracket of vector fields on M can be characterized in terms of a map, which is called the strong difference in [5]. Let $p_M^T : TM \to M$ be the bundle projection. Write

$$SM \subset TTM \times_M TTM$$

for the subset of all pairs (C_1, C_2) satisfying

$$Tp_M^T(C_1) = p_{TM}^T(C_2)\,, \quad Tp_M^T(C_2) = p_{TM}^T(C_1)\,. \tag{29}$$

Every such pair determines the strong difference, [5],

$$\sigma_M(C_1, C_2) = C_2 \div C_1 \in TM\,.$$

In the case $M = \mathbb{R}^m$ we have $TT\mathbb{R}^m = \overset{4}{\times}\mathbb{R}^m$, $(C_1, C_2) \in S\mathbb{R}^m$ is of the form $C_1 = (x, x_1, x_2, x_3)$, $C_2 = (x, x_2, x_1, x_4)$ and

$$\sigma_{\mathbb{R}^m}(C_1, C_2) = (x, x_4 - x_3)\,. \tag{30}$$

If X_1, X_2 are two vector fields on M, we have $(TX_2 \circ X_1, TX_1 \circ X_2) : M \to SM$ and the bracket $[X_1, X_2]$ satisfies

$$[X_1, X_2] = \sigma_M \circ (TX_2 \circ X_1, TX_1 \circ X_2)\,. \tag{31}$$

Further, we write

$$\Delta E = \{(Z_1, Z_2) \in TE \times_M TE, (Tq(Z_1), Tq(Z_2)) \in SM\}\,. \tag{32}$$

The formal bracket of E can be defined as a map $B : \Delta E \to E$ satisfying, for every $\varrho_1, \varrho_2 \in C^\infty E$,

$$[\![\varrho_1, \varrho_2]\!] = B(T\varrho_2 \circ q\varrho_1, T\varrho_1 \circ q\varrho_2)\,. \tag{33}$$

One verifies easily that for every transitive Lie algebroid there exists the formal bracket. — The concept of formal bracket will be discussed in more details in a next paper.

6. The Lie algebroid $\mathcal{W}^F E$. Assume E is transitive. To deduce that

$$\mathcal{W}^F E := J^r TM \times_{FTM} FE \tag{34}$$

is a Lie algebroid, we use the fact that E is locally of the form $T\mathbb{R}^m \times \mathfrak{g}$. We have $J^r T\mathbb{R}^m = \mathbb{R}^m \times (\mathbb{D}_m^r)^m = T\mathbb{R}^m \times (N_m^r)^m$, where N_m^r is the nilpotent part of $\mathbb{D}_m^r$. But $(N_m^r)^m$ is canonically identified with the Lie algebra $\mathfrak{g}_m^r$. On the other hand,

$$F(T\mathbb{R}^m \times \mathfrak{g}) = \mathbb{R}^m \times A^m \times \mathfrak{g} \otimes A = T\mathbb{R}^m \times N^m \times \mathfrak{g} \otimes A\,.$$

Since $t : \mathbb{D}_m^r \to A$ maps N_m^r into N, we have

$$J^r T\mathbb{R}^m \times_{FT\mathbb{R}^m} F(T\mathbb{R}^m \times \mathfrak{g}) = T\mathbb{R}^m \times \mathfrak{g}_m^r \times \mathfrak{g} \otimes A\,. \tag{35}$$

As a vector space, $\mathfrak{g}_m^r \times \mathfrak{g} \otimes A$ coincides with Lie $(W_H^A G)$. Hence the bracket (10) defines the structure of Lie algebroid on (35). Thus, we have proved

Proposition 4. *If E is transitive, $\mathcal{W}^F E$ has a canonical structure of Lie algebroid. In the case of the Lie algebroid LP,*

$$\mathcal{W}^F(LP) = L(W^F P)\,. \tag{36}$$

Example 2. In the case $F = J^r$, we have $\mathcal{W}^{J^r} E = J^r E$ and we rededuce the well known fact that $J^r(LP)$ is the Lie algebroid of $W^r P$, [7].

7. F-prolongation of an action. Consider the canonical action φ of the Lie algebroid LP on an associated vector bundle $D = P[S, \ell]$ from Example 1. Comparing (13), (22) and (28), we obtain

Proposition 5. *In this case, the prolongation $\mathcal{F}\Phi$ of the formal version Φ of φ is the formal version of the induced canonical action of $L(\mathcal{W}^F P)$ on FD.* □

Remark 2. We remark that the functorial prolongation of the formal action can be applied even to the T^A-prolongation from Section 3. Under the assumptions of Proposition 5, one deduces easily that $\mathcal{T}^A\Phi$ is the formal version of the induced canonical action of $L(T^A P \to T^A M)$ on the vector bundle $T^A D \to T^A M$.

References

[1] Doupovec M., Kolář I., *Iteration of fiber product preserving bundle functors*, Monatsh. Math. **134** (2001), 39–50.

[2] Fernandez R. L., *Invariants of Lie algebroids*, Differential Geometry and its Applications **19** (2003), 223–243.

[3] Kolář I., *On the geometry of fiber product preserving bundle functors*, Differential Geometry and its Applications, Proceedings, Silesian University at Opava (2002), 85–92.

[4] Kolář I., *On the flow prolongation of vector fields*, Proceedings of Geometrical Seminar, Kazan **24** (2003), 69–80.

[5] Kolář I., Michor P. W., Slovák J., *Natural Operations in Differential Geometry*, Springer-Verlag, 1993.

[6] Kolář I., Mikulski W. M., *On the fiber product preserving bundle functors*, Differential Geometry and its Applications **11** (1999), 105–115.

[7] Kumpera A., Spencer D., *Lie equations, I*, Princeton University Press, 1972.

[8] Mackenzie K., *Lie Groupoids and Lie Algebroids in Differential Geometry*, Cambridge University Press, 1987.

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