The Structure of n-tuple groupoids and the Poincaré double group

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Part I

The Double Paradigm

Double Categories

Definition

A (small) **double category** *is an internal category in* **Cat***.*

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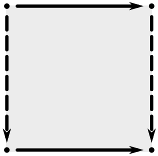
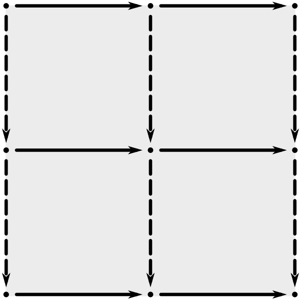
A general element in a double category is a square :



And it composes associatively in two directions, with two different units.

Moreover the two compositions interchange , i.e. the following diagram has a unique composition :

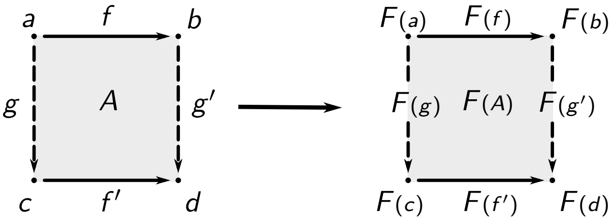


Double Functors

Definition

A **double functor** *is an internal functor in* **Cat**

It maps squares to squares respecting boundaries, units and composition.



Double Natural Transformations

Definition

A **horizontal double natural transformation** *is an internal natural transformation in* **Cat**

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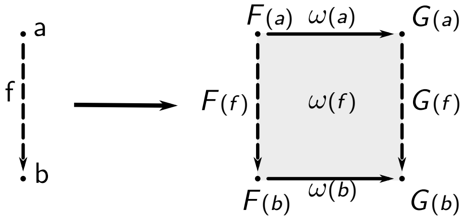
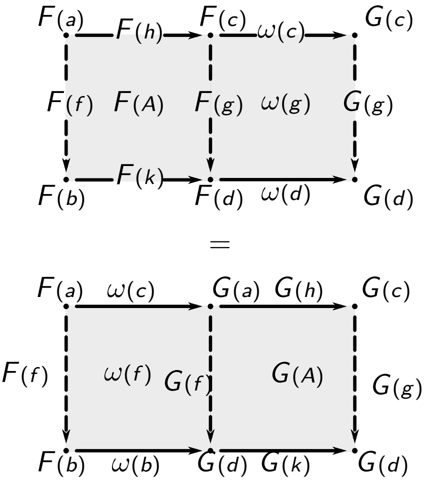
It associates squares to vertical morphisms :



In such a way that it intertwines functors horizontally :



Definition

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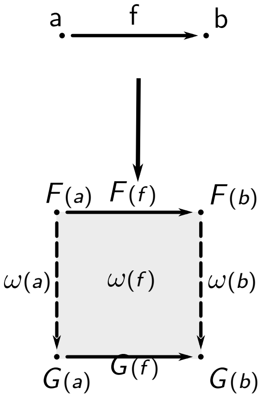
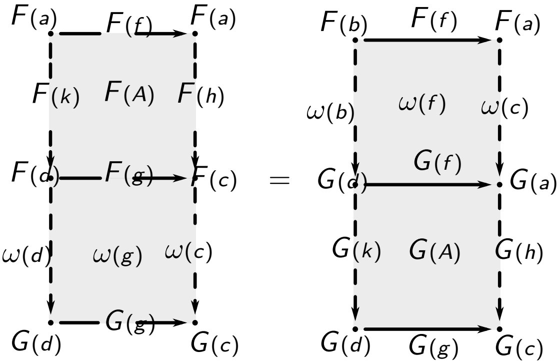


In such a way that it intertwines functors vertically :



Double Comparison

Definition

A **double comparison** *is an internal comparison in* **Cat**.

