Neurogeometry
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Some elements of neurogeometry

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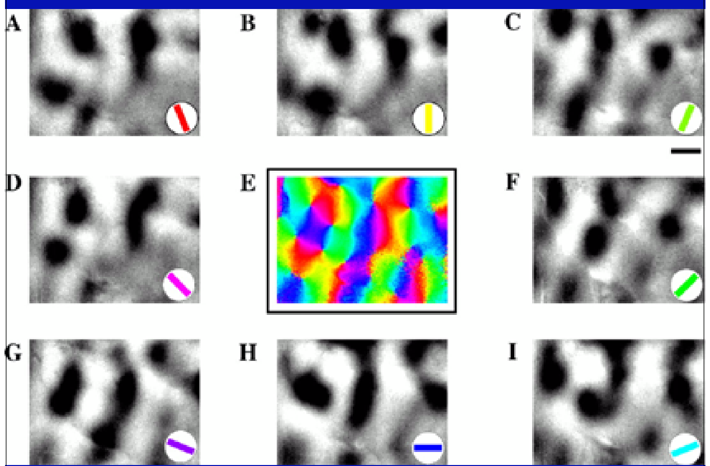
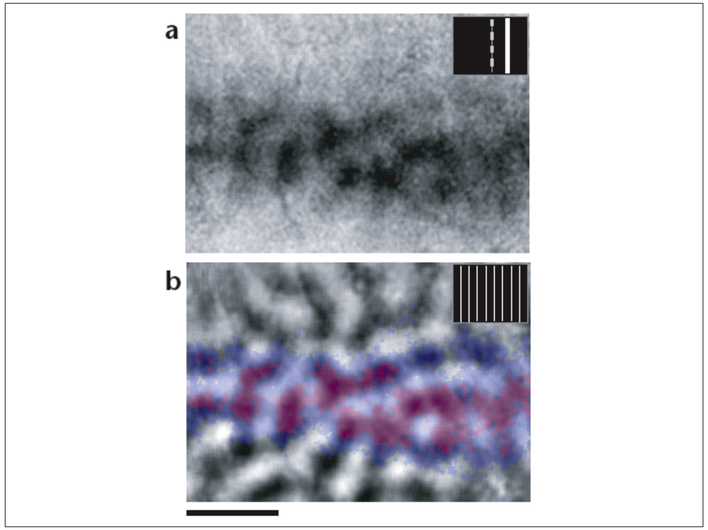
The pinwheel structure of V1

- Let us begin with an image of the structure of V1 of the tree-shrew or tupaya (W. Bosking).
- The method (Bonhöffer & Grinvald, ~ 1990) of *in vivo optical imaging* based on activity-dependent intrinsic signals allows to acquire images of the activity of the superficial cortical layers.

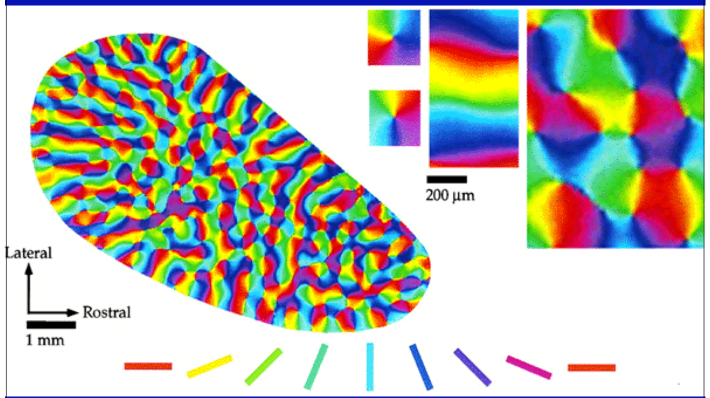


- At a certain resolution and with a population coding, a "point" corresponds to a small assembly of neurons with approximatively the same receptive field and the same preferred orientation.
- It codes a contact element (a, p).

- The following picture shows
- (a) the sub-population (stripe) of V1 neurons activated by a long line stimulus located at a precise (vertical) position (scale bar = 1mm).
- (b) the embedding of the stripe in the population of V1 neurons responding to the same vertical orientation but at different positions.

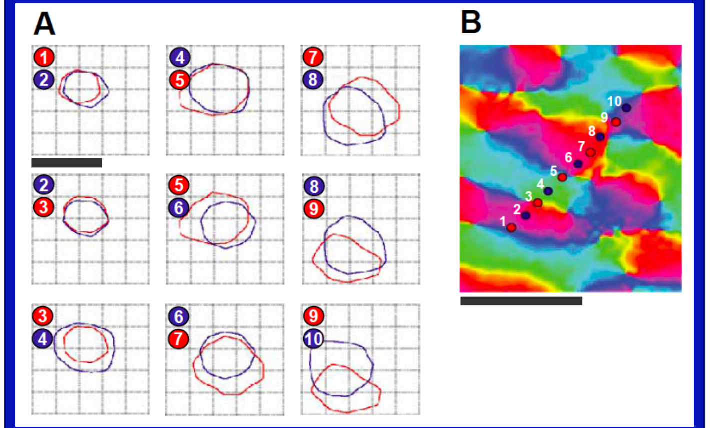


- In the following picture the orientations are coded by colors and iso-orientation lines are therefore coded by monocolor lines.
- The cortical layer is reticulated by a network of singular points which are the centers of the pinwheels.
- Locally, around these singular points all the orientations are represented by the rays of a "wheel" and the local wheels are glued together in a global structure.

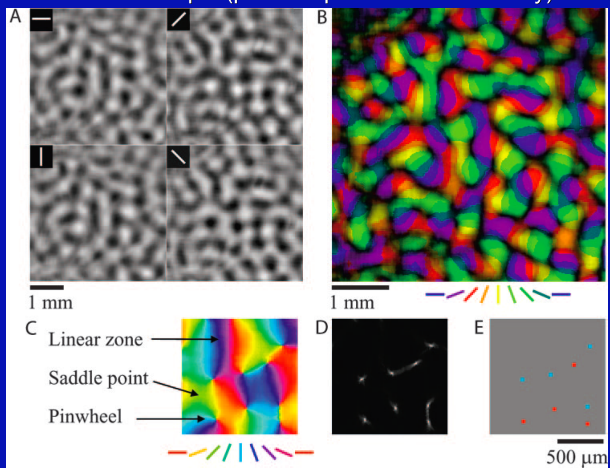


- There are 3 classes of points :
  - regular points where the orientation field is locally trivial;
  - singular points at the center of the pinwheels;
  - saddle-points localized near the centers of the cells of the network.
- Two adjacent singular points are of opposed chirality (CW and CCW).
- It is like a field in *W* generated by topological charges with « field lines » connecting charges of opposite sign.

