Workshop on Neurogeometry and other (related) problems

CONSTRUCTIONS OF METRICS WITH SPECIAL HOLONOMIES VIA GEOMETRICAL FLOWS

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Let (M^n, g) be a Riemannian manifold, $p \in M^n$.

 $Hol_{p}(M^{n}) = \{P_{\gamma}|\gamma(t), 0 \leq t \leq 1 - a \text{ loop in } M^{n}, \gamma(0) = \gamma(1) = p\},$

where P_{γ} — parallel transport along the curve γ with respect to the Levi-Civita connection.

$$Hol_{\rho}(M^n) \subset Iso(T_{\rho}M^n) = SO(n).$$

Theorem (de Rham, 1952)

Let M — be a complete Riemannian manifold and $Hol(M) = G_1 \times G_2$. Then $M = M_1 \times M_2$, where $Hol(M_1) = G_1$ and $Hol(M_2) = G_2$.

Theorem (E. Cartan, 1926)

Let $M^n = G/H$ be a symmetrical space (G — Lee group generated by all symmetrical reflections that turn over the geodesics pass through point $p \in M$, H is a stabilizer of p). Then H = Hol(M).

Theorem (Berger, 1955)

Let M^n be a simply connected irreducible non-symmetrical Riemannian manifold. Then one of the following cases holds: 1.Hol(g) = SO(n), 2.Hol(g) = U(m), $n = 2m \ge 4$, 3.Hol(g) = SU(m), $n = 2m \ge 4$, 4.Hol(g) = Sp(m), $n = 4m \ge 8$, 5.Hol(g) = Sp(m)Sp(1), $n = 4m \ge 8$, $6.Hol(g) = G_2$, n = 7, 7.Hol(g) = Spin(7), n = 8.

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The Riemannian cone over Riemannian manifold (M, ds^2) is

$$C(M) = \mathbb{R}_+ imes M$$

with metric

$$dt^2 + t^2 ds^2$$
.

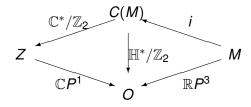
M is 3-Sasakian if there are 3 unit Killing vector fields ξ_i , i = 1, 2, 3 on M such that $[\xi_i, \xi_j] = 2\epsilon_{ijk}\xi_k$ and that (1, 1)-tensor field $\Phi_i = \nabla \xi_i$ satisfies $(\nabla_X \Phi_i)(Y) = g(Y, \xi_i)X - g(X, Y)\xi_i$.

Theorem (Boyer-Galicki)

Let M^n be a complete 3-Sasakian manifold. Then C(M) is hyperkahler, i.e. $Hol(C(M)) = Sp(\frac{n+1}{4})$.

There is a diagram, which called a 'diamond diagram', that connect different geometries:

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We suppose that M is a 7-dimensional 3-Sasakian manifold and that quaternionic Kähler orbifold O of M is Kähler. For example, SU(3)/U(1) is a such manifold.

We consider a deformation of the cone metric:

$$g(t) = dt^{2} + A_{1}(t)^{2}\eta_{1}^{2} + A_{2}(t)^{2}\eta_{2}^{2} + A_{3}(t)^{2}\eta_{3}^{2}$$

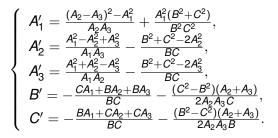
$$B(t)^{2}(\eta_{4}^{2} + \eta_{5}^{2}) + C(t)^{2}(\eta_{6}^{2} + \eta_{7}^{2})$$

and want to find metric g such that Hol(g) = Spin(7).

$$Sp(2) \subset SU(4) \subset Spin(7) \subset SO(8)$$

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$\mathit{Hol}(g) \subset \mathit{Spin}(7) \hspace{0.1in} \Leftrightarrow \hspace{0.1in} abla \Psi = 0 \hspace{0.1in} \Leftrightarrow \hspace{0.1in} d\Psi = 0 \hspace{0.1in} \Leftrightarrow$



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Space \mathcal{M}_1^8 :

(1)
$$A_1(0) = A_2(0) = A_3(0) = 0$$
, $|A'_1(0)| = |A'_2(0)| = |A'_3(0)| = 1$;
(2) $B(0) \neq 0$, $B'(0) = 0$;
(3) $C(0) \neq 0$, $C'(0) = 0$;

(4) functions A_1, A_2, A_3, B, C have definite signs on $(0, \infty)$.