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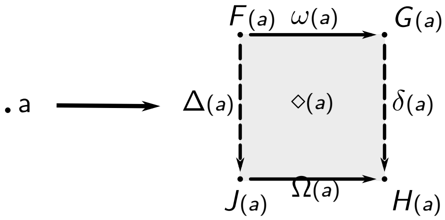
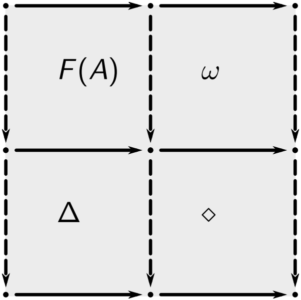
It associates squares to objects :



In such a way that all the following are equal :



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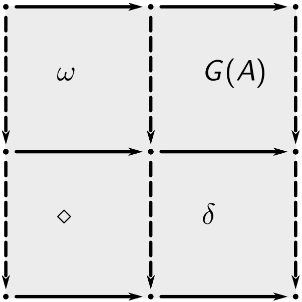

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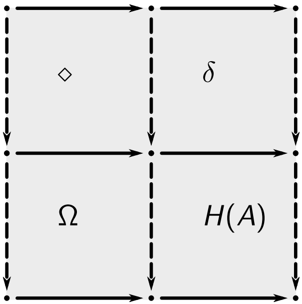

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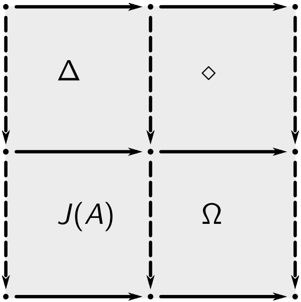

Degeneracies

As there are 2 degenerate versions of a segment, there are 2 degenerate forms of categories :

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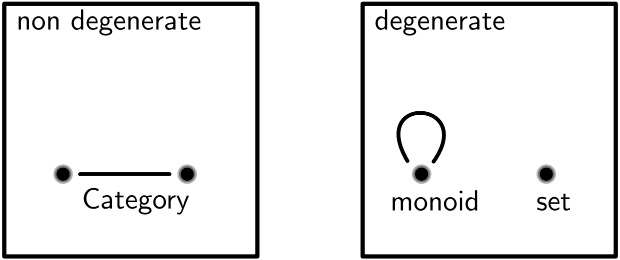

As there are 7 degenerate squares, there are 7 degenerate forms of double categories, only 4 of which are new :

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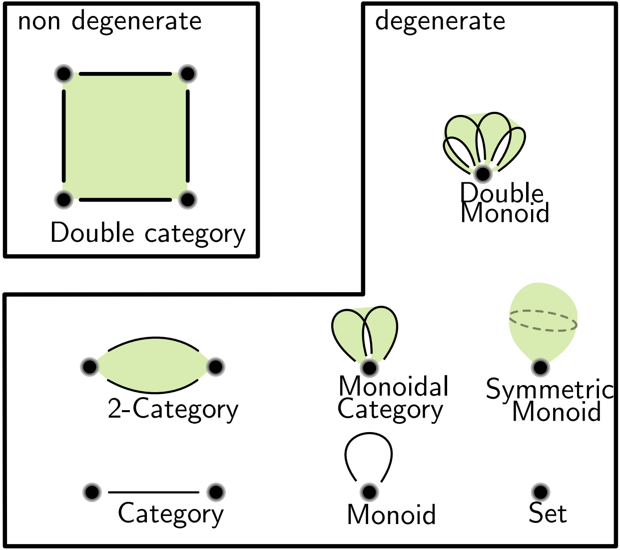


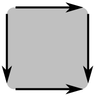
Part II

The Structure of Double Groups

Definition

A **double groupoid** *is a (strict) double category where every square has both horizontal and vertical inverses. A* **double group** *is a double groupoid with a single object.*

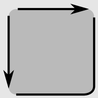
To save time and space, we will from now on not draw the objects :



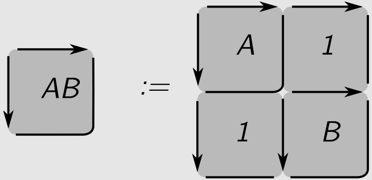
The Core

Definition

The **core groupoid** $\tau_\lrcorner$ *of a double groupoid* τ *is the diagonal groupoid of elements of* τ *whose targets are identities. These are squares of the form :*

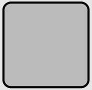


whose multiplication is defined by :



Definition

The **core bundle** $\tau_\bullet$ *of a double groupoid* τ *is the sub groupoid of* $\tau_{\lrcorner}$ *whose boundaries are all identities. It is a group bundle over the objects whose elements are squares are of the form :*