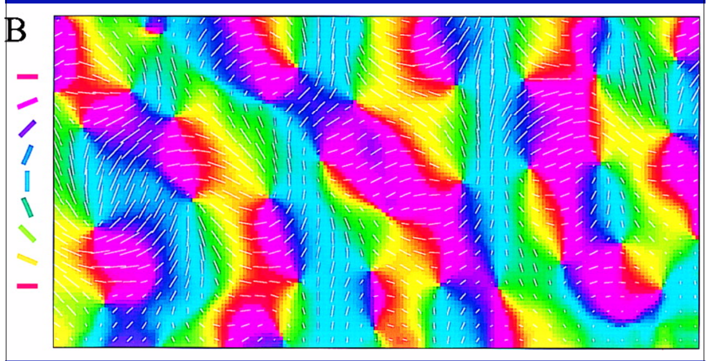
- Overlapping of receptive fields (Yu *et al.*)



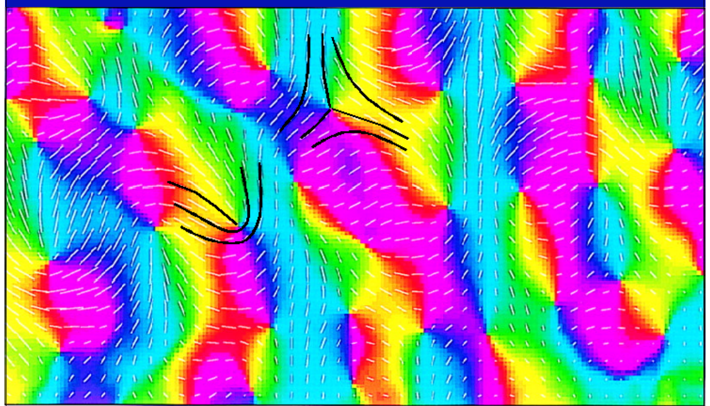
- Another example (primate: prosimian Bush Baby)



- In the following picture due to Shmuel (cat's area 17), the orientations are coded by colors but are also represented by white segments.



- We observe very well the two types of generic singularities of 1D foliations in the plane.



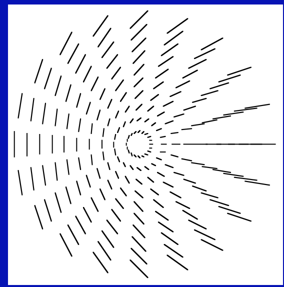
- They arise from the fact that, in general, the direction θ in V1 of a ray of a pinwheel is not the orientation p_θ associated to it in the visual field.
- When the ray spins around the singular point with an angle φ, the associated orientation rotates with an angle $\varphi/2$. Two diametrally opposed rays correspond to orthogonal orientations.
- There are two cases.

- If the orientation p_θ associated with the ray of angle θ is $p_\theta = \alpha + \theta/2$ (with $p_0 = \alpha$), the two orientations will be the same for

$$p_\theta = \alpha + \theta/2 = \theta$$

that is for $\theta = 2\alpha$.

- As α is defined modulo π, there is only one solution : end point.

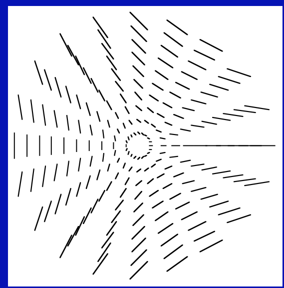


- If the orientation p_θ associated with the ray of angle θ is $p_\theta = \alpha - \theta/2$, the two orientations will be the same for

$$p_\theta = \alpha - \theta/2 = \theta$$

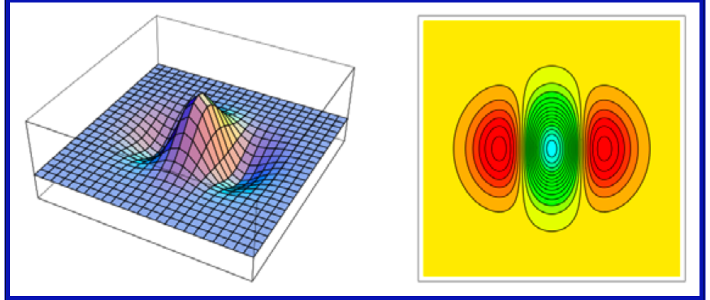
that is for $\theta = 2\alpha/3$.

- As α is defined modulo π, there are three solutions : triple point.



Supplementary structures

- Receptive profiles of simple V1 neurons have a characteristic shape (wavelets).



- Along a vertical penetration inside the cortical layer the phase changes.
- Along the rays of the pinwheels the spatial frequency changes.
- Direction of orientation.
- Ocular dominance.
- Gluing the two parts of V1 (the two visual hemifields) through the corpus callosum.
- Feed back from other areas onto V1, etc.
- But even the basic structure is non trivial.

Wolf-Geisel model

- Fred Wolf and Theo Geisel modeled the pinwheel network using a complex field

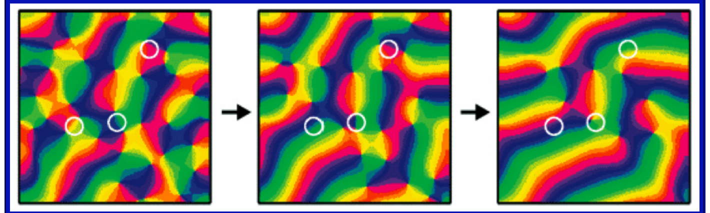
$$z(a) \quad \left(a = \rho e^{i\theta}, z = re^{i\varphi}\right)$$

  where the spatial phase $\varphi(a)$ codes the preferred orientation and the module $r(a)$ codes the orientation selectivity.
- Singularities are *zeroes* of this field.

- They study the evolution of pinwheels under learning dynamics.
- Starting with $z_0(a) \approx 0$ one applies Hebb's law according to which stimuli strengthen the connections they activate.
- Hence a PDE of evolution (ξ = noise)

$$\frac{\partial z(a,t)}{\partial t} = F\left(z\left(a,t\right)\right) + \xi$$

- Evolution of pinwheels.



- Let us suppose that the maximal selectivity = 1. The functional architecture is a section of the fibration

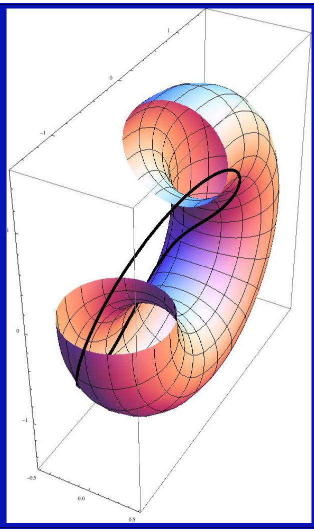
$$\pi : \mathbb{C} \times \mathbb{D} \to \mathbb{C} \ \left(a = \rho e^{i\theta}, z = re^{i\varphi}\right)$$

- Let us take e.g.

$$\varphi = \theta,\ r = \tfrac{1}{2}\rho$$

- Above a small circle C_ρ around $a = 0$ we have the torus

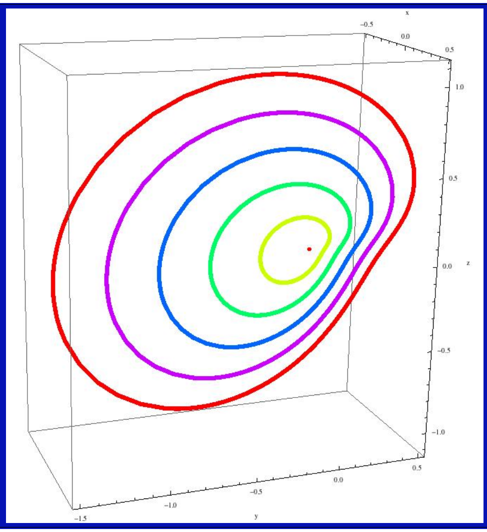
$$C_\rho \times \Sigma_{\rho/2} \to C_\rho$$



- The lift of C_ρ is the curve Γ_ρ

$$\left(\frac{1}{2}\rho\sin(\theta), \rho\left(1-\frac{1}{2}\cos(\theta)\right)\cos(\theta), \rho\left(1-\frac{1}{2}\cos(\theta)\right)\sin(\theta)\right)$$

- As orientation selectivity vanishes at 0, when $\rho \to 0$ we have also $\Gamma_\rho \to 0$
- The projection is locally a diffeomorphism.