Space \mathcal{M}_2^8 :

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(1)
$$A_1(0) = 0, |A'_1(0)| = 4;$$

(2) $A_2(0) = -A_3(0) \neq 0, A'_2(0) = A'_3(0),$
(3) $B(0) \neq 0, B'(0) = 0;$
(4) $C(0) \neq 0, C'(0) = 0;$
(5) functions A_1, A_2, A_3, B, C have definite signs on $(0, \infty).$

Theorem (Bazaikin-M.)

For $0 \leq \alpha < 1$ every metric from the family

$$g_{\alpha} = \frac{r^4(r^2 - \alpha^2)(r^2 + \alpha^2)}{r^8 - 2\alpha^4(r^4 - 1) - 1} dr^2 + \frac{r^8 - 2\alpha^4(r^4 - 1) - 1}{r^2(r^2 - \alpha^2)(r^2 + \alpha^2)} \eta_1^2 + r^2(\eta_2^2 + \eta_3^2)$$

+
$$(r^2 + \alpha^2)(\eta_4^2 + \eta_5^2) + (r^2 - \alpha^2)(\eta_6^2 + \eta_7^2),$$

is complete smooth Riemannian SU(4)-holonomy metric on the square of the canonical complex line bundle over the space of flags in \mathbb{C}^3 . Metric g_0 is isometric to the Calabi metric with holonomy SU(4); and metric g_1 is isometric to the Calabi metric to the Calabi metric with holonomy $Sp(2) \subset SU(4)$ on the $T^*\mathbb{C}P^2$.

Let us now consider the deformation of the cone C(M) as an evolution of the M under some specific geometric flow.

We want to find some reasonable geometric flow on arbitrary 3-Sasakian 7-dimensional manifold $(M, \bar{g}(t))$ such that cone C(M) with metric $dt^2 + \bar{g}(t)$ has special holonomy group.

$$rac{\partial}{\partial t}ar{g}(t) = RHS(ar{g})$$

Dirac flow

Theorem

Let S^3 be a 3-dimensional sphere with conformally round metric

$$\bar{g}(t) = f^2(t)(\eta_1^2 + \eta_2^2 + \eta_3^2)$$

which satisfies the following flow

$$rac{\partial}{\partial t}ar{g}(t)=\sqrt{ar{ extsf{Ric}}-4K}.$$

Then the cone $C(S^3)$ with metric $dt^2 + \bar{g}(t)$ is isometric to the space with constant curvature K for $K \in \{-1, 0, +1\}$.

Note that for K = +1 at $t = 2\pi$ sphere S^3 is collapsed and the cone $C(S^3)$ with such metric turns out to be a sphere S^4 .

If we consider a metric

$$\bar{g}(t) = A_1^2(t)\eta_1^2 + A_2^2(t)(\eta_2^2 + \eta_3^2).$$

Then a Ricci flow

$$ar{g}(t) = -2 extsf{Ric}(ar{g}(t))$$

for this metric will be equivalent to the system

$$\left\{ egin{array}{l} rac{A_1'}{A_1} \cdot rac{dt}{d au} = -4rac{A_1^2}{A_2^4}, \ rac{A_2'}{A_2} \cdot rac{dt}{d au} = -rac{4}{A_2^2}(2-rac{A_1^2}{A_2^2}). \end{array}
ight.$$

This system can be integrable:

$$-\frac{1}{16}\beta - \frac{\sqrt{2}}{128}c_1 \arctan(\frac{4\sqrt{2}\beta}{\sqrt{c_1^2 - 32\beta^2}}) = t + c_2,$$

where $\alpha = A_1^2 = \frac{1}{8}\beta(\beta' + 16)$ and $\beta = A_2^2$.

Theorem

Let S^3/\mathbb{Z}_2 be a 3-dimensional projective space with metric

$$ar{g}(t) = A_1^2(t)\eta_1^2 + A_2^2(t)(\eta_2^2 + \eta_3^2)$$

satisfying

$$rac{\partial}{\partial t}ar{g}(t)=\sqrt{ extsf{det}(ar{ extsf{Ric}})}ar{ extsf{Ric}}^{-1}$$

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with $A_1(0) = 0$, $A'_1(0) = 2$ and $A_2(0) \neq 0$. Then the metric $dt^2 + \bar{g}(t)$ is isometric to the Eguchi-Hanson metric.

Thanks for your attention.

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