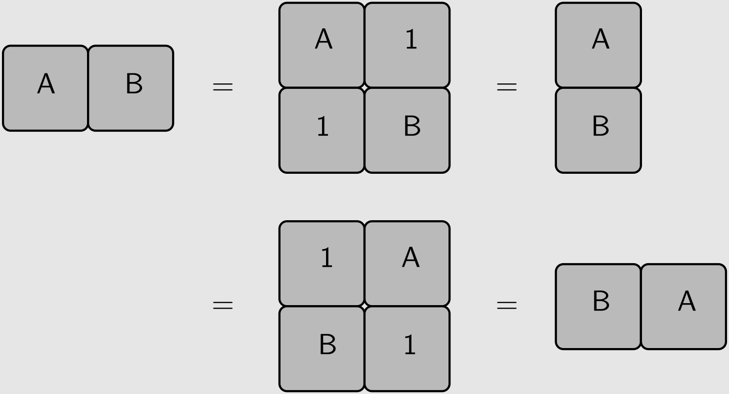


Lemma

The core bundle of a double groupoid is an abelian group bundle over its objects.

Proof.

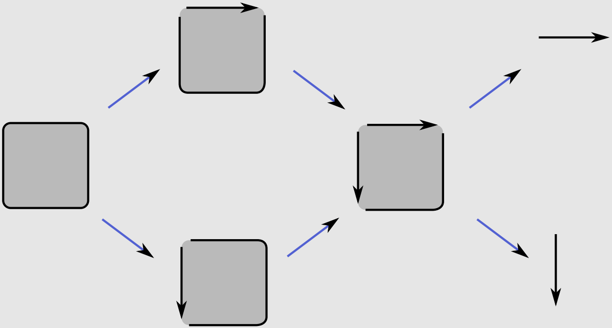
This is the celebrated Eckmann-Hilton argument, which goes as follows :



□

Definition

The **core diagram** *of a double groupoid is the following diagram of groupoids :*



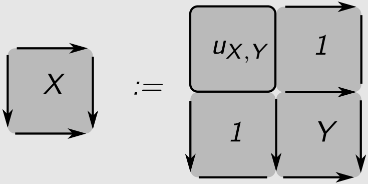
Definition

A double groupoid is **slim** *if its core bundle is the trivial bundle. It is* **exclusive** *if its core groupoid is equal to its core bundle.*

Slim double groupoids have at most one square per boundary condition, as the following lemma shows :

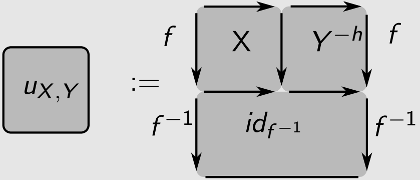
Lemma

Let τ be a double groupoid, $X, Y \in \tau$ with the same boundary. Then there exist a unique element $u_{X,Y}$ in the core bundle of τ such that :



Proof.

Defining $u_{X,Y}$ by :

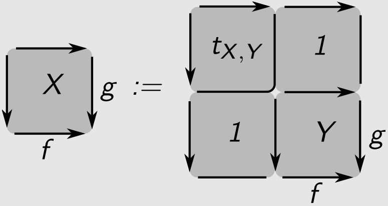


and using inveses to isolate X yields the claim. □

Exclusive double groupoids have their boundary fixed by the knowledge of one of its horizontal and one of its vertical component, as shows the following lemma shows :

Lemma

Let τ be a double groupoid, $X, Y \in \tau$ with the same targets. Then there exist a unique element $t_{X,Y}$ in the core groupoid of τ such that :



Proof.

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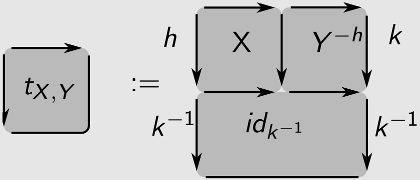


and using inveses to isolate X yields the claim. □

Bicrossed products

When both conditions are present, a familiar structure of group is recovered : bicrossed products.

Theorem (Andruskiewitsch Natale '09)

Maximal, slim and exclusive double groups are equivalent to bicrossed products of groups.