- But many experiments show that orientation selectivity doesn't vanish at singular points.

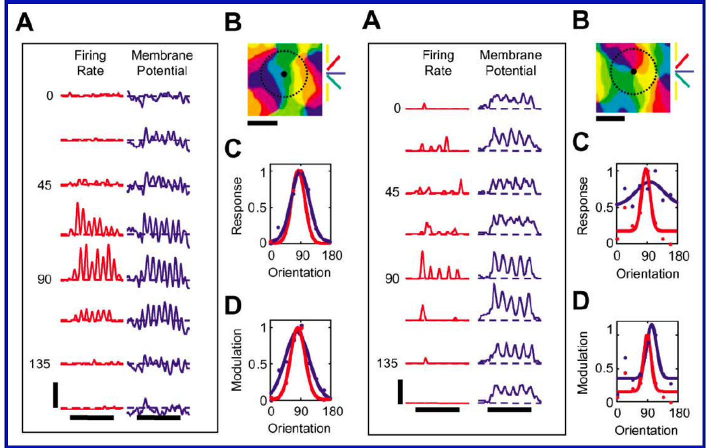
Structure near pinwheel centers

- P. E. Maldonado *et al.* have analyzed the fine-grained structure of orientation maps at the singularities. They found that

  « orientation columns contain sharply tuned neurons of different orientation preference lying in close proximity ».

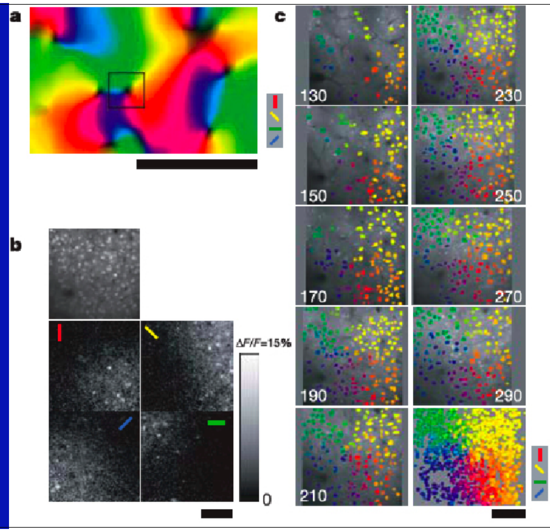
- James Schummers has shown that
  - « neurons near pinwheel centers have subthreshold responses to all stimulus orientations but spike responses to only a narrow range of orientations ».

- Far from a pinwheel, cells « show a strong membrane depolarization response only for a limited range of stimulus orientation, and this selectivity is reflected in their spike responses ».
- At a pinwheel center, on the contrary, only the spike response is selective. There is a strong depolarization of the membrane for all orientations.



Micro structure

- The spatial (50μ) and depth resolutions of optical imaging is not sufficient.
- Two-photon calcium imaging *in vivo* (confocal biphotonic microscopy) provides functional maps at single-cell resolution.
  - Kenichi Ohki, *et al*.



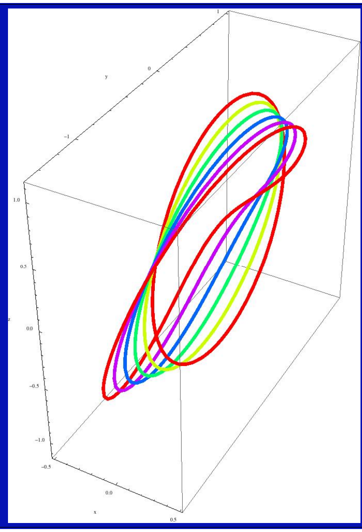
- A simple model would be r = cst = 1.
- The lift of C_ρ would then be the curve Γ_ρ

$$\left(\frac{1}{2}\rho\sin(\theta),\left(1-\frac{1}{2}\rho\cos(\theta)\right)\cos(\theta),\left(1-\frac{1}{2}\rho\cos(\theta)\right)\sin(\theta)\right)$$

- When $\rho \longrightarrow 0$ we have

$$\Gamma_\rho \longrightarrow (0,\cos(\theta),\sin(\theta))$$

- The projection is no longer a local diffeomorphism. Exceptional fiber.



Blowing-up models

- All orientations must be present with a good selectivity at the singularities.
- In fact it is a 3D abstract space

$$V = \mathbb{R}^2 \times \mathbb{P}^1$$

which is implemented into 2D neural layers.

- How ?

- An idea could be to use the concept of blowing-up.
- The blowing-up of a point $O = (0, 0)$ in the plane associates to every point

$$a = (x,y) \neq (0,0)$$

the line Oa.

- One gets the map

$$\begin{array}{rcl}\delta : \mathbb{R}^2 - \{O\} & \rightarrow & \mathbb{P}^1 \\ a = (x,y) & \mapsto & \delta(a) = p = \frac{y}{x}\end{array}$$

- The graph of δ is a helicoidal ruled surface H in

$$V = \mathbb{R}^2 \times \mathbb{P}^1$$

which is isomorphic to $\mathbb{R}^2 - \{O\}$ through the projection π .

- Its closure is a helicoid with an exceptional fiber

$$\pi^{-1}(O) = \Delta \simeq \mathbb{P}^1$$

- As the inverse image of O by π is

$$\Delta = \mathbb{P}^1$$

the blowing-up is in some sense of intermediary dimension between 2D and 3D. It is an unfolding of a 2D orientation wheel along a third dimension.

- In a second step, one can localize the blowing-up model of a pinwheel and restrict it to a neighborhood U of O.
- One can then take the germ, that is the limit w.r.t the filter of neighborhoods.
- In the germ, $p = \frac{dy}{dx}$ is in the kernel of the 1-form

$$\omega = dy - pdx$$

- On can then "compactify" the fiber (à la Kaluza-Klein) and pull it down in the base space.
- One gets that way a model for a single pinwheel.

- In this perspective a pinwheel is like a "fat point".
- In a letter (1986) concerning singularities of analytic functions, P. Deligne introduced the idea of substituting to a point $a = 0$, a small disk D with boundary $\partial D = \Delta$ and consider the space

$$\widetilde{\mathbb{C}} = \mathbb{C}^* \cup D$$

with the topology of the real blowing-up on

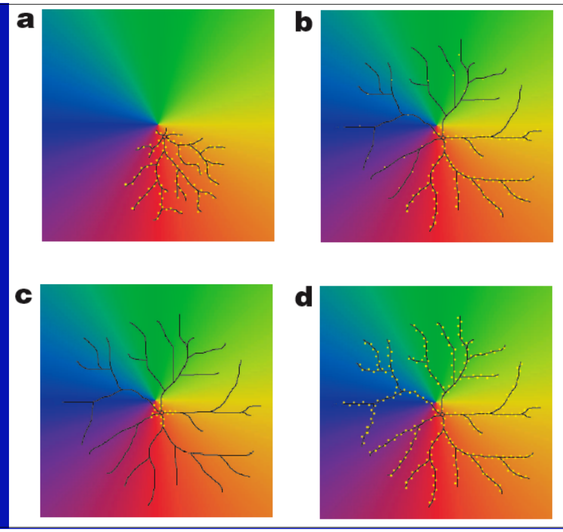
$$\mathbb{C}^* \cup \Delta$$

- In a third step, one can blow-up in parallel several points a_i and glue the local pinwheels (a_i, Δ_i) using a field endowing the a_i with topological charges (chirality).
- One gets that way a model of a network of pinwheels.

Singular connections

- A hypothesis could be to consider that the pinwheel structure with its field of preferred orientations is a solution of a differential equation with singularities.
- But for developing such an idea, one would have to look at the micro dendritic structures at singular points.

- Connectivity implementing the sharp orientation tuning near the centre.
- Dendritic trees near the centre C (few tens μ) in an iso-orientation domain *D* (yellow dots = excitatory synapses).
  - (a) d.t. biased towards *D*.
  - (b) d.t. symmetric, but excitatory inputs biased towards *D*.
  - (c) d.t. sym., excit. inputs sym. but local and therefore inside *D* (good segregation near *C*).
  - (d) d.t. sym., excit.inputs sym. and integrated uniformly over a large dendritic area.



Towards continuous models

- In a fourth step, one can go in a different direction and consider networks of singular points a_i with a mesh $\to 0$.
- The idea is that one could recover the fibration

$$V = \mathbb{R}^2 \times \mathbb{P}^1$$

and its contact structure by blowing-up in parallel all the points of the plane.

- It is possible to use non standard analysis (Robinson-Luxemburg).
- In his last paper (edited on 1992 by Jean-Pierre Ramis) Jean Martinet proposed to interpret "fat points" using non standard analysis : take for D an infinitesimal disk with only one standard point (the center).

- One restricts the infinitesimal model to the monads

$$\mu(a) = \{(x + dx, y + dy)\}$$

of the standard points a of the plane.

- When one blows-up a in the monad, one gets an exceptional fiber Δ^* whose standard points correspond to standard orientations.
- We work now in the fibration $V = \mathbb{R}^2 \times \mathbb{P}^1$

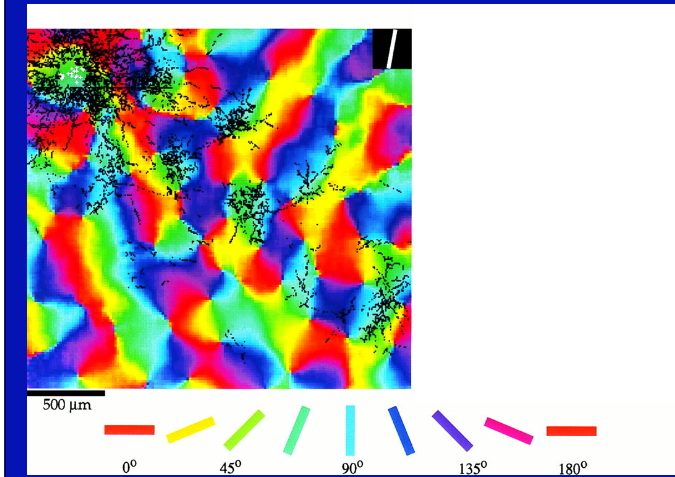
Functional architecture

- The "local" "vertical" retino-geniculo-cortical connections inside the pinwheels (hypercolumns) are not sufficient for perception.
- A functional architecture is necessary.
- FA : activation = to do geometry.

- To implement a global coherence of contours, the visual system must be able to compare two retinotopically neighboring hyper-columns P_a et P_b over two neighboring points a and b.

- This is a process of parallel transport implemented by the lateral ("horizontal") cortico-cortical connections.
- Cortico-cortical connections connect neurons coding contact elements (a, p) and (b, p) such that p is approximately the orientation of the line ab .

- The next slide shows how a marker (biocytin) injected locally in a zone of specific orientation (green-blue) diffuses via horizontal cortico-cortical connections.
- The key fact is that the long range diffusion is highly anisotropic and restricted to zones of the same orientation (the same color) as the initial one.



- W. Bosking :
  - « The system of long-range horizontal connections can be summarized as preferentially linking neurons with co-oriented, co-axially aligned receptive fields ».
- So, the well known Gestalt law of “good continuation” is neurally implemented.
- In fact, a certain amount of curvature is allowed in alignements.
- Neural origin of geometry.

- These experimental results mean essentially that the contact structure of the fiber bundle

$$\pi : V = R \times P \rightarrow R$$

is neurally implemented with
- dimensional collapse,
- discretization,
- population coding.

The contact structure of V1

- The simplest model of the functional architecture of V1 is the space of 1-jets of curves C in R.
- If C is curve in R (a contour), it can be lifted to V. The lifting Γ is the map (1-jet)

$$j : C \to \quad V = R \times P$$

  wich associates to every point a of C the pair (a, p_a) where p_a is the tangent of C at a.
- Legendrian lift.

- Conversely, if $\Gamma = (a, p) = (x, y(x), p(x))$ is a curve in V, the projection $a = (x, y(x))$ of Γ is a curve C in R. But Γ is the lifting of C iff $p(x) = y'(x)$.
- This is an integrability condition. It says that to be a coherent curve in V, Γ must be an integral curve of the contact structure of the fibration π.
- Legendrian curves generalize lifts of graphs.

- The condition is that at every point of Γ the tangent vector t is in the kernel of the differential 1-form

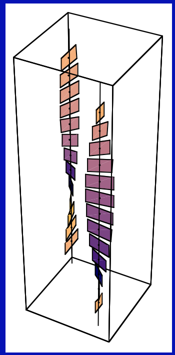
$$\omega = dy - pdx$$

  This kernel is the contact plane of V at (a, p).
- The underlying neural functional micro connectivity is expressed geometrically by a differential form.

- The vertical component p' of the tangent vector is the curvature :

$$p = y' \Rightarrow p' = y''$$

- The 2D contact distribution is not integrable. It has no integral surfaces but only integral curves.
- Indeed, $\omega \wedge d\omega = \text{volume form}$ while Frobenius integrability condition is $\omega \wedge d\omega = 0$.



- V1 is like a Lie-Cartan neural machine : 2D neural implementation of (at least) a contact structure.
- Understanding the geometrical content of functional architecture for understanding the neural origin of "external" geometry.
- Translate visual problems into problems of contact geometry.
- Even if the mathematical tools are rather elementary, the fact that they are neurally implemented is highly non trivial.

- For instance developement and learning can be translated into a problem of deformation of an initial functional architecture into a contact structure.
- We will focus on the problem of illusory contours (David Mumford).

- Some bibliography
  - 1999. *Vers une Neurogéométrie., Mathématiques, Informatique et Sciences Humaines*, 145, 5-101, EHESS, Paris.
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  - 2006. Neurogéométrie des architectures fonctionnelles de la vision, *Journée annuelle de la SMF*, 24 juin 2006, 69-128.
  - 2008, *Neurogéométrie de la vision. Modèles mathématiques et physiques des architectures fonctionnelles*, Les Editions de l'Ecole Polytechnique, Paris.

Towards neurogeometry

- The apparently trivial condition

$$\omega = dy - pdx = 0$$

contains in fact a rich geometry.

- It results from the action of a group.

Contact structure and Heisenberg group

- The contact structure on V is a left-invariant distribution of planes for a group structure which is the <u>polarized</u> Heisenberg group :

$$(x, y, p).(x', y', p') = (x + x', y + y' + px', p + p')$$

- If $t = (\xi, \eta, \pi)$ are the tangent vectors of $\mathfrak{V} = T_0V$, the Lie algebra of V has the Lie bracket

$$[t, t'] = [(\xi, \eta, \pi), (\xi', \eta', \pi')] = (0, \xi'\pi - \xi\pi', 0)$$

- $(0, 0, 0)$ is the neutral element.
- If $v = (x, y, p)$, v^{-1} (or $-v$ in additive notation) is $(-x, -y + px, -p)$.