Let's recall what the bicrossed product of groups is :

Definition

Let H and K be groups, then a bicrossed product $H \bowtie K$ is a group defined on the set $H \times K$ by the following multiplication :

$$(h,k)(h',k') = (h(k \triangleright h'), k^{h'}k')$$

where $\triangleright$ is a left action of K on H and $(\ )^K$ is a right action of H on K such that :

$$k \triangleright (hh') = (k \triangleright h)(k^h \triangleright h')$$
$$(kk')^h = k^{k' \triangleright h} k'^h$$
$$k \triangleright 1 = 1$$
$$1^h = 1$$

Other names for the bicrossed products of groups are :
knit product, Zappa-Szep product or matched pairs of groups.
They emerge in the following situations:

Lemma

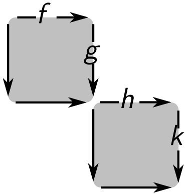
Let H an K be subgroups of G such that every element $g \in G$ can be uniquely written as :

$$g = hk$$

Then there exist a bicrossed product on $H \times K$ such that

$$G \simeq H \bowtie K$$

In the case that every pair of arrows sharing a corner bound a square (maximality), then a diagonal groupoid exists, given by :



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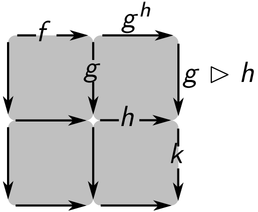

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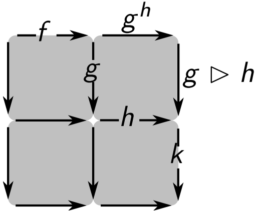
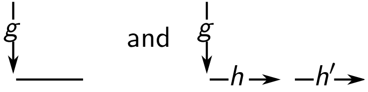


which is the bicrossed product, with conditions given by (for the right action):



Removing the assumption of being slim, we get :

Theorem (Majard '11)

Maximal exclusive double groups are equivalent to semi-direct products of an abelian group with a matched pair of groups.

Examples

The Poincaré Group

The first interesting example is the Poincare Group. Indeed, the Iwasawa decomposition of $SO(3,1)$ tells us that

Lemma

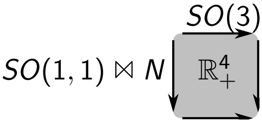
The poincare group is isomorphic to the following group :

$$Poinc \simeq (SO(3) \bowtie (SO(1,1) \bowtie N)) \ltimes \mathbb{R}^4_+$$

where :

$$N := exp\left\{ \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & a & b \\ a & -a & 0 & 0 \\ b & -b & 0 & 0 \end{bmatrix} \quad | a, b \in \mathbb{R} \right\}$$

It therefore corresponds to a unique double group :



Hyperplatonic solids

Another interesting connection is given by hyperplatonic solids, or platonic solids in 4 dimension. These are given by finite subgroups of SO(4). Or $SU(2) \times SU(2)$ is a double cover of $SO(4)$

Part III

General Case

N-tuple Categories

Definition

*An n-**tuple category** is an internal category in the category of $(n-1)$-tuple categories.*

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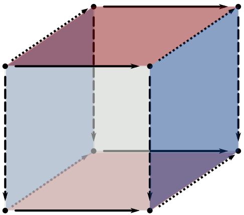
Its elements are n-cubes that compose associatively and with unit in all n directions and the interchange law is valid for any pair of directions.



Definition

*An n-**tuple groupoid** is an n-tuple category whose n-cubes are invertible in all directions. An n-**tuple group** is an n-tuple groupoid on one object.*

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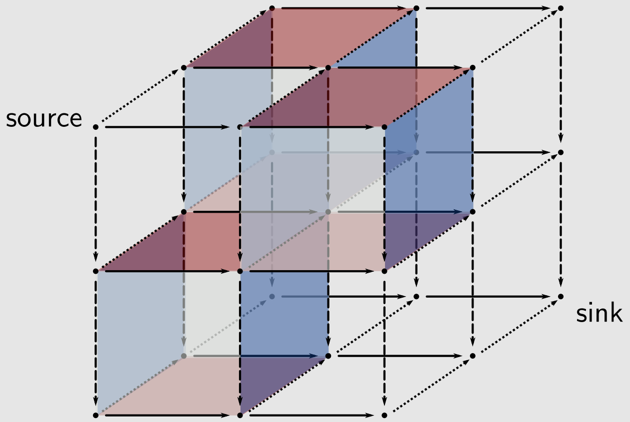
Definition