- The Lie algebra $\mathfrak{V} = T_0 V$ is spanned by
  $X_1 = \partial_x + p\partial_y = (1, p, 0)$,
  $X_2 = \partial_p = (0, 0, 1)$,
  and
  $[X_1, X_2] = -X_3 = -\partial_y = (0, -1, 0)$
  (other brackets $= 0$).
- The contact plane are spanned by X_1 and X_2, and the contact distribution is therefore <u>bracket generating</u> (Hörmander condition).

- A consequence is <u>Chow theorem</u> : two points of V can always be joined by an integral curve.

- In matrix terms, $v = (x, y, p)$ and $t = (\xi, \eta, \pi)$ can be written

$$\begin{pmatrix} 1 & p & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & \pi & \eta \\ 0 & 0 & \xi \\ 0 & 0 & 0 \end{pmatrix}$$

- So the inner automorphisms are :

$$\begin{array}{llcl} A_v : & v' & \mapsto & v.v'.v^{-1} \\ & (x', y', p') & \mapsto & (x', y' + px' - p'x, p') \end{array}$$

- The tangent map of A_v at 0 is :

$$Ad_v = \begin{pmatrix} 1 & 0 & 0 \\ p & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}$$

$$Ad_v(t) = (\xi, p\xi + \eta - x\pi, \pi)$$

- This yields the <u>adjoint representation</u> of the Lie group V on its Lie algebra $\mathfrak{V} = T_0 V$.

- For the coadjoint representation, take the basis $\{dx, dy, dp\}$ for the 1-forms of $\mathfrak{V}^*$:

$$\gamma = \mu dx + \lambda dy + \nu dp \in \mathfrak{V}^*$$

- We get, using $\langle Ad_v^*(\gamma), t\rangle = \langle \gamma, Ad_{-v}(t)\rangle$

$$Ad_v^*(\gamma) = \mu' dx + \lambda' dy + \nu' dp$$

$$\begin{cases} \mu' = \mu - \lambda p \\ \lambda' = \lambda \\ \nu' = \nu + \lambda x \end{cases}$$

- Orbits :
  - If $\lambda \neq 0$, planes $\lambda = \mathrm{cst}$.
  - If $\lambda = 0$, every point of the $(\mu, 0, \nu)$ plane is a degenerate orbit.

Unitary irreducible representations

- The unitary irreducible representations (unirreps) of this group are given by the Stone - von Neumann theorem.

- The unirreps of V are either trivial ones of dimension 1 multiplying $z \in \mathbb{C}$ by

$$\pi_{\mu,\nu}(x, y, p) = e^{i(\mu x + \nu p)}$$

or infinite dimensional ones operating in the Hilbert space $L^2(\mathbb{R})$

$$\pi_\lambda(x, y, p)\, u(s) = e^{i\lambda(y + xs)} u(s + p), \text{ with } \lambda \neq 0$$

- Kirillov : they correspond to the orbits of the coadjoint representation of V.
- Planes $\lambda = \mathrm{cst}$ for $\lambda \neq 0$ correspond to

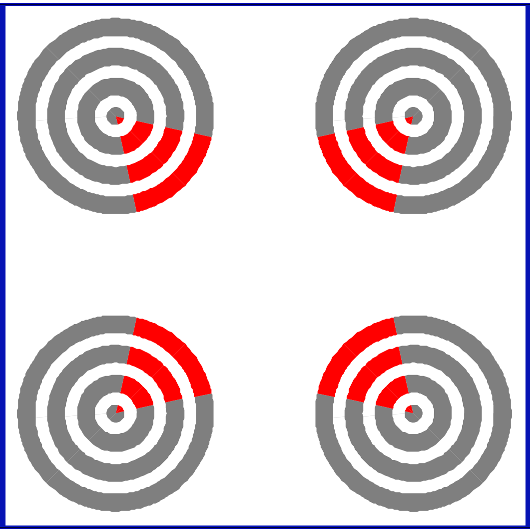
$$\pi_\lambda(x, y, p)\, u(s) = e^{i\lambda(y + xs)} u(s + p), \text{ with } \lambda \neq 0$$

- Points of the $(\mu, 0, \nu)$ plane for $\lambda = 0$ correspond to

$$\pi_{\mu,\nu}(x, y, p) = e^{i(\mu x + \nu p)}$$

The neurogeometrical problem of illusory contours

- A typical example of the problems of neuro-geometry is given by well known Gestalt phenomena such as Kanizsa illusory contours.
- The visual system (V1 with some feedback from V2) constructs very long range and crisp virtual contours.
- They are in fact boundaries of virtual *surfaces* but we will restrict to the 1D problem.



Sub-Riemannian geometry

- In this neuro-geometrical framework, we can easily interpret the variational process giving rise to illusory contours.
- The idea is to use a geodesic model in the sub-Riemannian geometry associated to the contact structure.
- This generalizes the "elastica" model proposed by David Mumford.

- We need also metrics and geodesics for analyzing diffusion and computing the heat kernel for this specific functional architecture.

- If $\mathcal{K}$ is the contact structure on V and if one considers only curves Γ in V which are integral curves of $\mathcal{K}$, then metrics $g_{\mathcal{K}}$ defined only on the planes of the distribution $\mathcal{K}$ are called <u>sub-Riemannian</u> metrics.

- In a Kanizsa figure, two pacmen of respective centers a and b with a specific aperture angle define two elements (a, p) and (b, q) of V.
- An illusory contour interpolating between (a, p) and (b, q) is
  - 1. a curve C from a to b in R with tangent p at a and tangent q at b ;
  - 2. a curve minimizing an "energy" (variational problem), that is a geodesic for some sub-Riemannian metric.

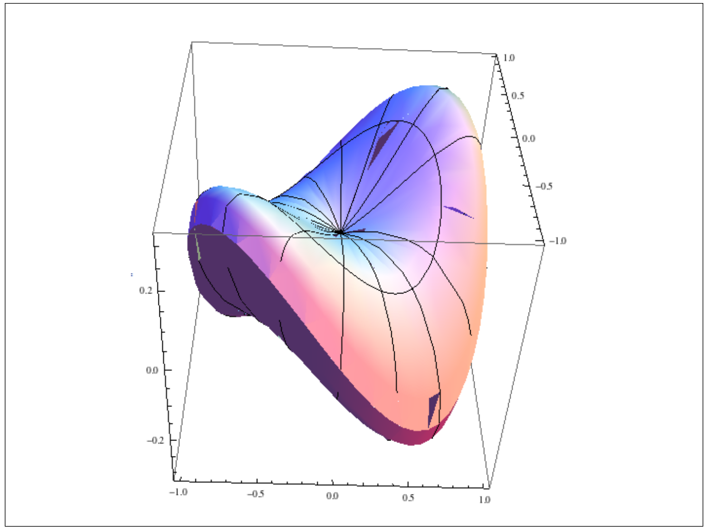
- It is natural to take on the contact planes the metric making orthonormal their generators :

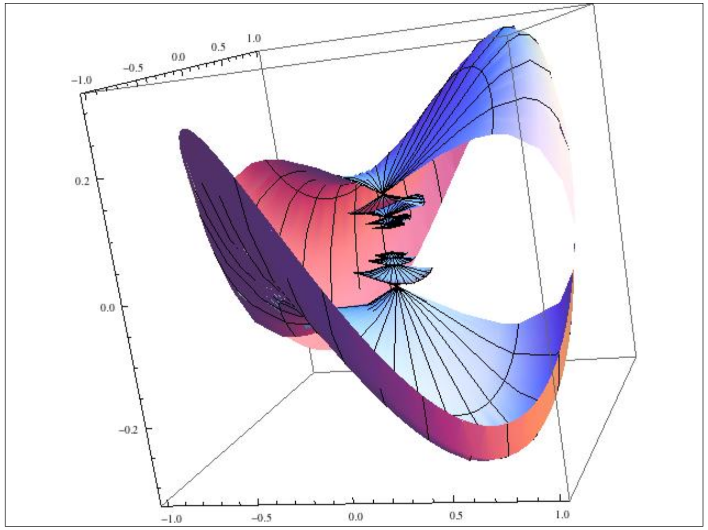
$$X_1 = \partial_x + p\partial_y , \quad X_2 = \partial_p .$$

- It is the Euclidean metric for X_2 whose Euclidean norm is 1, but not for X_1 whose Euclidean norm is $(1 + p^2)^{1/2}$ and not 1.

- We compute the sub-Riemannian sphere S and the wave front W (geodesics of SR length 1) (it is a variant of Beals, Gaveau, Greiner computations).

- Sphere $S(v, r) = \{\, w : d(v, w) = r$ (geodesics of length r that are global minimizers)$\}$.
- Wave front $W(v, r) = \{\, w : \exists$ a geodesic $\gamma : v \to w$ of length r (not necessarily a global minimizer)$\}$.
- Cut locus of $v = \{\, w : w$ end point of a geodesic $\gamma : v \to w$ which is no longer globally minimizing $\}$.
- Conjugate locus of v = caustic = Σ_v = $\{$ singular locus of the exponential $\mathscr{E}_v$ $\}$.





- Geodesics are projections on $V = \mathbb{R}^3$ of Hamiltonian trajectories of an Hamiltonian H defined on the cotangent bundle $T^*\mathbb{R}^3$.
- It is a consequence of Pontryagin maximum principle.

- H corresponds to the kinetic energy (ξ, η, π are the conjugate momenta of x, y, p).

$$H(x, y, p; \xi, \eta, \pi) =$$

$$1/2[((\xi, \eta, \pi)(X_1))^2 + ((\xi, \eta, \pi)(X_2))^2]$$

with $X_1 = (1, p, 0)$ and $X_2 = (0, 0, 1)$

$$H(x, y, p, \xi, \eta, \pi) = \frac{1}{2}\left[(\xi + p\eta)^2 + \pi^2\right]$$

- Hamilton equations are

$$\begin{cases} \dot{x}(s) = \frac{\partial H}{\partial \xi} = \xi + p\eta \\ \dot{y}(s) = \frac{\partial H}{\partial \eta} = p(\xi + p\eta) = p\dot{x}(s) \text{ i.e. } p = \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} \\ \dot{p}(s) = \frac{\partial H}{\partial \pi} = \pi \\ \dot{\xi}(s) = -\frac{\partial H}{\partial x} = 0 \\ \dot{\eta}(s) = -\frac{\partial H}{\partial y} = 0 \\ \dot{\pi}(s) = -\frac{\partial H}{\partial p} = -\eta(\xi + p\eta) = -\eta\dot{x}(s) \end{cases}$$

- The momenta ξ and η are constant since H is independent of x and y.

$$x_1 = \frac{|\sin(\varphi)|}{\varphi}\cos(\theta)$$

$$p_1 = \frac{|\sin(\varphi)|}{\varphi}\sin(\theta)$$

$$y_1 = \frac{\varphi + 2\sin^2(\varphi)\cos(\theta)\sin(\theta) - \cos(\varphi)\sin(\varphi)}{4\varphi^2}$$

Contact structure and Euclidean group

- With Alessandro Sarti and Giovanna Citti, we emphasized the fact that it is more natural to work with the fibration $\pi : V = \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2$ endowed with the contact form

$$\omega = -\sin(\theta)dx + \cos(\theta)dy$$

which is $\cos(\theta)(dy - pdx)$

- No privileged x-axis.

- The contact planes are spanned by

$$X_1 = \cos(\theta)\,\partial_x + \sin(\theta)\,\partial_y$$
$$X_2 = \partial_\theta$$

with Lie bracket

$$[X_1, X_2] = \sin(\theta)\,\partial_x - \cos(\theta)\,\partial_y = -X_3$$

- (Tangent vectors are interpreted as oriented derivatives.)
- This is a non-holonomic basis.

- V becomes the Euclidean group, which is the semidirect product $G =$

$$E(2) = SO(2) \ltimes \mathbb{R}^2$$

$$\begin{pmatrix} x_1 \\ y_1 \\ \theta_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2\cos(\theta_1) - y_2\sin(\theta_1) \\ y_1 + x_2\sin(\theta_1) + y_2\cos(\theta_1) \\ \theta_1 + \theta_2 \end{pmatrix}$$

- This group is not nilpotent and its tangent cone is the polarized Heisenberg group.

- By left invariance, the basis at 0

$$\{\partial_x, \partial_y, \partial_\theta\}_0$$

left translates into the non-holonomic basis

$$\{\cos(\theta)\,\partial_x + \sin(\theta)\,\partial_y = X_1, -\sin(\theta)\,\partial_x + \cos(\theta)\,\partial_y = X_3, \partial_\theta = X_2\}_q$$

and the covector at 0

$$\omega_0 = dy$$

left translates into the contact form ω.

Sub-Riemannian geometry of the Euclidean group E(2)

- For the non nilpotent Euclidean group, Andrei Agrachev and his group at the SISSA (Yuri Sachkov, Ugo Boscain, Igor Moiseev) solved the problem of SR geodesics and Sachkov compared it with Mumford's elastica model.

- One works with the fibration $V = \mathbb{R}^2 \times \mathbb{S}^1$ where the Legendrian lifts are solutions of the control system :

$$\begin{cases} \dot{x} = u_1 \cos(\theta) \\ \dot{y} = u_1 \sin(\theta) \\ \dot{\theta} = u_2 \end{cases}$$

- Let

$$p = (p_x, p_y, p_\theta) \in T_q^* V$$

be the momenta covectors.

- The Hamiltonian on T^*V for geodesics is

$$H(p,q) = \frac{1}{2}\left(u_1^2 + u_2^2\right) = \frac{1}{2}\left(\left(p_x \cos(\theta) + p_y \sin(\theta)\right)^2 + p_\theta^2\right)$$

and corresponds to the X_1 , X_2 basis .

- For θ small $= p$ and momenta ξ, η, π, we find again the polarized Heisenberg case :

$$H(x,y,p,\xi,\eta,\pi) = \frac{1}{2}\left[(\xi + p\eta)^2 + \pi^2\right]$$

- Hamilton equations are therefore :

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p_x} = p_x \cos^2(\theta) + p_y \cos(\theta)\sin(\theta) \\ \dot{y} = \frac{\partial H}{\partial p_y} = p_y \sin^2(\theta) + p_x \cos(\theta)\sin(\theta) \\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = p_\theta \end{cases}$$

$$\begin{cases} \dot{p}_x = -\frac{\partial H}{\partial x} = 0 \\ \dot{p}_y = -\frac{\partial H}{\partial y} = 0 \\ \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \left(p_x \cos(\theta) + p_y \sin(\theta)\right)\left(-p_x \sin(\theta) + p_y \cos(\theta)\right) \end{cases}$$

- The system can be explicitely integrated via elliptic functions.
- The sub-Riemannian geodesics are the projections of the integral curves on V.

Noncommutative harmonic analysis and SR geometry

- Using this geometrical analysis of the functional archictecture of V1, it is interesting to study the diffusion (heat kernel) and advection-diffusion (Fokker-Planck) processes on this subriemannian geometry of E(2).