An arrangement of n-cubes combinatorially equivalent to the one given by excluding the subspaces $x_i = \frac{1}{2}$ *for all* $i \in [n]$ *from* $[0,1]^n \subset \mathbb{R}^n$ *is called a* **barycentric subdivision** *of the n-cube.*

Depth

The **depth** of a cube in a barycentric division has to do with its position with respect to the source object.
For example, here are the cubes of depth 1 in a barycentric division of dimension 3.



The structure of *n*-tuple groups

Definition

Let τ be an n-tuple groupoid and define

$$\begin{aligned}\tau_{\lrcorner} :&= \{n\text{-}cubes\ whose\ recursive\ targets\ are\ identities\} \\ &= \{X \in \tau \mid t_i(X) = \iota_{\hat{i}}(t_{[n]}(X))\}\end{aligned}$$

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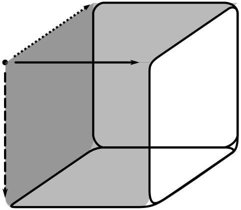
For example, in dimension 3:



Definition

For $u \in \tau_\lrcorner$ *and* $X \in \tau$ *such that* $t_{[n]}(u) = s_{[n]}(X)$ *define the* **transmutation of** X **by** u, *denoted* $u \cdot X$, *to be the n-cube accepting a barycentric subdivision with* u *of depth 0,* X *of depth* n *and all others identities, as defined above.*

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For $u \in \tau_\lrcorner$ and $X \in \tau$ such that $t_{[n]}(u) = s_{[n]}(X)$ define the **transmutation of** X **by** u*, denoted $u \cdot X$, to be the n-cube accepting a barycentric subdivision with u of depth 0, X of depth n and all others identities, as defined above.*

Lemma

Let $u, v \in \tau_\lrcorner$, then $u \cdot v \in \tau_\lrcorner$. Moreover $(\tau_\lrcorner, \cdot, \imath)$ is a groupoid, , called the **core groupoid***.*

Lemma

Let $X, Y \in \tau$ such that $t_i(X) = t_i(Y), \forall i$, then there exists a unique element $u_{XY} \in \tau_\lrcorner$ such that

$$X = u_{XY} \cdot Y$$

Definition

Let $\tau_\bullet$ be the sub groupoid of $\tau_\lrcorner$ composed of n-cubes whose boundaries are all identities. It is called the **core bundle**.

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Definition

A n-tuple groupoid is **slim** *if its core bundle is trivial*

Corrolary

A n-tuple groupoid is **slim** *iff there is at most one n-cube per boundary condition.*

Definition

A n-tuple groupoid is **exclusive** *if* $\tau_{\lrcorner} = \tau_{\bullet}$.

Definition

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Corrolary

A n-tuple groupoid is exclusive if and only if the boundary of its n-cubes are determined by one of their boundaries of each type.

Definition

An n-tuple groupoid τ is **maximal** *if for any $(f_1, f_2, \cdots, f_n)$ s.t. $f_i \in \tau_i$ and $t_i(f_i) = t_j(f_j) \quad \forall i, j \in \mathbb{Z}_n$ there exists $X \in \tau$ s.t.*

$$s_{1\cdots(i-1)} t_{(i+1)\cdots n}(X) = f_i$$

Definition

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$$s_{1\cdots(i-1)}t_{(i+1)\cdots n}(X) = f_i$$

Definition

An n-tuple groupoid is **maximally exclusive** *if*

- *all boundary i-tuple groupoids are slim for* $i > 1$
- *all boundary double groupoids are exclusive*
- *it is maximal*

Theorem (Majard '11)

Maximally exclusive n-groups are equivalent to semi-direct products of an abelian group with an iterated bicrossed product of groups.

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For example in dimension 3, groups of the form:

$$G \simeq (H_1 \bowtie H_2 \bowtie H_3) \ltimes A$$

Conclusion

Thank you !