- For the Heisenberg group, R. Beals, B. Gaveau, P. Greiner, D-Ch Chang, constructed the heat kernel.
- The problem is rather difficult since there are singularities (cut points) in every neighborhood of each point (B. Gaveau, IHP, 26-10-2005).

- One can use the non-commutative Fourier transform defined on the dual of the group G.
- For the polarized Heisenberg group V (1-jet space), the dual V^* of V is the set of unitary irreducible representations (unirreps) of V in the Hilbert space of functions

$$\{u(s)\} = L^2(\mathbb{R}, \mathbb{C}).$$

- We have seen that the unirreps of V are infinite dimensional ones (Stone - Von Neumann).

$$\pi_\lambda(x,y,p)\,u(s) = e^{i\lambda(y+xs)}u(s+p), \text{ with } \lambda \neq 0$$

- For $\lambda = 0$ they degenerate into trivial representations of dimension 1 : multiplication by

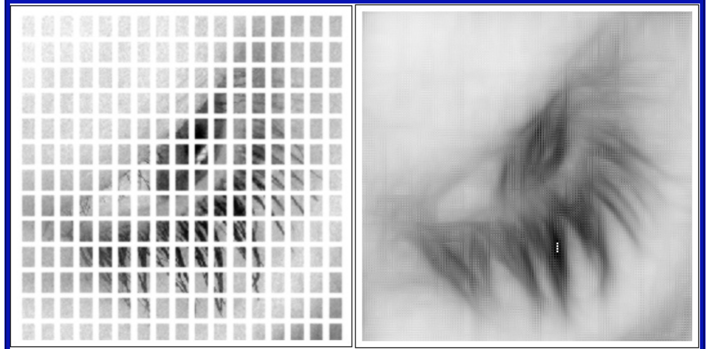
$$\pi_{\mu,\nu}(x,y,p) = e^{i(\mu x+\nu p)}$$

- Recently (2008), Andrei Agrachev, Ugo Boscain, Jean-Paul Gauthier and Francesco Rossi have found the heat kernel for $G = SE(2)$ and other unimodular groups.
- The hypo-elliptic Laplacian is the sum of squares of the bracket generating Lie subalgebra :

$$\Delta_K = X_1^2 + X_2^2$$

- The subriemannian diffusion on G is highly anisotropic since it is restricted to an angular diffusion of θ and a spatial diffusion only along the X_1 direction.
- It is a diffusion constrained by the "good continuation" constraint.
- Example :

- Completion image : Jean-Paul Gauthier.



- The dual G^* of G is the set of unitary irreducible representations of G in the Hilbert space $\{\psi(\theta)\} = \mathcal{H} = L^2(S^1, \mathbb{C})$
- If the elements of G are

$$g = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & x \\ \sin(\alpha) & \cos(\alpha) & y \\ 0 & 0 & 1 \end{pmatrix}$$

then the unirreps are parametrized by a positive real λ :

$$\begin{array}{llllll} \mathcal{X}^\lambda & : & G & \rightarrow & \mathcal{U}(\mathcal{H}) & \\ & & g & \mapsto & \mathcal{X}^\lambda(g) : & \mathcal{H} \rightarrow \mathcal{H} \\ & & & & & \psi(\theta) \mapsto e^{i\lambda(x\sin(\theta)+y\cos(\theta))}\psi(\theta+\alpha) \end{array}$$

- This means that to every element g of G one associates an automorphism $\mathcal{X}^\lambda(g)$ of the Hilbert space $\mathcal{H}$.
- Such an automorphism associates to each function $\psi(\theta)$ in $\mathcal{H}$ another function in $\mathcal{H}$.

- There exists a measure on G^*, the Plancherel measure, given by $dP(\lambda) = \lambda d\lambda$, which enables to make integrations.
- To compute the Fourier transform of the sub-Riemannian Laplacian we have to look at the action of the differential of the unirreps on the left-invariant vector fields X.
- These X are given by the left translation of vectors of the Lie algebra $\mathfrak{g}$ of G.

- By definition,

$$d\mathcal{X}^\lambda : X \rightarrow d\mathcal{X}^\lambda(X) := \left.\frac{d}{dt}\right|_{t=0} \mathcal{X}^\lambda\left(e^{tX}\right)$$

and

$$\widehat{X_i}^\lambda = d\mathcal{X}^\lambda(X_i)$$

- It is easy to apply these formulas.

$$\psi(\theta) \mapsto e^{i\lambda(x\sin(\theta)+y\cos(\theta))}\psi(\theta+\alpha)$$

$$\begin{aligned}
X_1 &= (1,0,0) \\
e^{tX_1} &= (t,0,0) \\
\mathcal{X}^\lambda\left(e^{tX_1}\right)\psi(\theta) &= e^{i\lambda t\sin(\theta)}\psi(\theta) \\
\widehat{X_1}^\lambda\psi(\theta) &= d\mathcal{X}^\lambda(X_1)\psi(\theta) = \left.\frac{d}{dt}\right|_{t=0}\mathcal{X}^\lambda\left(e^{tX_1}\right)\psi(\theta) \\
&= \left.\frac{d}{dt}\right|_{t=0} e^{i\lambda t\sin(\theta)}\psi(\theta) = i\lambda\sin(\theta)\psi(\theta)
\end{aligned}$$

$$\psi(\theta) \mapsto e^{i\lambda(x\sin(\theta)+y\cos(\theta))}\psi(\theta+\alpha)$$

$$\begin{aligned}
X_2 &= (0,0,1) \\
e^{tX_2} &= (0,0,t) \\
\mathcal{X}^\lambda\left(e^{tX_2}\right)\psi(\theta) &= \psi(\theta+t) \\
\widehat{X_2}^\lambda\psi(\theta) &= d\mathcal{X}^\lambda(X_2)\psi(\theta) = \left.\frac{d}{dt}\right|_{t=0}\mathcal{X}^\lambda\left(e^{tX_2}\right)\psi(\theta) \\
&= \left.\frac{d}{dt}\right|_{t=0}\psi(\theta+t) = \frac{d\psi(\theta)}{d\theta}
\end{aligned}$$

- The GFT of the sub-Riemannian Laplacian is therefore the Hilbert sum (integral on λ with the Plancherel measure $dP(\lambda) = \lambda d\lambda$) of the $\widehat{\Delta_K}^\lambda$

  with

$$\widehat{\Delta_K}^\lambda\psi(\theta) = \left(\left(\widehat{X_1}^\lambda\right)^2 + \left(\widehat{X_2}^\lambda\right)^2\right)\psi(\theta) = \frac{d^2\psi(\theta)}{d\theta^2} - \lambda^2\sin^2(\theta)\psi(\theta)$$

  which is the *Mathieu equation*.

- The heat kernel is

$$P(g,t) = \int_{G^*} Tr\left(e^{t\widehat{\Delta_K}^\lambda}\mathcal{X}^\lambda(g)\right)dP(\lambda),\ t \geq 0$$

- For small angles we find the equation

$$\hat{\Delta}^{\lambda} : y''(s) - \lambda^2 s^2 y(s)$$

which gives the Mehler kernel.

Confluence

- We can construct an interpolation between the E(2) model and the H(3) model.
- It corresponds to a confluence of singularities between the two associated equations.
- See e.g. Dominique Manchon.

$$X_1 = \cos(\theta)\,\partial_x + \sin(\theta)\,\partial_y$$
$$X_2 = \partial_\theta$$
$$X_3 = -\sin(\theta)\,\partial_x + \cos(\theta)\,\partial_y$$
$$[X_1, X_2] = -X_3$$
$$[X_2, X_3] = X_1$$
$$[X_1, X_3] = 0$$
$$E(2) \text{ with } S^1 = \frac{\mathbb{R}}{2\pi\mathbb{Z}}$$
$$X_1(\psi(\theta)) = i\lambda \sin(\theta)\,\psi(\theta)$$
$$X_2(\psi(\theta)) = \psi'(\theta)$$
$$\hat{\Delta}^{\lambda} : \psi''(\theta) - \lambda^2 \sin^2(\theta)\,\psi(\theta)$$
$$\psi''(\theta) + (\mu - \lambda^2)\sin^2(\theta)\,\psi(\theta) = 0$$
$$\sin^2(\theta) \to t$$
$$t(1-t)\,y''(t) + \tfrac{1}{2}(1-2t)\,y'(t) + \\ + \tfrac{1}{4}(\mu - \lambda^2 t)\,y(t) = 0$$
3 sing.: $0, 1$ regular, ∞ irregular

$$X_1^0 = \partial_x + p\partial_y$$
$$X_2^0 = \partial_p$$
$$X_3^0 = \partial_y$$
$$[X_1^0, X_2^0] = -X_3^0$$
$$[X_2^\alpha, X_3^\alpha] = 0$$
$$[X_1^\alpha, X_3^\alpha] = 0$$
$$H(3) \text{ with } S_0^1 = \mathbb{R}$$
$$X_1^0(y(s)) = i\lambda s y(s)$$
$$X_2^0(y(s)) = y'(s)$$
$$\hat{\Delta}^{\lambda} : y''(s) - \lambda^2 s^2 y(s)$$
$$y''(s) + (\mu - \lambda^2)\,s^2 y(s) = 0$$
$$s^2 \to t$$
$$t y''(t) + \tfrac{1}{2} y'(t) + \\ + \tfrac{1}{4}(\mu - \lambda^2 t)\,y(t) = 0$$
2 sing.: 0 regular, $\alpha^{-2} = \infty$ irregular
CONFLUENCE

$$X_1^\alpha = \cos(\theta)\,\partial_x + \tfrac{1}{\alpha}\sin(\alpha\theta)\,\partial_y$$
$$X_2^\alpha = \partial_\theta$$
$$X_3^\alpha = -\alpha\sin(\alpha\theta)\,\partial_x + \cos(\theta)\,\partial_y$$
$$[X_1^\alpha, X_2^\alpha] = -X_3^\alpha$$
$$[X_2^\alpha, X_3^\alpha] = \alpha^2 X_1^\alpha$$
$$[X_1^\alpha, X_3^\alpha] = 0$$
$$E_\alpha(2) \text{ with } S_\alpha^1 = \frac{\mathbb{R}}{2\pi\alpha^{-1}\mathbb{Z}}$$
$$X_1^\alpha(\psi(\theta)) = i\lambda\alpha^{-1}\sin(\alpha\theta)\,\psi(\theta)$$
$$X_2^\alpha(\psi(\theta)) = \psi'(\theta)$$
$$\hat{\Delta}^{\lambda} : \psi''(\theta) - \frac{\lambda^2}{\alpha^2}\sin^2(\alpha\theta)\,\psi(\theta)$$
$$\psi''(\theta) + \left(\mu - \frac{\lambda^2}{\alpha^2}\sin^2(\alpha\theta)\right)\psi(\theta) = 0$$
$$\frac{\sin^2(\alpha\theta)}{\alpha^2} \to t$$
$$t(1-\alpha^2 t)\,y''(t) + \tfrac{1}{2}(1-2\alpha^2 t)\,y'(t) + \\ + \tfrac{1}{4}(\mu - \lambda^2 t)\,y(t) = 0$$
3 sing.: $0, \alpha^{-2}$ regular, ∞ irregular

$X_1 = \cos(\theta)\,\partial_x + \sin(\theta)\,\partial_y$	$X_1^\alpha = \cos(\theta)\,\partial_x + \frac{1}{\alpha}\sin(\alpha\theta)\,\partial_y$	$X_1^0 = \partial_x + p\partial_y$
$X_2 = \partial_\theta$	$X_2^\alpha = \partial_\theta$	$X_2^0 = \partial_p$
$X_3 = -\sin(\theta)\,\partial_x + \cos(\theta)\,\partial_y$	$X_3^\alpha = -\alpha\sin(\alpha\theta)\,\partial_x + \cos(\theta)\,\partial_y$	$X_3^0 = \partial_y$
$[X_1, X_2] = -X_3$	$[X_1^\alpha, X_2^\alpha] = -X_3^\alpha$	$[X_1^0, X_2^0] = -X_3^0$
$[X_2, X_3] = X_1$	$[X_2^\alpha, X_3^\alpha] = \alpha^2 X_1^\alpha$	$[X_2^\alpha, X_3^\alpha] = 0$
$[X_1, X_3] = 0$	$[X_1^\alpha, X_3^\alpha] = 0$	$[X_1^\alpha, X_3^\alpha] = 0$
$E(2)$ with $S^1 = \frac{\mathbb{R}}{2\pi\mathbb{Z}}$	$E_\alpha(2)$ with $S_\alpha^1 = \frac{\mathbb{R}}{2\pi\alpha^{-1}\mathbb{Z}}$	$H(3)$ with $S_0^1 = \mathbb{R}$
$X_1(\psi(\theta)) = i\lambda\sin(\theta)\,\psi(\theta)$	$X_1^\alpha(\psi(\theta)) = i\lambda\alpha^{-1}\sin(\alpha\theta)\,\psi(\theta)$	$X_1^0(y(s)) = i\lambda s y(s)$
$X_2(\psi(\theta)) = \psi'(\theta)$	$X_2^\alpha(\psi(\theta)) = \psi'(\theta)$	$X_2^0(y(s)) = y'(s)$
$\hat{\Delta}^\lambda : \psi''(\theta) - \lambda^2\sin^2(\theta)\,\psi(\theta)$	$\hat{\Delta}^\lambda : \psi''(\theta) - \frac{\lambda^2}{\alpha^2}\sin^2(\alpha\theta)\,\psi(\theta)$	$\hat{\Delta}^\lambda : y''(s) - \lambda^2 s^2 y(s)$
$\psi''(\theta) + (\mu - \lambda^2)\sin^2(\theta)\,\psi(\theta) = 0$	$\psi''(\theta) + \left(\mu - \frac{\lambda^2}{\alpha^2}\sin^2(\alpha\theta)\right)\psi(\theta) = 0$	$y''(s) + (\mu - \lambda^2)\, s^2 y(s) = 0$
$\sin^2(\theta) \to t$	$\frac{\sin^2(\alpha\theta)}{\alpha^2} \to t$	$s^2 \to t$
$t(1-t)\,y''(t) + \frac{1}{2}(1-2t)\,y'(t) +$	$t(1-\alpha^2 t)\,y''(t) + \frac{1}{2}(1-2\alpha^2 t)\,y'(t) +$	$ty''(t) + \frac{1}{2}y'(t) +$
$+\frac{1}{4}(\mu - \lambda^2 t)\,y(t) = 0$	$+\frac{1}{4}(\mu - \lambda^2 t)\,y(t) = 0$	$+\frac{1}{4}(\mu - \lambda^2 t)\,y(t) = 0$
3 sing.: $0, 1$ regular, ∞ irregular	3 sing.: $0, \alpha^{-2}$ regular, ∞ irregular	2 sing.: 0 regular, $\alpha^{-2} = \infty$ irregular